PSET 3 solutions

1)

$$f(x,y) = x^4 + y^4 - 4xy$$

$$\nabla f(x,y) = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle$$

Critical point has $\nabla f = 0$

$$\left.\begin{array}{l} x^3 - y = 0 \\ y^3 - x = 0 \end{array}\right\} \Rightarrow y = x^3$$

So $\quad x^9 - x = 0$

$$x(x^8 - 1) = 0$$

Factoring

$$x(x^4-1)(x^4+1) = 0$$

$$x(x^2-1)(x^2+1)(x^4+1) = 0$$

$$x(x-1)(x+1)(x^2+1)(x^4+1) = 0 \quad \Rightarrow \quad x=0 \text{ or } x=1 \text{ or } x=-1$$

Critical points are at

$$(0,0),\ (1,1),\ (-1,-1)$$

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

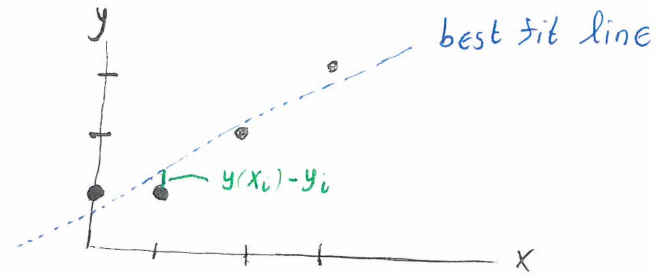
$\det H = 144x^2y^2 - 16.$

At $(0,0)$, $\det H = -16 < 0.$ Saddle

At $(1,1)$ $\det H > 0$ & $f_{xx} > 0$ Min

At $(-1,-1)$ $\det H > 0$ & $f_{xx} > 0$ Min.

2)



Let best fit line have form $y = mx + b$

$$\min_{m,b} \sum_{i=1}^{4} (y_i - y(x_i))^2$$

Want to solve $$\min_{m,b} (\underbrace{1}_{y_1} \underbrace{- b}_{-y(x_1)})^2 + (\underbrace{1}_{y_2} \underbrace{- m - b}_{-y(x_2)})^2 + (\underbrace{2}_{y_3} \underbrace{- 2m - b}_{-y(x_3)})^2 + (\underbrace{3}_{y_4} \underbrace{- 3m - b}_{-y(x_4)})^2$$

Let $f(m,b) = (1-b)^2 + (1-m-b)^2 + (2-2m-b)^2 + (3-3m-b)^2$

Simplify by negating inside of each square

$$f(m,b) = (b-1)^2 + (m+b-1)^2 + (2m+b-2)^2 + (3m+b-3)^2$$

$$\begin{aligned} \partial_m f &= 2(m+b-1) + 2(2m+b-2)2 + 2(3m+b-3)3 \\ &= 4 \cdot (7m + 3b - 7) \end{aligned}$$

$$\begin{aligned} \partial_b f &= 2(b-1) + 2(m+b-1) + 2(2m+b-2) + 2(3m+b-3) \\ &= 2(6m + 4b - 7) \end{aligned}$$

Critical point where $\partial_m f = 0$ $\partial_b f = 0$

$$\begin{aligned} 7m + 3b &= 7 \\ 6m + 4b &= 7 \end{aligned} \implies m = b = 7/10$$

Best fit line is $\boxed{y = \frac{7}{10}x + \frac{7}{10}}$

3)

Choose variables r, h

Surface area $= 2\pi r^2 + 2\pi r h$
Volume $= \pi r^2 h$

max $\pi r^2 h$ subject to $2\pi r^2 + 2\pi r h = S$

Simplify

max $r^2 h$ subject to $r^2 + rh = \hat{S}$ where $\hat{S} = S/2\pi$

$$\mathcal{L}(r,h,\lambda) = r^2 h + \lambda(r^2 + rh - \hat{S})$$

$$\partial_r \mathcal{L} = 2rh + \lambda(2r+h) = 0 \implies \lambda = \frac{-2rh}{2r+h}$$

$$\partial_h \mathcal{L} = r^2 + \lambda r = 0 \implies \lambda = -r$$

So

$$-r = \frac{-2rh}{2r+h}$$

$$1 = \frac{2h}{2r+h}$$

$$2r+h = 2h$$

$$\boxed{2r = h}$$ ← height is twice radius!

Plugging into constraint

$$2\pi r^2 + 2\pi r(2r) = S$$

$$6\pi r^2 = S \implies r = \sqrt{S/6\pi}$$

$$\boxed{r = \sqrt{S/6\pi} \quad h = 2\sqrt{\frac{S}{6\pi}}}$$

4)

a) $\min \; I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3$ subject to $I_1 + I_2 + I_3 = I$

Remove constraint, $I_3 = I - I_1 - I_2$

$$\min \; \underbrace{I_1^2 R_1 + I_2^2 R_2 + (I - I_1 - I_2)^2 R_3}_{\text{Call this } f(I_1, I_2)} \quad \text{without constraint}$$

$$\partial_{I_1} f = 2 I_1 R_1 - 2(I - I_1 - I_2) R_3 = 0$$

$$\partial_{I_2} f = 2 I_2 R_2 - 2(I - I_1 - I_2) R_3 = 0$$

We see that at a critical point

$$2 I_1 R_1 = 2(I - I_1 - I_2) R_3 = 2 I_2 R_2$$

So $\quad I_1 R_1 = I_3 R_3 = I_2 R_2.$

4 b)

$$\min \underbrace{I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3}_{f(I_1, I_2, I_3)} \quad st \quad \underbrace{I_1 + I_2 + I_3 - I}_{g(I_1, I_2, I_3)} = 0$$

$$\mathcal{L}(I_1, I_2, I_3, \lambda) = I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3 + \lambda(I_1 + I_2 + I_3 - I)$$

$$\left.\begin{aligned} \partial_{I_1} \mathcal{L} &= 2 I_1 R_1 + \lambda = 0 \\ \partial_{I_2} \mathcal{L} &= 2 I_2 R_2 + \lambda = 0 \\ \partial_{I_3} \mathcal{L} &= 2 I_3 R_3 + \lambda = 0 \end{aligned}\right\} \quad I_1 R_1 = I_2 R_2 = I_3 R_3 = -\lambda.$$