07/03/2013

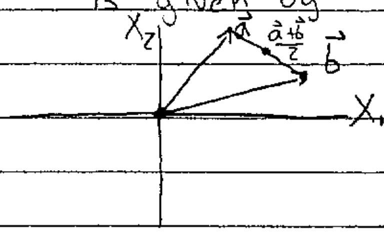
Midpoint Formula

Geometric Proofs

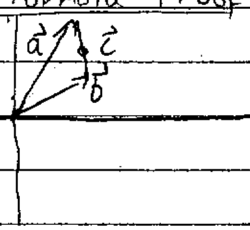
Lines and Planes

Midpoint Formula

The midpoint in the line connecting vectors $\vec{a}$ & $\vec{b}$ is given by $\frac{\vec{a}+\vec{b}}{2}$.

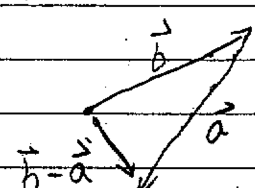


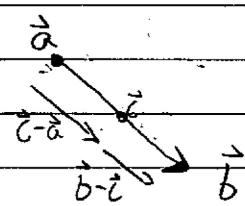
Midpoint Formula Proof:



$\Rightarrow$ where $\vec{c}$ is the unknown midpoint.

The line connecting $\vec{a}$ to $\vec{b}$ is given by $\vec{b}-\vec{a}$





The line connecting $\vec{a}$ to $\vec{c}$ is given by $\vec{c}-\vec{a}$.

The line connecting $\vec{c}$ to $\vec{b}$ is given by $\vec{b}-\vec{c}$.

Because $\vec{c}$ is the midpoint, the magnitude and direction of $\vec{c}-\vec{a}$ and $\vec{b}-\vec{c}$ are equal.

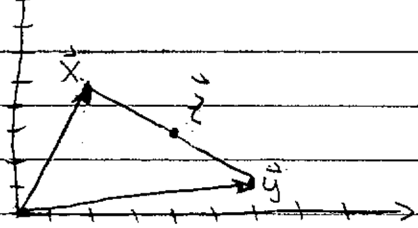
Therefore, $\vec{c}-\vec{a}=\vec{b}-\vec{c}$

$$2\vec{c} = \vec{a}+\vec{b}$$

$$\boxed{\vec{c} = \frac{\vec{a}+\vec{b}}{2}}$$

Example

$\vec{x}=\langle 2, 5\rangle$ & $\vec{y}=\langle 6, 1\rangle$



$$\vec{z} = \frac{\vec{x}+\vec{y}}{2} = \frac{\langle 2,5\rangle+\langle 6,1\rangle}{2} = \langle 8/2, 6/2\rangle$$

$$\vec{z} = \langle 4, 3\rangle$$

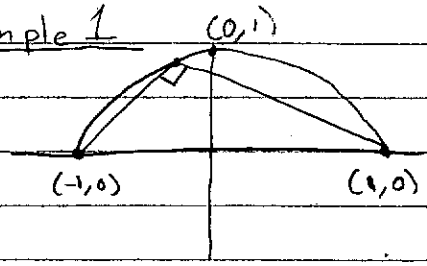
Geometric Proofs with Vectors

Steps (structure):

- Label important points (as few as possible)
- Identify important vectors
- Interpret given and goals in terms.
- Manipulate algebra

Tools: $+, -, \cdot, \times$
scalar
$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
$\vec{a} \times \vec{b} = 1 \Rightarrow \vec{a} \perp \vec{b}$
$\vec{a} \cdot \vec{a} = |\vec{a}|^2$

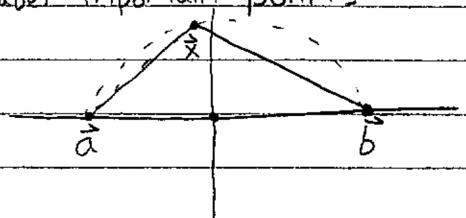
Example 1



Show that the lines connecting any point of the semicircle of radius 1 to (1,0) and (-1,0) are perpendicular.

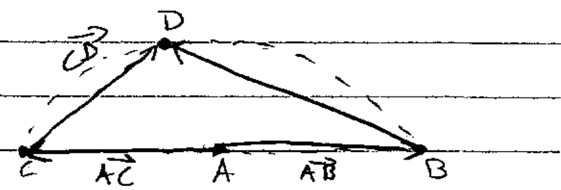
Step 1

- Label important points



Better

- Few points.

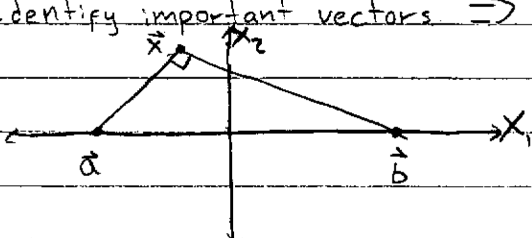


Worse

- Unnecessary point A
- Vector named with two letters
- Algebra will be messy.

Step 2

- Identify important vectors $\Rightarrow \vec{a}, \vec{b}, \vec{x}, \vec{x}-\vec{a}, \vec{x}-\vec{b}$



$\vec{a} = \langle -1, 0 \rangle = \hat{x}_1 = -\vec{b}$
$\vec{b} = \langle 1, 0 \rangle = -\hat{x}_1 = -\hat{a}$
(Draw the vectors)

Step 3

- Interpret given and goals in terms of important vectors.

Given: $|\vec{x}| = |\vec{a}| = |\vec{b}| = 1$

Goal: $(\vec{x} - \vec{a}) \cdot (\vec{x} - \vec{b}) = 0$

Step 4

- Manipulate algebra

$(\vec{x}-\vec{a})\cdot(\vec{x}-\vec{b})$

$=\vec{x}\cdot\vec{x} - \vec{x}\cdot\vec{b} - \vec{x}\cdot\vec{a} + \vec{a}\cdot\vec{b}$

$=\vec{x}\cdot\vec{x} - \cancel{\vec{x}\cdot\vec{b}} + \cancel{\vec{x}\cdot\vec{b}} + \vec{a}\cdot\vec{b}$

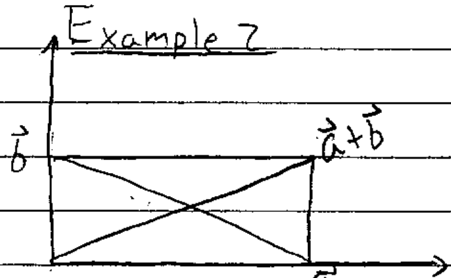
$=\vec{x}\cdot\vec{x} + \langle -1,0\rangle\cdot\langle 1,0\rangle$

$=\vec{x}\cdot\vec{x} - 1$ => Given $|\vec{x}|=1$

$1-1$

$=0$

Example 2



Show that a rectangle is a square IFF its diagonals are perpendicular.

- Important vectors:

$\vec{a},\ \vec{b},\ \vec{a}+\vec{b},\ \vec{a}-\vec{b}$

Given: $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$

Goal: $|\vec{a}|=|\vec{b}|$

Note: notice

Algebra:

$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$

$\vec{a}\cdot\vec{a} + \cancel{\vec{a}\cdot\vec{b}} - \cancel{\vec{a}\cdot\vec{b}} - \vec{b}\cdot\vec{b}=0$

$\vec{a}\cdot\vec{a}=\vec{b}\cdot\vec{b}$

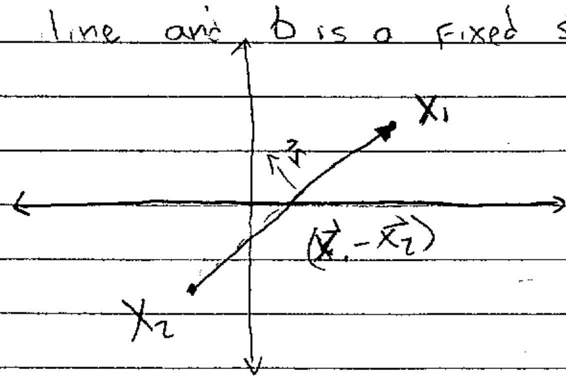
$\boxed{|\vec{a}|=|\vec{b}|}$

Lines and Planes (Implicitly)

A 2d line ~~on a plane~~ can be defined by the formula:

$$\vec{n}\cdot\vec{x}=b,$$

where $\vec{n}$ is the vector normal to the line, $\vec{x}$ is a point on the line and b is a fixed scalar product.



Take 2 arbitrary points, The vector that joins the two points and thus lies on the line:

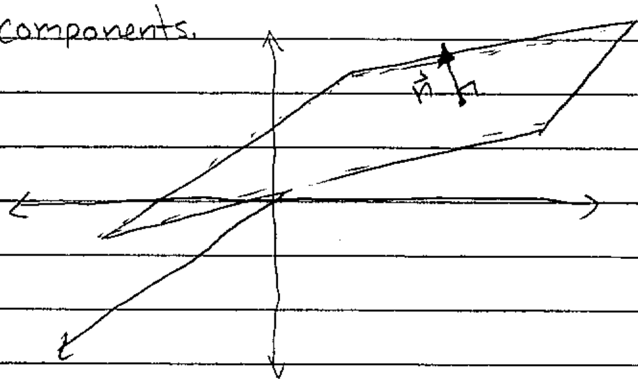
$\vec{n}\cdot\vec{X}_1=b$

$\vec{n}\cdot\vec{X}_2=b$

$\vec{n}\cdot\vec{X}_1=\vec{n}\cdot X_2$

$(\vec{X}_1-\vec{X}_2)\cdot\vec{n}=0$

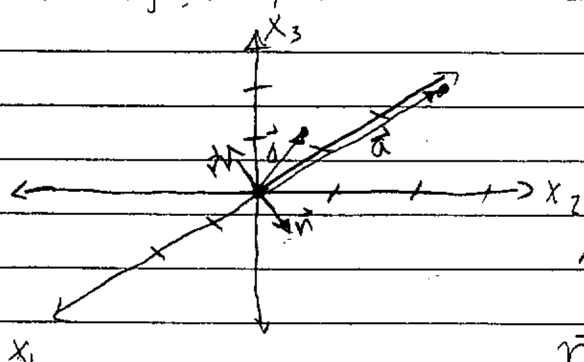
A plane on 3D coordinates can also be defined by the formula $\vec{n} \cdot \vec{x} = b$, however each vector now contains 3 components.



Example 1 implicit

Finding the equation of plane with points $(0, 2, 3)$, $(1, 1, 2)$ and $(0, 0, 0)$.

Since we know we need to obtain the normal vector $\vec{n}$, we can obtain it by finding the cross product of two vectors made by the points.



$\vec{a} = \langle 0, 2, 3 \rangle$

$\vec{b} = \langle 1, 1, 2 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\hat{i} - \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}\hat{j} + \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}\hat{k}$$

$\vec{n} = \vec{a} \times \vec{b} = \langle 1, 3, -2 \rangle$

$\vec{n} \cdot \vec{x} = b$; use any point on the plane

$b = \langle 1, 3, -2 \rangle \cdot \langle 0, 2, 3 \rangle = (1 \cdot 0) + (3 \cdot 2) + (-2)(3)$

$b = 0 + 6 - 6 = 0$

$b = 0$

$\vec{n} \cdot \vec{x} = 0$

$\langle 1, 3, -2 \rangle \cdot \langle X_1, X_2, X_3 \rangle = 0$

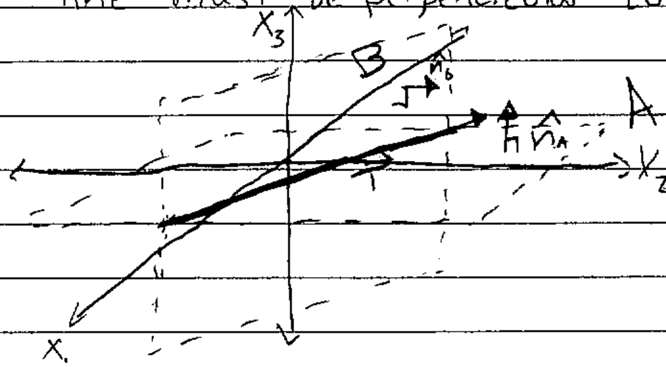
$\boxed{X_1 + 3X_2 - 2X_3 = 0}$

*Note: If the plane contains the origin as a point, $b = 0$

Intersection of Planes

non-parallel

The intersection between two planes creates a line. This line must be perpendicular to both planes.



The direction of the line can be found by crossing the normal vector of each plane.

$$\vec{n}_A \times \vec{n}_B = \vec{T}$$

To define the line implicitly, either normal vector can be used. However, the scalar term, b, must match the equation.

$$\vec{n}_A \cdot \vec{X}_0 = b_A$$

$$\vec{n}_B \cdot \vec{X}_0 = b_B$$

, where $\vec{X}_0$ is a point on the line that satisfies both equations.

Example 1

Determine the line of intersection L of the the following planes
Plane $H = \langle 2, 3, -1 \rangle \cdot \vec{x} = -3$ and Plane $G \langle 4, 5, 1 \rangle \cdot \vec{x} = 1$

First, arbitrarily chose a value for one coordinate and solve for the other two values to find a point on the line.

$$H \Rightarrow 2X_1 + 3X_2 - X_3 = -3 \Rightarrow \text{choose } X_1 = 1$$

$$G \Rightarrow 4X_1 + 5X_2 + X_3 = 1$$

<u>H</u> $2(1) + 3X_2 - X_3 = -3$

$3X_2 - X_3 = -5$

$X_3 = 3X_2 + 5 \longrightarrow$

<u>G</u> $4(1) + 5X_2 + X_3 = 1$

$5X_2 + X_3 = -3$

$5X_2 + 3X_2 + 5 = -3$

$8X_2 = -8$

$X_2 = -1$

$X_3 = 3(-1) + 5 = 2 \longleftarrow$

$$\vec{X}_0 = \langle 1, -1, 2 \rangle$$

Plug into our into the implicit plane formula, using either normal vector and find the scalar product.

$$\vec{n}_H \cdot \vec{X}_0 = \langle 2, 3, -1 \rangle \cdot \langle 1, -1, 2 \rangle = 2 - 3 - 2 = -3$$

Then our line is defined as $\boxed{\vec{n}_H \cdot \vec{X} = -3}$