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Analysis I
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Week 13 — Summary — Implicit Function Theorem

Reading: XVIII.4

139. Let $f : J_1 \times J_2 \to \mathbb{R}$ be a function of two real variables defined on a product of open intervals J_1, J_2. Assume f is C^p. Let $(a, b) \in J_1 \times J_2$, $f(a, b) = 0$, $D_2 f(a, b) \neq 0$. Then the map

$$\psi : J_1 \times J_2 \to \mathbb{R} \times \mathbb{R}$$
$$(x, y) \mapsto (x, f(x, y))$$

is locally C^p invertible at (a, b).

140. Let S be the set of (x, y) such that $f(x, y) = 0$. Then there exists an open set $U_1 \subset \mathbb{R}^2$ containing (a, b) such that $\psi(S \cap U_1)$ consists of all numbers $(x, 0)$ for x in some open interval around a.

141. Implicit Function Theorem: Let $f : J_1 \times J_2 \to \mathbb{R}$ be a function defined on the product of two open intervals. Assume f is C^p. Let $(a, b) \in J_1 \times J_2$, $f(a, b) = 0$, and $D_2 f(a, b) \neq 0$. Then there exists an open interval $J \subset \mathbb{R}$ containing a and a C^p function $g : J \to \mathbb{R}$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in J$.

142. Implicit Function Theorem in higher dimensions: Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}$ be a C^p function on U. Let $(a, b) = (a_1, \ldots, a_{n-1}, b) \in U$ and assume that $f(a, b) = 0$ but $D_n f(a, b) \neq 0$. Then there exists an open ball $V \subset \mathbb{R}^{n-1}$ centered at (a) and a C^p function $g : V \to \mathbb{R}$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in V$.

143. Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}$ be a C^p function. Let $P \in U$ and assume that $f(P) = 0$ but $\text{grad} f(P) \neq 0$. Let $w \in \mathbb{R}^n$ be perpendicular to $\text{grad} f(P)$. Let S be the set of points X such that $f(X) = 0$. Then there exists a C^p curve $\alpha : J \to S$ defined on an open interval J containing the origin such that $\alpha(0) = P$ and $\alpha'(0) = w$.