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Analysis I
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Day 15 — Summary — Limits in normed vector spaces and function spaces

88. Functions of multiple variables may have a limit in each variable separately but not in all variables together.

89. Pointwise convergence vs. uniform convergence vs L_1 convergence vs L_2 convergence.

90. The space of bounded maps from one normed vector space to another is complete with respect to the sup norm.

91. The uniform limit of continuous functions is continuous.

92. Limits do not interchange in general. That is, $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) \neq \lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$ in general.

93. If $\lim_{x\to x_0} f(x,y)$ exists for all y, and $\lim_{y\to y_0} f(x,y)$ exists uniformly for all x, then

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = \lim_{y\to y_0}\lim_{x\to x_0} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y).$$

88)

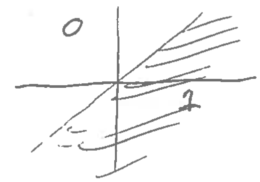
$f(x,y)$ has multiple senses of limits

$\lim_{x\to 0} f(x,0)$, $\lim_{y\to 0} f(0,y)$, $\lim_{(x,y)\to(0,0)} f(x,y)$

may all be equal or different or nonexistent

Exercise: $\lim_{x\to 0^+} f(x,0) \neq \lim_{y\to 0^+} f(0,y)$

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq x \\ 1 & \text{if } y > x \end{cases}$$



$\lim_{x\to 0} f(x,0) = 1 \neq \lim_{y\to 0} f(0,y) = 0$

Example: $\lim_{x\to 0} f(x,0) = \lim_{y\to 0} f(0,y) \neq \lim_{(x,y)\to(0,0)} f(x,y)$

Example $\lim_{\varepsilon\to 0} f(\varepsilon x, \varepsilon y) = 0 \ \forall\, x,y$ but $\lim_{(x,y)\to 0} f(x,y) \neq 0$

89) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f : \mathbb{R} \to \mathbb{R}$

$f_n \to f$ (convergence of functions) can happen in many ways

- pointwise $\forall x \; f_n(x) \to f(x)$ as $n \to \infty$
- uniformly $\|f_n - f\|_\infty \to 0$ as $n \to \infty$
- in L_1 $\int_\Omega |f_n - f| \, dx \to 0$ as $n \to \infty$
- in L_2 $\int_\Omega |f_n - f|^2 dx \to 0$ as $n \to \infty$

Exercise: $f_n \to f$ pointwise but not uniformly

$f_n \to f$ pointwise but not in L_1

$f_n \to f$ pointwise but not in L_2

$f_n \to f$ uniformly but not in L_1

$f_n \to f$ in L_1 but not L_2

$f_n \to f$ in L_2 but not L_1

$f_n \to f$ in L_1 but not pointwise

90) Space of bdd maps from a set to a normed, complete vector space is complete wrt sup norm.

$$B(S,V) = \{ f : S \to V \mid \exists M \text{ st } \|f(x)\|_V \le M \ \forall x \in S \}$$

$B(S,V)$ is a vector space.

$B(S,V)$ has ~~the~~ norm $\|f\|_\infty = \sup_{x\in S} \|f(x)\|_V$

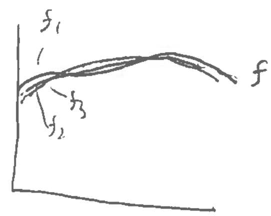
$B(S,V)$ is complete wrt $\|\cdot\|_\infty$ means

f_n Cauchy wrt $\|\cdot\|_\infty \Rightarrow \exists f$ st $\|f_n - f\|_\infty$

Special case: If $V = \mathbb{R}$ this is just L_∞ is complete under L_∞ norm

If $S = \mathbb{Z}^+$, $V = \mathbb{R}$, this is just l_∞ is complete under l_∞ norm

Picture



Each f_i lives within small band of f for arb large i.

Claim $B(S,V)$ complete wrt $\|\cdot\|_\infty$

Proof: ~~As~~ Let $f_n : S \to V$ be Cauchy wrt $\|\cdot\|_\infty$

$\forall \varepsilon \, \exists N$ st $n,m \ge N \Rightarrow \|f_n - f_m\|_\infty < \varepsilon/2$ $(*)$

At each x, $f_n(x)$ is Cauchy. Let $f(x) = \lim_{n\to\infty} f(x)$, exists by completeness of V.

~~Fix ε. choose an x_0. $\exists N$ st $n \ge N \Rightarrow \|f_n(x_0) - f(x)\|_V < \varepsilon/2$~~

Fix ε. Let N be as per $(*)$. Fix $x \in S$. We will show $\|f_n(x) - f(x)\| < \varepsilon$

select $m \ge N$ (may depend on x) st $|f(x) - f_m(x)| < \varepsilon/2$

So $|f(x) - f_n(x)| \le |f(x) - f_m(x)| + |f_m(x) - f_n(x)|$

$\le \varepsilon/2 + \varepsilon/2 = \varepsilon$.

can also show $\|f\|_\infty \le \|f_N\| +$