CS3000: Algorithms & Data
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Lecture 16:

- Shortest Paths and Dijkstra’s Algorithm
- Correctness of Dijkstra’s Algorithm

Mar 18, 2019

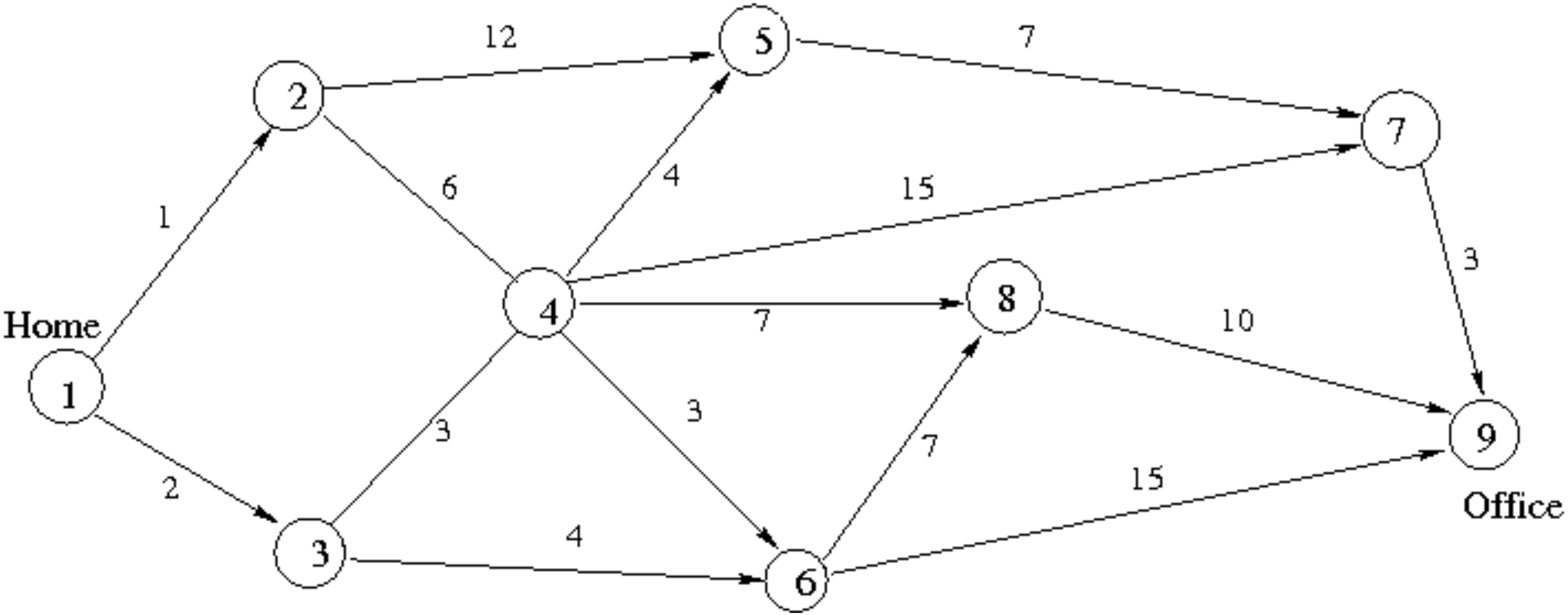
Shortest Paths

Weighted Graphs

- **Definition:** A weighted graph $G = (V, E, \{w(e)\})$
  - V is the set of vertices
  - $E \subseteq V \times V$ is the set of edges
  - $w_e \in \mathbb{R}$ are edge weights/lengths/capacities
  - Can be directed or undirected

- **Today:**
  - Directed graphs (one-way streets)
  - Strongly connected (there is always some path)
  - Non-negative edge lengths ($\ell(e) \geq 0$)

Activity: Find the shortest path



Shortest Paths

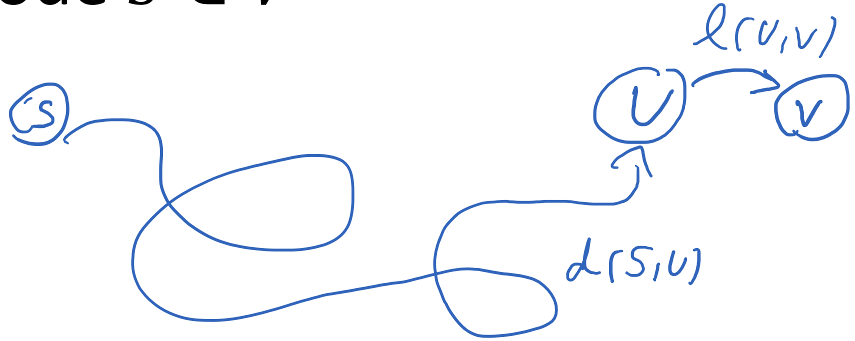
- The length of a path $P = v_1 - v_2 - \cdots - v_k$ is the sum of the edge lengths

- The distance $d(s, t)$ is the length of the shortest path from s to t
- **Shortest Path:** given nodes $s, t \in V$, find the shortest path from s to t
- **Single-Source Shortest Paths:** given a node $s \in V$, find the shortest paths from s to **every** $t \in V$

Structure of Shortest Paths

distance from s to v is no more than distance to u + length of u→v edge

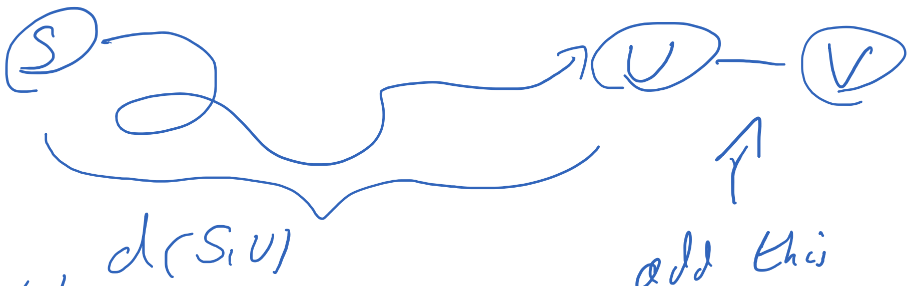
- If $(u, v) \in E$, then $d(s, v) \leq d(s, u) + \ell(u, v)$ for every node $s \in V$



Can get to v via u

(maybe we can do better on another path)

- If $(u, v) \in E$, and $d(s, v) = d(s, u) + \ell(u, v)$ then there is a shortest $s \rightsquigarrow v$-path ending with (u, v)



There is a path from s to u w/ length

add this edge

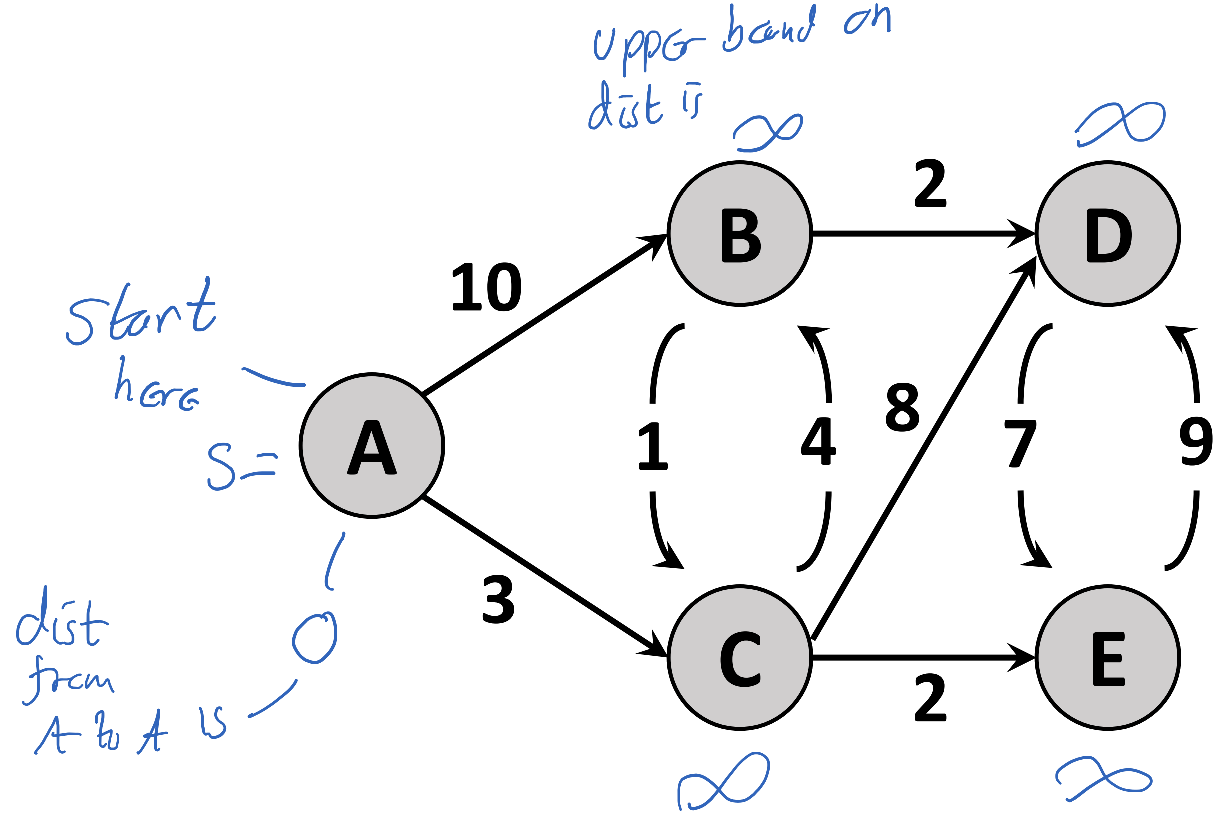
Dijkstra's Algorithm

- **Dijkstra's Shortest Path Algorithm** is a modification of BFS for non-negatively weighted graphs
- **Informal Version:**
  - **Maintain a set S of explored nodes**
  - **Maintain an upper bound on distance**
    - If u is explored, then we know $d(u)$ (Key Invariant)
    - If u is explored, and (u, v) is an edge, then we know $d(v) \leq d(u) + \ell(u, v)$
  - **Explore the "closest" unexplored node**
  - **Repeat until we're done**

upper bound on $d(s,v)$ at i^{th} step

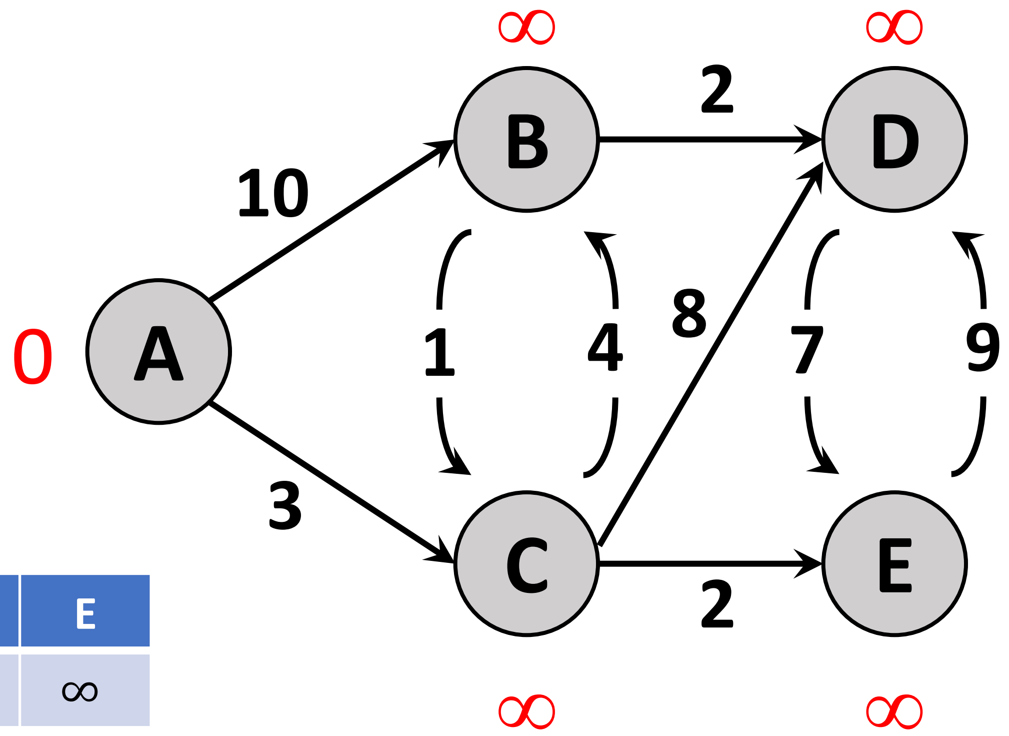
$d_i(v)$

Dijkstra's Algorithm: Demo



Dijkstra's Algorithm: Demo

Initialize



upper band
on dist of
u from S
at iter 0

	A	B	C	D	E
$d_0(u)$	**0**	∞	∞	∞	∞

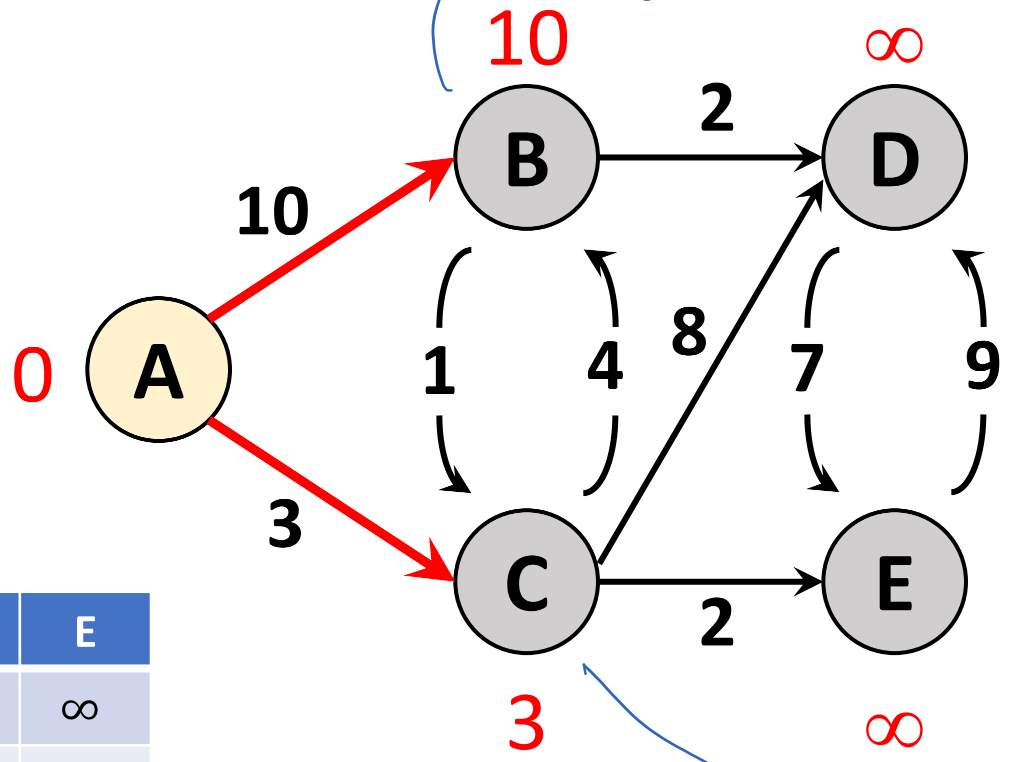
$S = \{\}$

Dijkstra's Algorithm: Demo

Explore A

Declare A explored

B is no more than 10 away



	A	B	C	D	E
$d_0(u)$	0	∞	∞	∞	∞
$d_1(u)$	0	10	3	∞	∞

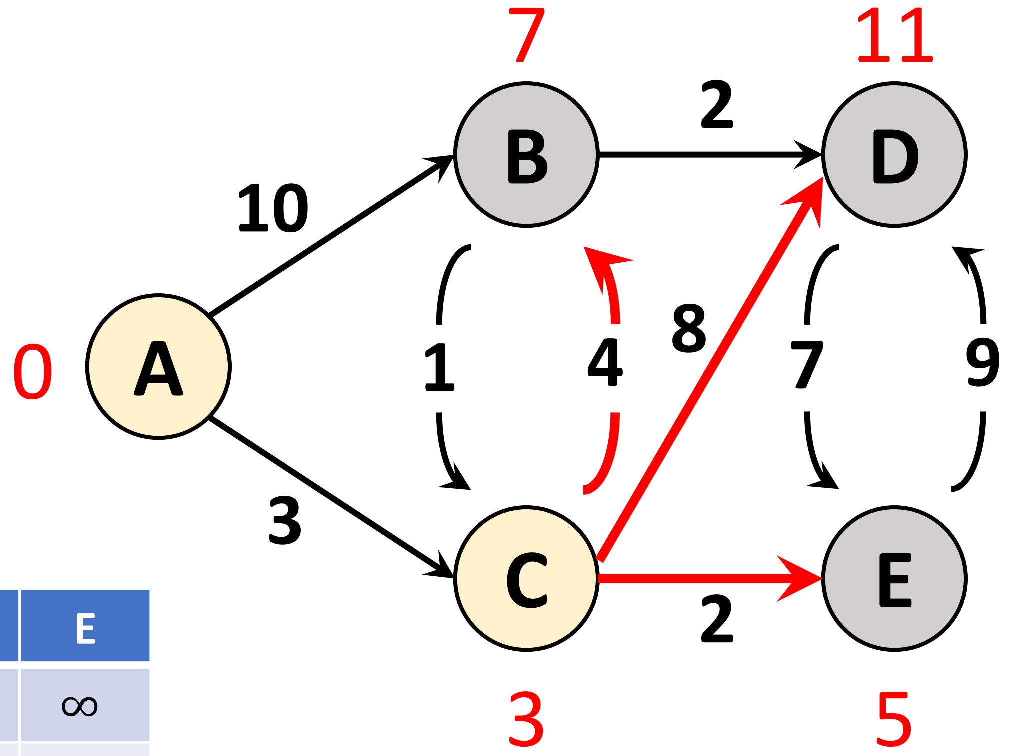
This node is the closest unexplored node to A

$S = \{A\}$

C is no more than 3 away

Dijkstra's Algorithm: Demo

Explore C



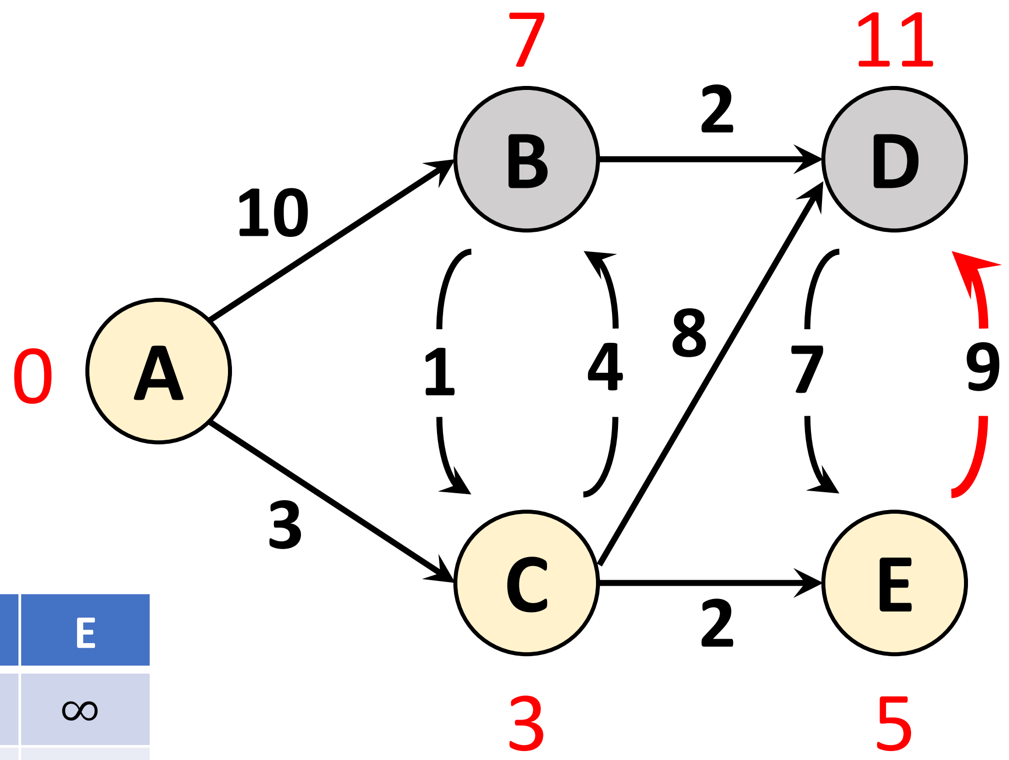
	A	B	C	D	E
$d_0(u)$	0	∞	∞	∞	∞
$d_1(u)$	0	10	3	∞	∞
$d_2(u)$	0	7	3	11	5

explore
E next

$$S = \{A, C\}$$

Dijkstra’s Algorithm: Demo

Explore E

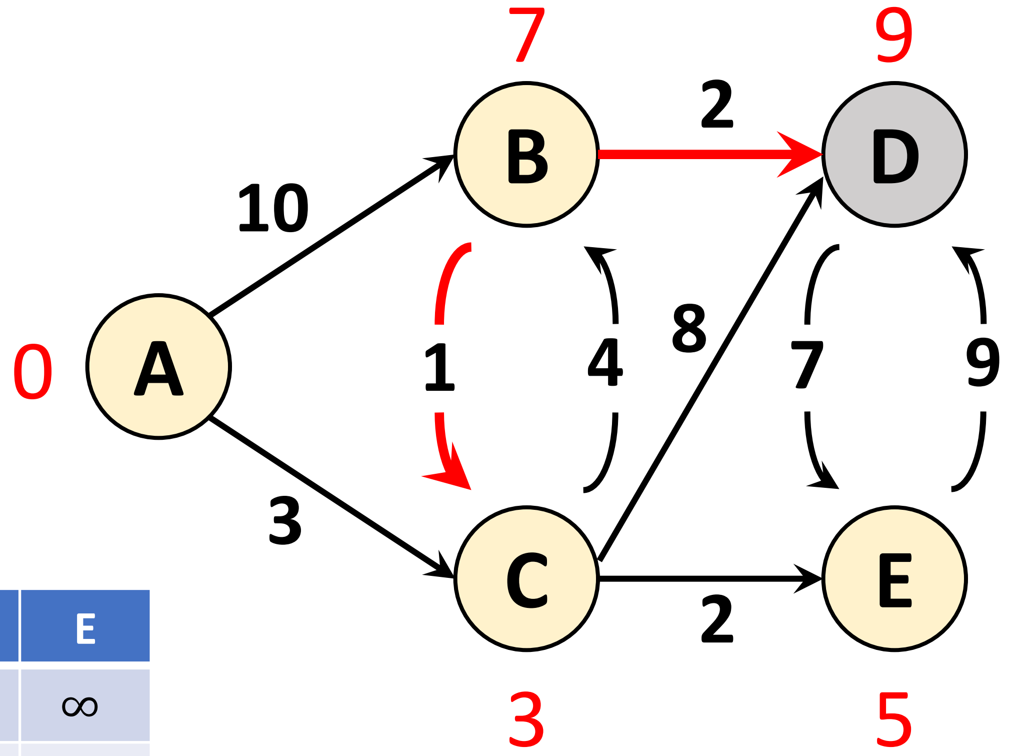


	A	B	C	D	E
$d_0(u)$	0	∞	∞	∞	∞
$d_1(u)$	0	10	3	∞	∞
$d_2(u)$	0	7	3	11	5
$d_3(u)$	0	7	3	11	5

$$S = \{A, C, E\}$$

Dijkstra's Algorithm: Demo

Explore B

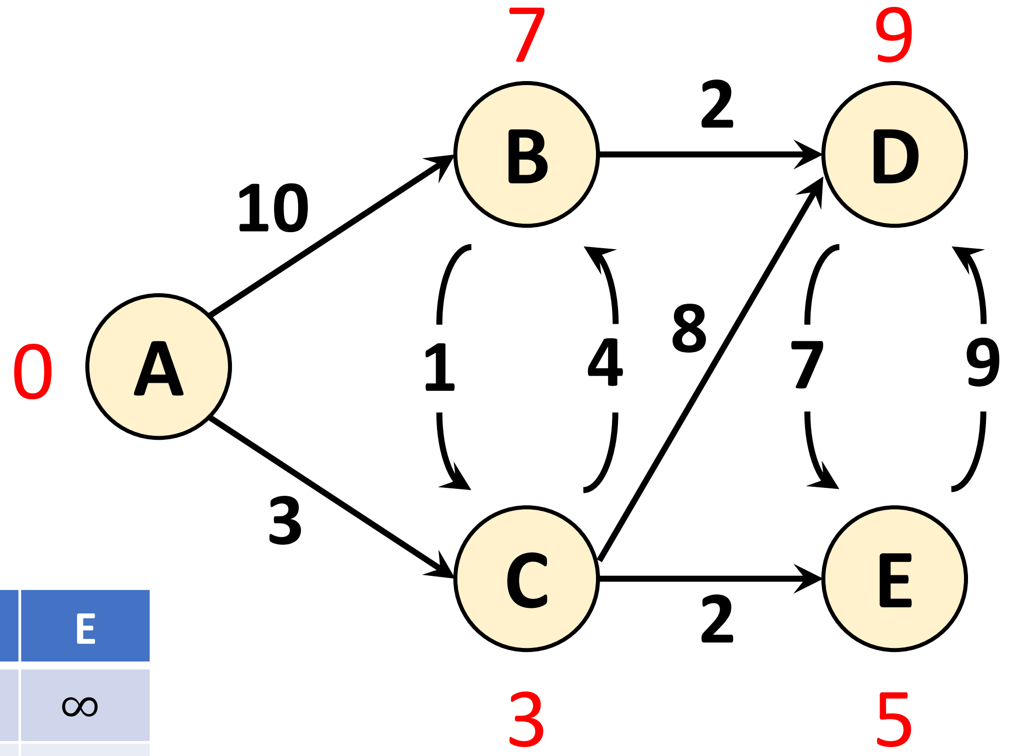


	A	B	C	D	E
$d_0(u)$	0	∞	∞	∞	∞
$d_1(u)$	0	10	3	∞	∞
$d_2(u)$	0	7	3	11	5
$d_3(u)$	0	7	3	11	5
$d_4(u)$	0	7	3	9	5

$$S = \{A, C, E, B\}$$

Dijkstra's Algorithm: Demo

Don't need to explore D

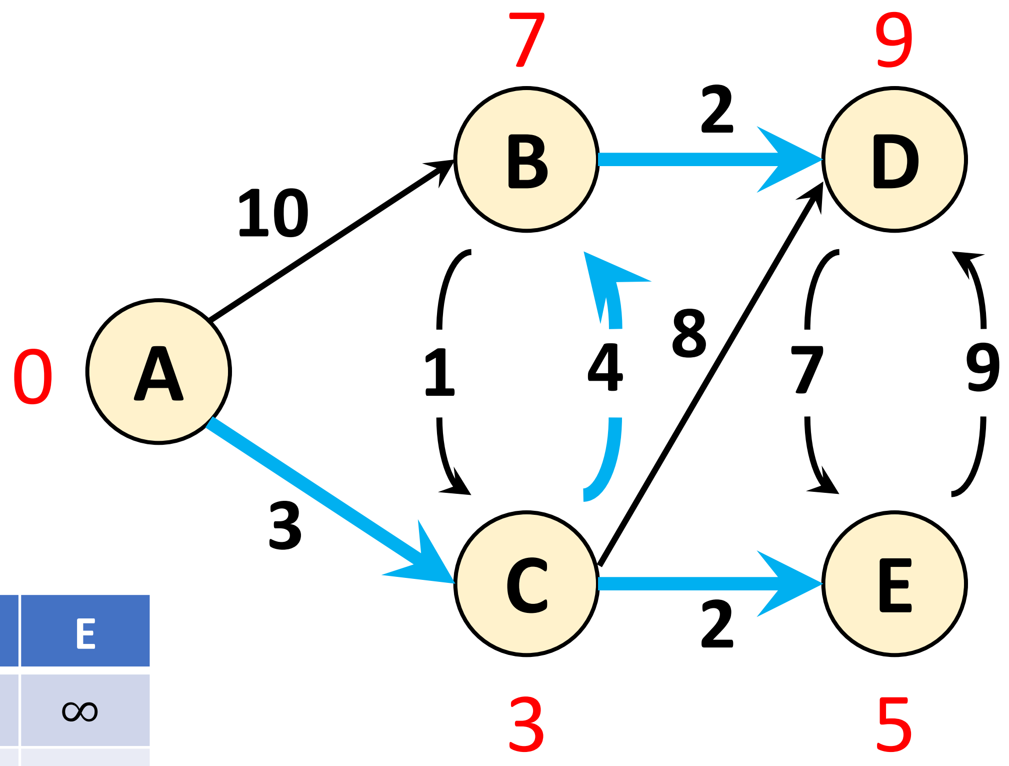


	A	B	C	D	E
$d_0(u)$	0	∞	∞	∞	∞
$d_1(u)$	0	10	3	∞	∞
$d_2(u)$	0	7	3	11	5
$d_3(u)$	0	7	3	11	5
$d_4(u)$	0	7	3	9	5

$$S = \{A, C, E, B, D\}$$

Dijkstra's Algorithm: Demo

Maintain parent pointers so we can find the shortest paths



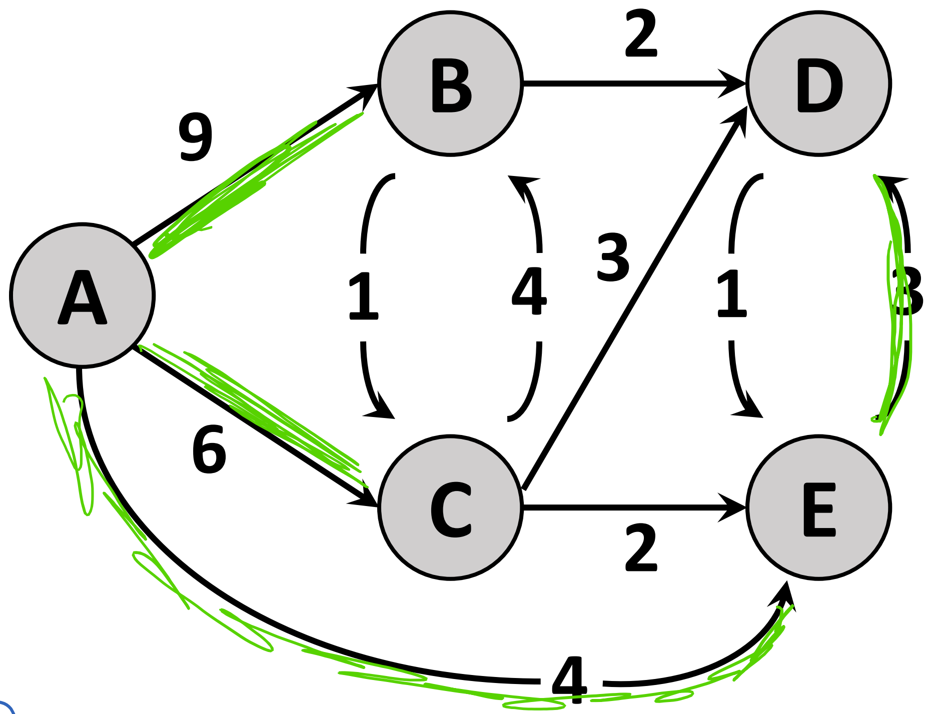
	A	B	C	D	E
$d_0(u)$	0	∞	∞	∞	∞
$d_1(u)$	0	10	3	∞	∞
$d_2(u)$	0	7	3	11	5
$d_3(u)$	0	7	3	11	5
$d_4(u)$	0	7	3	9	5

Execute Dijkstra's Algorithm: Activity

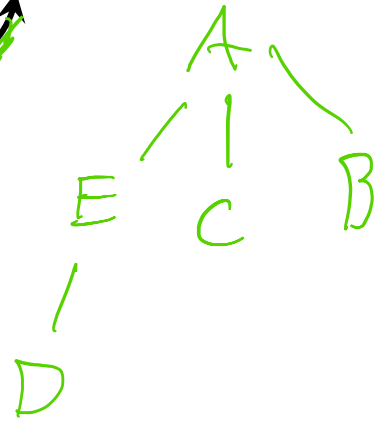
	A	B	C	D	E
$d_0(u)$	**0**	∞	∞	∞	∞
$d_1(u)$	**0**	9	6	∞	4
$d_2(u)$	**0**	9	6	7	4
$d_3(u)$	**0**	9	6	7	4
$d_4(u)$	**0**	9	6	7	4
$d_5(u)$	**0**	9	6	7	4

found shortest path to E
then to C
then to D
then to B

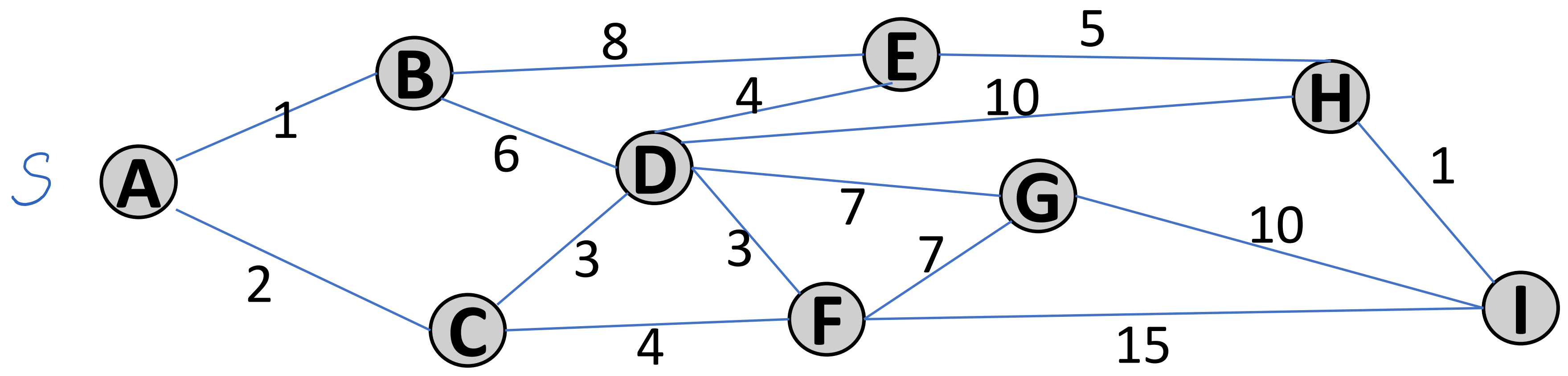
- For each row, find closest unexplored node to A by current estimates
- Look at its neighbors and update their path length if possible
- Repeat



Draw closest Path tree.



Execute Dijkstra's Algorithm: Activity

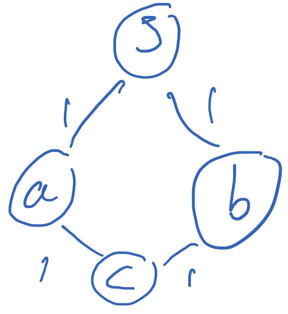


	A	B	C	D	E	F	G	H	I
$d_0(u)$	0	∞	∞	∞	∞	∞	∞	∞	∞
$d_1(u)$	0	1	2	–	–	–	–	–	–
$d_2(u)$	0	1	2	7	9	–	–	–	–
$d_3(u)$	0	1	2	5	9	6	–	–	–
$d_4(u)$	0	1	2	5	9	6	12	15	–
$d_5(u)$	0	1	2	5	9	6	12	15	21
$d_6(u)$	0	1	2	5	9	6	12	14	21
$d_7(u)$	0	1	2	5	9	6	12	14	21
$d_8(u)$	0	1	2	5	9	6	12	14	15
$d_9(u)$	0	1	2	5	9	6	12	14	15

Activity:

Build a graph where there is a tie in one row of table.

Do you get something different based on how you break tie.



No, in terms of distances

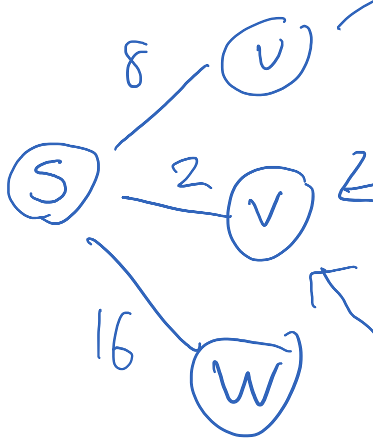
Maybe in terms of tree

Correctness of Dijkstra

- **Warmup 0:** initially, $d_0(s)$ is the correct distance

Trivial (any path $s \to v_1 \to \cdots \to s$ would have at least 0 distance)
no neg path lengths

- **Warmup 1:** after exploring the second node v, $d_1(v)$ is the correct distance



Selected because it had shortest edge from s

Any other path has some distance ≥ 0 added to dist from $s \to v$ or $s \to w$ which was bigger than $s \to v$

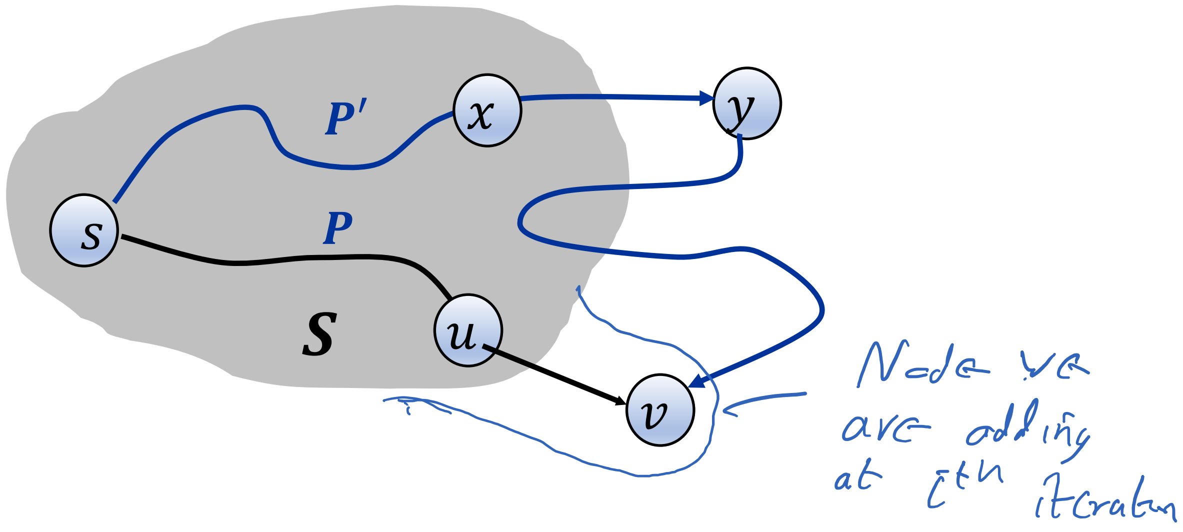
Correctness of Dijkstra

- **Invariant:** after we explore the i-th node, $d_i(v)$ is correct for every $v \in S$ ← set of explored nodes

  This is what we will prove

- We just argued the invariant holds after we've explored the 1st and 2nd nodes

Correctness of Dijkstra



- **Invariant:** after we explore the i-th node, $d_i(v)$ is correct for every $v \in S$
- **Proof:**

- There is some path P of length $\ell(P) = d_i(v)$
- Consider any other path P' from s to v
  - P' leaves S using some edge $x \to y$
  - The part of P' from s to x is a shortest path, has length $d_i(x)$

$$
\begin{aligned}
\ell(P') &\geq (\text{distance to } x) + \ell(x \to y) \\
&= d_i(x) + \ell(x \to y) \\
&\geq d_i(y) && [\text{Because } x \text{ was explored}] \\
&\geq d_i(v) && [\text{Because we explored } v, \text{ not } y]
\end{aligned}
$$

- $\ell(P') \geq \ell(P)$

  $\therefore$ P is a shortest path

  $\therefore$ $d_i(v) = d(s,v)$ $\square$

Implementing Dijkstra

```
Dijkstra(G = (V,E,{ℓ(e)}, s):
  d[s] ← 0, d[u] ← ∞ for every u != s
  parent[u] ←⊥ for every u
  Q ← V                      // Q holds the unexplored nodes

  While (Q is not empty):
```
$u \leftarrow \operatorname{argmin}_{w \in Q} d[w]$ //Find closest unexplored

current estimate

```
    Remove u from Q

    // Update the neighbors of u
    For ((u,v) in E):
      If (d[v] > d[u] + ℓ(u,v)):
        d[v] ← d[u] + ℓ(u,v)
        parent[v] ← u

  Return (d, parent)
```

Implementing Dijkstra Naively

- Need to explore all n nodes
- Each exploration requires:
  - Finding the unexplored node u with smallest distance
  - Updating the distance for each neighbor of u
    - Lookup current distance
    - Possibly decrease distance