1 July 2013
Calculus 3, Interphase 2013
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Problem Set 1 [Revised]

Due: **8 June 2013** in class.

1. (10 points) Taylor Series
   (a) Directly compute the Taylor series about $x = 0$ for $\log(1 - x)$.
   (b) Use your answer to (a) to write down the Taylor series about $x = 0$ for $\log(1 + x)$.
   (c) Directly compute the second order Taylor series expansion of $g(x) = \frac{1}{1-\sin x}$ about $x = 0$.
   (d) Compute the second order Taylor series expansion of $g(x)$ about $x = 0$ by combining the Taylor series of $\sin x$ with that of $\frac{1}{1-x}$.
   (e) Use a Taylor series to compute $\lim_{x\to 0} \frac{1+x+\frac{1}{2}x^2-e^x}{x^3}$.

2. (10 points) Do the following series converge or diverge? Justify your answer.
   (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$
   (b) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \cdots$
   (c) $\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + \cdots$
   (d) $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots$
   (e) $\sum_{n=1}^{\infty} ne^{-n}$
   Hint: Use a comparison test against a slower decaying exponential

3. (20 points) Let $a = (1, 1, 1)$ and $b = (-1, 2, 2)$.
   (a) Find the angle between a and b.
   (b) Find the area of the parallelogram spanned by a and b.
   (c) Find the projection of b along a.
   (d) Find the plane that contains a and is perpendicular to b.
   (e) Find the plane going through a, b, and the origin.

4. (20 points) Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.
   (a) By direct calculation, show that $a \cdot (a \times b) = 0$.
   (b) By direct calculation, show that $a \times b = -(b \times a)$.

5. (20 points) Use vectors to prove the following:
   (a) Suppose ABCD is a quadrilateral. If the midpoint of AC equals the midpoint of BD, then ABCD is a parallelogram.
   (b) A parallelogram whose diagonals have equal length is a rectangle.

6. *Spam Filtering*. One way to filter spam is as follows. For a bunch of emails, have a human classify them as spam or not spam. For each email, compute m numerical features and combine them into an m-dimensional vector. Such features could include the fraction of letters that are upper case, the number of URLs, the number of dollar signs, etc. If there is a plane in m-dimensional space such that most of the spam are on one side and most of the non-spam are on the other side, we can use the plane to classify incoming emails.

   As a three-dimensional toy example, suppose the feature vectors of several spam messages are $(2, 0, 2)$, $(3, 1, 4)$, and $(1, 2, 4)$. Suppose the vectors of several non-spam messages are represented as $(2, 0, 1)$, $(0, 2, 3)$, $(1, 3, 2)$.

   (a) (15 points) Find a plane such that all the spam messages are on one side of the plane and all the non-spam messages are on the other side.
   Hint: Try sketching/visualizing the data in various ways.

   (b) (5 points) Based on your answer to (a), would you classify the following points as spam or not?

      i. $(2, 1, 1)$
      ii. $(0, 0, 3)$