23 September 2015
Analysis I
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Pledged HW 5

Time limit: 3 hours. You may not use your books, your homeworks, your notes, or any electronics during this timed homework. Please write the start and finish times on your paper. Each subproblem is worth 10 points. To receive full credit, you must name all major theorems and state definitions used in your arguments. All counter examples must be accompanied by a proof. You may cite results from class and well-known theorems.

This homework is pledged. On the first page, please write your signature and the Rice University pledge: "On my honor, I have neither given nor received any unauthorized aid on this homework."

Due: Tuesday, 29 September 2015 at the beginning of class.

1. Let $\{x_n\}$ and $\{y_n\}$ be nonnegative sequences of real numbers indexed by $n \in \mathbb{N}$.
   (a) Prove that $\limsup_{n\to\infty}(x_n y_n) \leq \limsup_{n\to\infty} x_n \limsup_{n\to\infty} y_n$
   (b) Provide an example for which the inequality is strict.
2. Let $f(x) = \begin{cases} x\sin(\log x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$
   (a) Plot f on $[0, \infty)$.
   (b) Is f uniformly continuous on $[0, \infty)$? Prove your answer.
3. Prove that there does not exist a $f : \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere and such that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x \geq 0$.
4. Prove that if $f : [0, 1] \to \mathbb{R}$ is monotonic increasing, then it is Riemann integrable.

1) Let $\{X_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be nonnegative sequences of real numbers.

a) Prove: $\limsup\limits_{n\to\infty} X_n y_n \leq \limsup\limits_{n\to\infty} X_n \limsup\limits_{n\to\infty} y_n$

Proof: If $\{X_n\}$ or $\{y_n\}$ is unbounded, the inequality holds trivially. We henceforth assume both sequences are bounded.

Recall $\limsup\limits_{n\to\infty} X_n = \lim\limits_{n\to\infty} \sup\limits_{m\geq n} X_m$, where the limit exists for any bounded sequence because $\sup\limits_{m\geq n} X_m$ is monotonic in n, and bounded.

Observe $\sup\limits_{m\geq n} X_m y_m \leq \sup\limits_{m\geq n} X_m \sup\limits_{m\geq n} y_m \quad \forall n$ by nonnegativity of the sequences

As $\lim\limits_{n\to\infty} \sup\limits_{m\geq n} X_m y_m$, $\lim\limits_{n\to\infty} \sup\limits_{m\geq n} X_m$, $\lim\limits_{n\to\infty} \sup\limits_{m\geq n} y_m$ all exist and are finite,

we conclude

$$\lim_{n\to\infty} \sup_{m\geq n} X_m y_m \leq \lim_{n\to\infty} \sup_{m\geq n} X_m \cdot \lim_{n\to\infty} \sup_{m\geq n} y_m \quad \blacksquare$$

b) Let $X_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

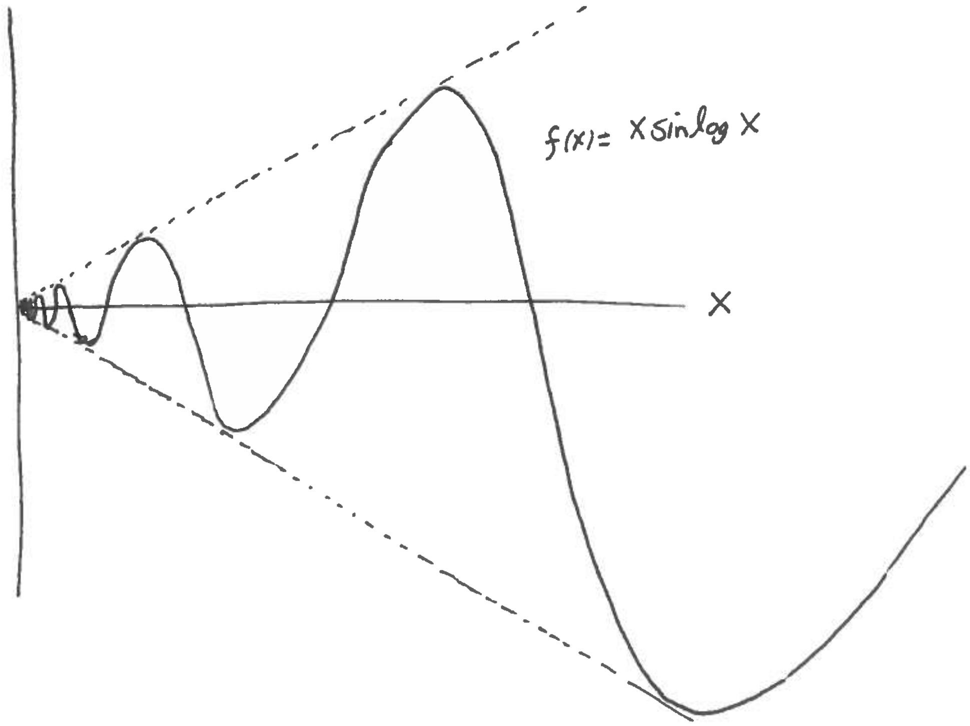
Let $y_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even.} \end{cases}$

$\limsup X_n = 1$, $\limsup y_n = 1$, yet $\limsup X_n y_n = 0$.

Let $f(x) = \begin{cases} x \sin(\log x) & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases}$

2)

a) Plot f on $[0,\infty)$



Claim:

b) Yes, f is uniformly continuous on $[0,\infty)$

Proof:

First, we show f is continuous at $x=0$.

Fix $\varepsilon>0$. Let $\delta=\varepsilon$. If $0 \leq x \leq \delta$, then $|f(x)-f(0)| = |x| \sin \log x \leq |x| \leq \delta = \varepsilon$.

For any $x \neq 0$, f is cnt at x bcs it is compo of two cnt fns

Hence, as f is continuous $\forall x$, we observe

f is uniformly continuous on $[0,1]$

3) There does not exist a $f: \mathbb{R} \to \mathbb{R}$ that is everywhere differentiable and such that $f'(x) < 0$ for $x < 0$ & $f'(x) > 0$ for $x \geq 0$.

Proof:

Assume such an f exists. Without loss of generality, let $f(0) = 0$. Note $f(-1) > 0$ because otherwise, the mean value theorem would guarantee $f'(z) = 0$ for some $z < 0$. Similarly $f(1) > 0$. As f is continuous, $\exists$ $y_1 < 0$, $y_2 > 0$ such that $f(y_1) = f(y_2) = \min(f(-1), f(-2))$, by the intermediate value theorem. Hence, by mean value theorem $\exists$ $x^* \in (y_1, y_2)$ such that $f'(x^*) = 0$, which is a contradiction $\blacksquare$

4) If $f: [0,1] \to \mathbb{R}$ is monotonic increasing, f is Riemann integrable

Proof: We will prove this by establishing the Darboux criterion:

$\forall \varepsilon \ \exists$ partition P such that $U_0^1(f,P) - L_0^1(f,P) < \varepsilon$.

Fix $\varepsilon > 0$. Let P be a uniform partition

$$P = \{0 = x_0, \delta = x_1, 2\delta = x_2, \dots, n\delta = x_n = 1\}$$

where $\delta < \dfrac{\varepsilon}{f(1) - f(0)}$.

Note: $$U_0^1(f,P) = \sum_{i=0}^{n-1} \max_{[x_i, x_{i+1}]} f(x)\, \delta = \sum_{i=0}^{n-1} f(x_{i+1})\, \delta.$$

$$L_0^1(f,P) = \sum_{i=0}^{n-1} \min_{[x_i, x_{i+1}]} f(x)\, \delta = \sum_{i=0}^{n-1} f(x_i)\, \delta$$

We now compute,

$$U_0^1(f,P) - L_0^1(f,P) = \sum_{i=0}^{n-1} \left(f(x_{i+1}) - f(x_i)\right) \delta$$

$$= [f(1) - f(0)]\, \delta$$

$$< (f(1) - f(0)) \frac{\varepsilon}{f(1) - f(0)} = \varepsilon \quad \blacksquare$$