Lecture 6

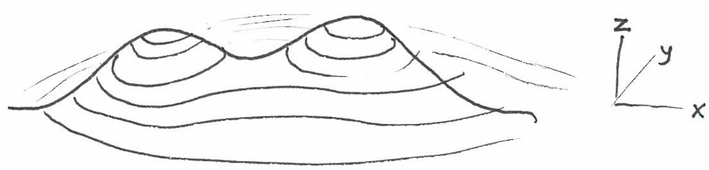
7/14/2014

Unconstrained Optimization

Constrained Optimization w/ Lagrange Multipliers

Activity:

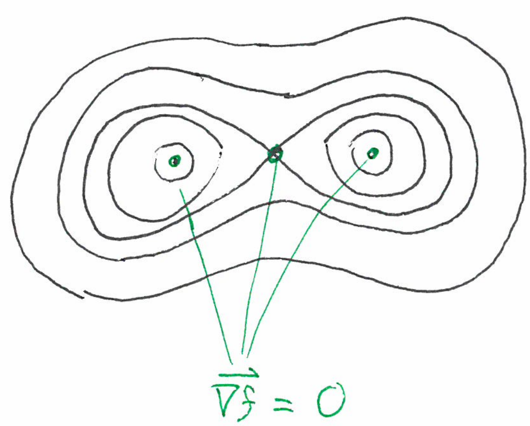
Let $z = f(x,y)$ describe height of two near by mountains



a) - Plot level sets of f. Make sure to include the most interesting level set.

b) - Draw where $\vec{\nabla} f = 0$.

a)



b)

At two peaks, $\partial_x f = 0$
$\partial_y f = 0$

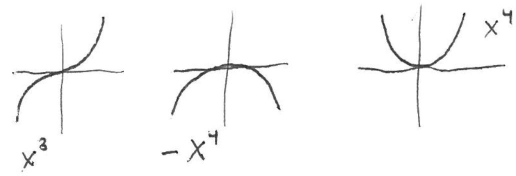
At middle-point, ∇f must be $\perp$ to both directions along level set. Only option is 0.

Optimization in 1d:

If $f'(x)=0$ and $f''(x)>0$, then x is local min

If $f'(x)=0$ & $f''(x)<0$, then x is local max

If $f'(x)=0$ & $f''(x)=0$ Inconclusive



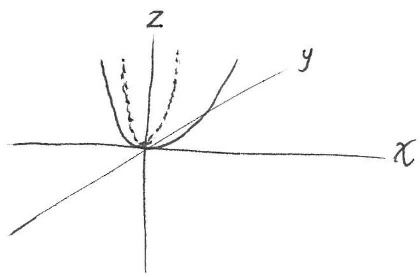
Activity: $f(0,0) = 0$ and

Suppose $\vec{\nabla} f(0,0) = 0$ and $\partial_{xx} f(0,0) = 2$

$\partial_{yy} f(0,0) = 2$

Sketch what this means visually.

Is $(0,0)$ necessarily a minimizer?



Not necessarily minimizer

example $f = (x-y)^2 - xy$.

$\partial_{xx} f = 2$

$\partial_{yy} f = 2$

$\nabla f(0,0) = \langle 0,0 \rangle$

$f(\varepsilon,\varepsilon) = -\varepsilon^2 < 0.$ $\Rightarrow$ $(0,0)$ is not minimizer

curves down in $\langle 1,1 \rangle$ direction

~~Unconstrained Optimization Problems~~

~~To find min/max of~~ $\cancel{f(x,y)}$

- ~~Find critical points, where~~ $\cancel{\vec{\nabla} f(x,y) = 0}$
- ~~If necessary, use 2^{nd} deriv test to verify that it is max/min.~~

Example:

$$f(x,y) = x^2 - x + y^2 + y$$

$$\nabla f(x,y) = (2x-1,\ 2y+1) = (0,0) \implies \begin{matrix} x = \frac{1}{2} \\ y = -\frac{1}{2} \end{matrix}$$

Critical point $\left(\frac{1}{2}, -\frac{1}{2}\right)$

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ has pos det. Local min

Activity:

For what value of a is $(0,0)$ a min / max / saddle point of

$$f(x,y) = x^2 + y^2 + axy.$$

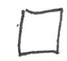
Activity:

Sketch

a) What 2d shape w/ perimeter 1 encloses the most area?

Circle

Square



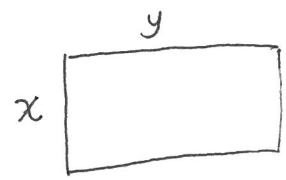
b) What rectangle w/ perimeter 1 ____ ____ ____ ____ ?

c) What polygon ____ ____ ____ ____ ____ ____ ____ ?

Does not exist

Activity:

Find rectangle of largest area given perimeter P



a) Write in form $\max f(x,y)$ subject to $g(x,y)=0$

$$\max\ xy \quad st \quad 2x+2y=P$$

b) Use constraint to write y in terms of x

$$y=\frac{P}{2}-x$$

c) Write in form $\max h(x)$

$$\max\ x\left(\frac{P}{2}-x\right) = \max\left(\frac{P}{2}x-x^2\right)$$

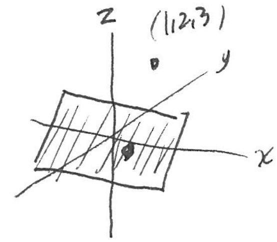
d) Solve,

$$\frac{d}{dx}\left(\frac{P}{2}x-x^2\right) = \frac{P}{2}-2x=0 \Rightarrow x=P/4 \Rightarrow y=P/4$$

$\Rightarrow$ Square!

Activity:

Find nearest point on plane $x+y+z=0$ to $(1,2,3)$



a) What is distance of $(1,2,3)$ to (x,y,z)?

$$\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$$

b) Write problem in form

$$\min \quad f(x,y,z) \quad \text{s.t.} \quad g(x,y,z)=0$$

$$\min \sqrt{(x-1)^2+(y-2)^2+(z-3)^2} \quad \text{st} \quad x+y+z=0$$

c) ~~Simplify~~ ~~objective~~ Use constraint to solve for z.
write problem in form

$$\min \quad h(x,y)$$

Note $z=-x-y$ $\qquad \min \sqrt{(x-1)^2+(y-2)^2+(-x-y-3)^2}$

d) Simplify + Solve

$$\min \underbrace{(x-1)^2+(y-2)^2+(x+y+3)^2}_{\tilde{h}(x,y)}$$

$$\vec{\nabla}\tilde{h} = \langle 2(x-1)+2(x+y+3),\ 2(y-2)+2(x+y+3) \rangle = 0$$

$$4x+2y+4=0 \qquad 2x+4y+2=0$$

$$\Rightarrow \quad x=-1 \quad y=0 \quad \Rightarrow z=1$$

Nearest point is $(-1,0,1)$

Activity:

Solve nearest point on plane $x+y+z=0$ to $(1,2,3)$ using Lagrange Multipliers

a) Write as $\min f(x,y,z)$ subject to $g(x,y,z)=0$

b) Write $\mathcal{L}(x,y,z,\lambda)$

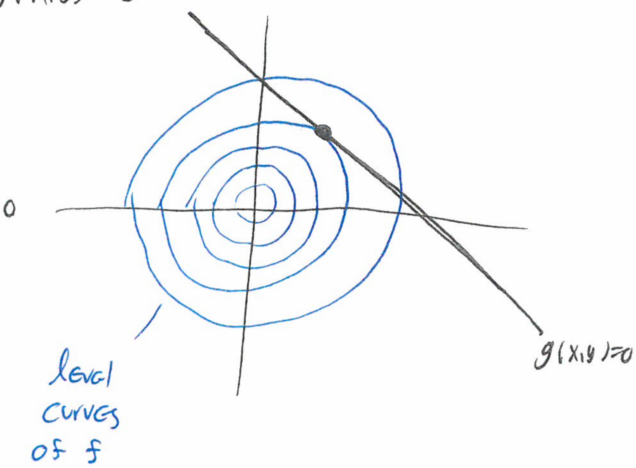
c) Set

$$\partial_x \mathcal{L} = 0$$
$$\partial_y \mathcal{L} = 0$$
$$\partial_z \mathcal{L} = 0$$
$$\partial_\lambda \mathcal{L} = 0$$

d) Solve.

Justification of Lagrange Multipliers

max/min $f(x,y)$ st $g(x,y)=0$



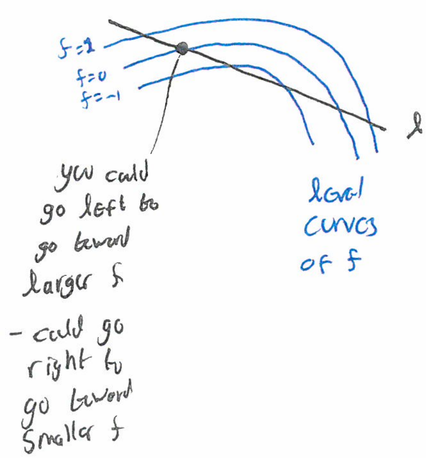
$\mathcal{L} = f(x,y) + \lambda g(x,y)$

$\nabla \mathcal{L} = 0 \Rightarrow \nabla f(x,y) + \lambda \nabla g(x,y) = 0$

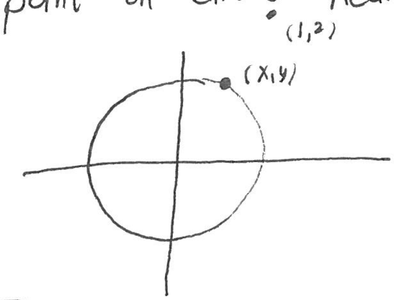
∇f is parallel to ∇g at constrained extremum

So level curves of f & g are tangent.

If level curves of f & g weren't tangent, you could move along constraint and increase OR decrease objective



Example: Find point on unit circle nearest $(1,2)$



$$\min \sqrt{(x-1)^2 + (y-2)^2} \quad \text{subject to} \quad x^2+y^2=1.$$

Simplify objective & put in standard form

$$\min \ (x-1)^2 + (y-2)^2 \quad \text{s.t.} \quad x^2+y^2-1=0$$

$$\mathcal{L}(x,y,\lambda) = (x-1)^2 + (y-2)^2 + \lambda(x^2+y^2-1) = 0$$

$$\partial_x \mathcal{L} = 2(x-1) + 2\lambda x = 0$$
$$\partial_y \mathcal{L} = 2(y-2) + 2\lambda y = 0$$
$$\partial_\lambda \mathcal{L} = x^2+y^2-1 = 0$$

So
$$(2+2\lambda)x = 2 \qquad \Rightarrow \qquad y = 2x.$$
$$(2+2\lambda)y = 4$$

Combine with $x^2+y^2=1 \Rightarrow x^2+4x^2=1 \Rightarrow 5x^2=1$
$\Rightarrow x^2 = 1/5$
$\Rightarrow x = \pm 1/5$

options: $\underbrace{\left(\frac{-1}{5}, \frac{-2}{5}\right)}_{\text{max dist}}$ & $\underbrace{\left(\frac{1}{5}, \frac{2}{5}\right)}_{\text{min dist.}}$