4 September 2014
Analysis I
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Day 4 — Summary — Squeeze theorem, limits and infinity, continuity and extrema

1. Squeeze theorem: Suppose $f(x) \leq g(x) \leq h(x)$ for x sufficiently close to a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists and is also equal to L.

2. $\lim_{x\to\infty} f(x) = L$ if for all ε, there exists a C such that $x > C \Rightarrow |f(x) - L| < \varepsilon$.
Corresponding definitions for $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$.

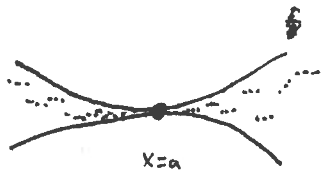
3. If $\lim_{x\to\infty} f(x) = L > 0$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty}(fg)(x) = \infty$.

4. Extreme value theorem: A continuous function over a closed bounded interval achieves its maximum and minimum.

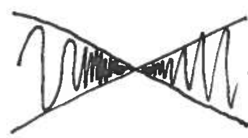
5. Let $a > 1, k \in \mathbb{N}$. $\lim_{n\to\infty} a^n/n^k = \infty$.

1) Squeeze Theorem

Idea: If a function is sandwiched between two functions that share a limit, then all three share the limit.

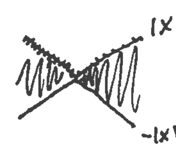
Picture: 

– this function is not continuous, but it has a limit as $x \to a$



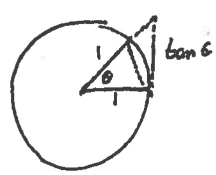
– this function gets arbitrarily steep, but still has a limit

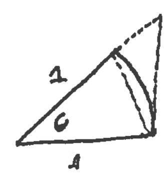
Application: $g(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\frac{1}{x} & \text{if } x \neq 0 \end{cases}$ is continuous at $x = 0$.



It is squeezed between $|x|$ and $-|x|$.
Hence it has limit 0

Application: Show $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Pf: 



Area of small triangle ≤ area of sector ≤ area of big triangle

$$\frac{\sin\theta}{2} \leq \frac{\theta}{2} \leq \tan\theta$$

$$\Rightarrow \quad 1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

$$\Rightarrow \quad 1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta.$$

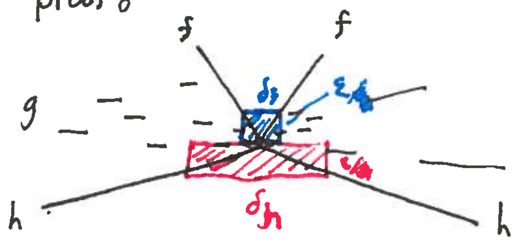
Squeezed $\frac{\sin\theta}{\theta}$ between 1 & $\cos\theta$ which $\to 1$ as $\theta \to 0$.

Formal Statement:

If $f(x) \le g(x) \le h(x)$ for x suff. near a

and if $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$, then $\lim\limits_{x \to a} g(x) = L$

Gist of proof:

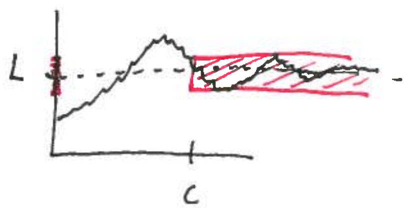


For a given ε, choose the smaller of the δ_f & δ_h corresponding to ε error from the limit. Because g squeezed between, its error is $\le \varepsilon$ within δ_f & δ_h.

2) $\lim_{x \to \infty} f(x) = L$

Idea: f ~~gets~~ is arbitrarily close to L for sufficiently large x

Picture:



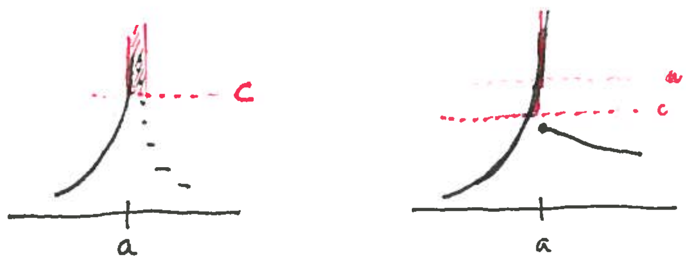
Some tail fits entirely in box of height ε for any ε.

Formal statement: $\forall \varepsilon \ \exists C \ \text{st} \ \forall x \geq C \quad |f(x) - L| \leq \varepsilon$

$\lim_{x \to a} f(x) = \infty$

Idea: f ~~gets~~ is arbitrarily large for x sufficiently near a

Picture:



$\lim = \infty$ $\qquad$ $\lim \neq \infty$

~~Set of~~ There are points arbitrarily near $x=a$ such that f is not larger than C

Formal statement: $\forall C \ \exists \delta \ \text{st} \ \forall |x-a| < \delta \quad f(x) \geq C.$

4) Extreme Value Theorem

A continuous function on a closed bounded interval achieves its max and min

Picture:

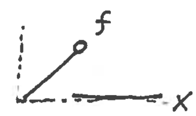
– if function gets very close to this y value, it must achieve it.

Consequence of theorem: $\min_{x \in [a,b]} f(x)$ – this minimization problem has a minimizer (an x that adopts the minimal value)

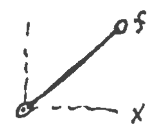
Non Examples:

$f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$ never achieves max on $[0,2]$

it is not continuous



$f(x) = x$ on $0 < x < 1$ never achieves max on $(0,1)$



Note: when max is not achieved we dont use word "max" we use "sup"

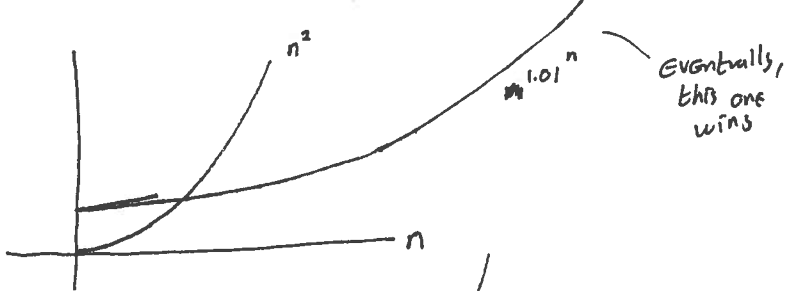
Proof: – Take subsequence such that f approachs supremum
– By Bolzano Weierstrass, $\exists$ conv. subseq
– Defn of continuity gives that max achieved.

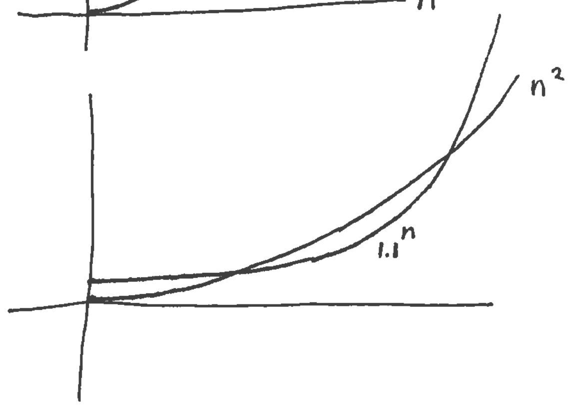
5) $a > 1$, $k \in \mathbb{N}$ $\lim_{n \to \infty} \frac{a^n}{n^k} = \infty$

Intuition: all (growing) exponentials grow faster than all polynomials

~~Picture:~~ Even if growth rate of exponential is small
1.000000001^n grows faster than $n^{(10^{10})}$

Picture:

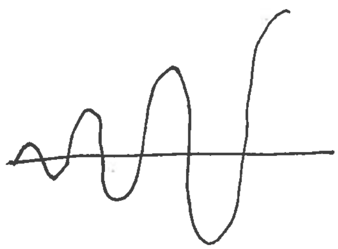




Significance: Theoretical Computer science is based on trying to find~~ing~~ algorithms that run in polynomial time. "Dumb" methods like exhaustive search take exponentially many operations. ~~Polyno~~ Algorithms that are polynomial time will outperform them for large problem sizes.

Proof: $\lim \frac{(1+b)^n}{n^k}$ Expand $(1+b)^n$ using binomial theorem. Keep one term to get bound.

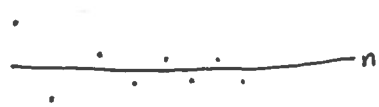
Activity: Draw and write down a smooth function that is not bounded but does not converge to ∞ as



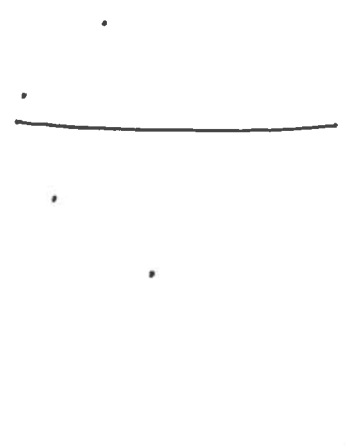
$f(x) = x\sin$

Activity: Draw and specify a sequence x_n that is never 0. yet converges to 0 yet $\frac{1}{x_n}$ does not have ∞ as a limit

x_n:



$\frac{1}{x_n}$:



$x_n = (-1)^n/n$

Questions: If I have a sequence of #'s in $\mathbb{R}$, is it true that there is a subsequence that converges (to a real number or $+\infty$ or $-\infty$)?

Yes. Either bounded or unbdd from above or unbdd from below.

Activity: What does it mean that a set S is unbound from above

$\nexists\, C$ such that $x \leq C \;\forall x \in S$

$\forall C \;\exists x \in S$ s.t. $x > C$.