Lecture 12   31 july 2013

Vector Fields
Line Integrals
Conservative Vector Fields
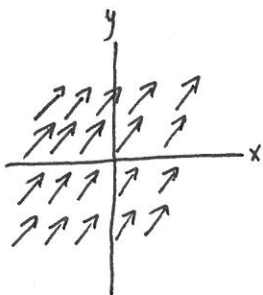Fundamental Theorem of Line Integrals

Vector Fields
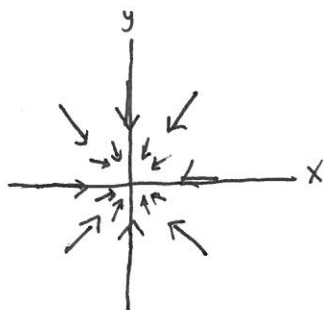
A vector field is a vector-valued function

(2D) $\quad \vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$

(3D) $\quad \vec{F}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$

Examples:

$\vec{F}(x,y) = \hat{\imath} + \hat{\jmath}$

$\vec{F}(x,y) = -x\,\hat{\imath} - y\,\hat{\jmath}$

$\vec{F}(x,y) = -y\,\hat{\imath} + x\,\hat{\jmath}$
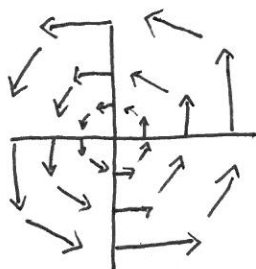
Example: Plot: $\vec{F}(x,y) = e^x \hat{\imath} + \hat{\jmath}$

Note: $\hat{\jmath}$ component always 1

$\hat{\imath}$ comp is positive, small for neg value of x
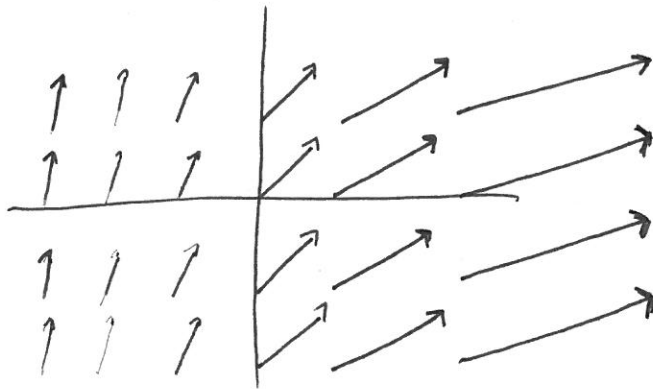
Example:
$$\vec{F} = \frac{(X-1)\hat{\imath} + (y+1)\hat{\jmath}}{\sqrt{(x-1)^2 + (y+1)^2}}$$
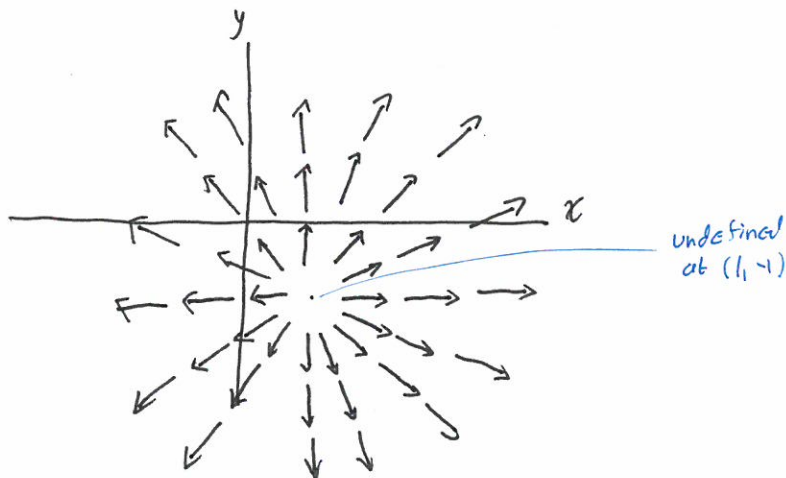
Plot

Denominator looks like a normalization.
$$\sqrt{(x-1)^2 + (y+1)^2} = \left| (x-1)\hat{\imath} + (y+1)\hat{\jmath} \right|$$

So $\vec{F}$ is always a unit vector.

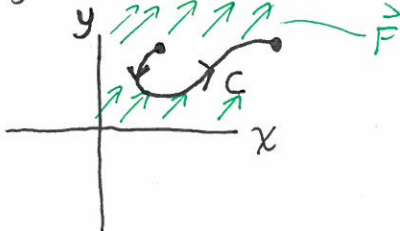To get a sense of direction, ignore denominator

$$(X-1)\hat{\imath} + (y+1)\hat{\jmath} = \langle X, y \rangle - \langle 1, -1 \rangle$$

is Vector from $\langle +1, -1 \rangle$ to $\langle X, y \rangle$
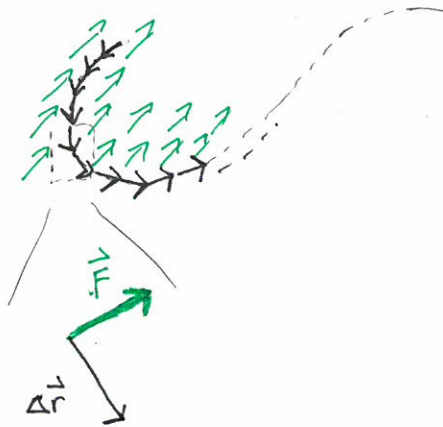

undefined at $(1, -1)$

Line Integral

$$\int_C \vec{F}(x,y) \cdot d\vec{r}$$ is   line integral of $\vec{F}$ over C

is like Riemann sum

$$\sum \vec{F}(x,y) \cdot \Delta\vec{r}$$

If $\vec{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$

write $$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C \langle P, Q \rangle \cdot \langle dx, dy \rangle$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$\int_C \vec{F}(x,y) \cdot d\vec{r} = \int \vec{F} \cdot \vec{T}\, ds$$

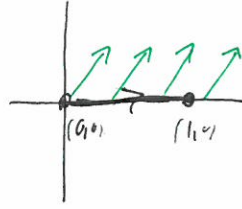unit tangent vector   arc length

Integral with respect to arc length of tangential component of the force

Example: $\vec{F} = \hat{\imath} + \hat{\jmath}$

C = line segment from $(0,0)$ to $(1,0)$

$d\vec{r}$ always in direction of $\hat{\imath}$

So $\int \vec{F} \cdot d\vec{r} \sim (\hat{\imath} + \hat{\jmath}) \cdot \hat{\imath} \cdot$ length of curve $= 1$

Evaluating line integral directly

$$\int_C \vec{F} \cdot d\vec{r}$$

Parameterize by $\vec{r}(t) = \langle x(t), y(t) \rangle$ $\quad a \leq t \leq b$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \cdot \frac{d\vec{r}}{dt}(t) \, dt$$
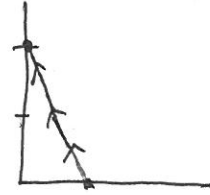
Doesnt depend on parameterization. All give same answer

Example: $\vec{F}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath}$

$C =$ line segment from $(1,0)$ to $(0,2)$

$\vec{r}(t) = \langle 1,0 \rangle + t\langle -1, 2 \rangle$ for $0 \le t \le 1$
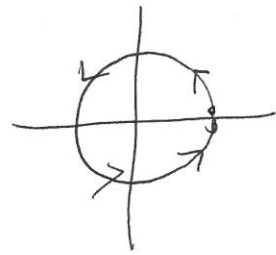
$\dfrac{d\vec{r}}{dt} = \langle -1, 2 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle 1-t, 2t \rangle \cdot \langle -1, 2 \rangle \, dt$$

$$= \int_0^1 (-1+t+4t)\,dt = \int_0^1 -1+5t \; dt$$

$$= -1 + \frac{5}{2} = \boxed{\frac{3}{2}}$$

Example: $\vec{F} = -y\,\hat{c} + x\,\hat{j}$

$C =$ Unit circle traversal
Canter clockwise (positive direction)

$\vec{r}(\theta) = \langle \cos\theta,\ \sin\theta \rangle \quad 0 \le \theta \le 2\pi$

$\dfrac{d\vec{r}}{dt} = \langle -\sin\theta,\ \cos\theta \rangle$

$\displaystyle \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \sin\theta,\ \cos t \rangle \cdot \langle -\sin\theta,\ \cos\theta \rangle\, dt$

$\displaystyle = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, dt = 2\pi.$

Conservative Vector Field

- $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ is <u>conservative</u> if $\vec{F}(x,y) = \nabla\varphi(x,y)$ for some $\varphi(x,y)$.

- $\vec{F}$ is conservative iff $\partial_y P(x,y) = \partial_x Q(x,y)$.

  why? $\vec{F} = \nabla\phi \Rightarrow \vec{F} = \underbrace{\partial_x\phi}_{P}\,\hat{\imath} + \underbrace{\partial_y\phi}_{Q}\,\hat{\jmath}$

  Recall $\phi_{xy} = \phi_{yx}$. So $\partial_y P = \partial_x Q$.

- To find φ from $\vec{F}$, integrate each component and combine

- Note φ defined up to a constant

Example:　　$\vec{F} = \underset{P}{\underbrace{x}}\,\hat{\imath} + \underset{Q}{\underbrace{y}}\,\hat{\jmath}$,　　Is it conservative? Find ϕ

$$\partial_y P = \partial_x Q = 0 \implies \text{Conservative}$$

$$\partial_x \phi = x \implies \phi = \tfrac{1}{2}x^2 + f(y) \quad\overset{\text{combine}}{\searrow}$$

$$\partial_y \phi = y \implies \phi = \tfrac{1}{2}y^2 + g(x) \quad\nearrow \qquad \phi = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2$$

is a potential function for $\vec{F}$

Example　　$\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath}$　　Is it conservative

$$\partial_y P = -1$$　　Not equal. Not conservative.

$$\partial_x Q = 1$$

Example: $\vec{F} = (x+y)\,\hat{\imath} + (y+x)\,\hat{\jmath}$ Is $\vec{F}$ conservative?
Find potential functions

$$\partial_y P = 1$$
$$\partial_x Q = 1 \quad \Rightarrow \text{ Conservative}$$

To find ψ:

$$\partial_x \psi = x+y \quad \Rightarrow \quad \psi = \tfrac{1}{2}x^2 + xy + f(y)$$
$$\partial_y \psi = x+y \quad \Rightarrow \quad \psi = xy + \tfrac{1}{2}y^2 + g(x)$$

Combining, $\psi = \tfrac{1}{2}x^2 + xy + \tfrac{1}{2}y^2$