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Analysis I
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Day 12 — Summary — Open sets and closed sets

1. Definition: A subset S of a normed vector space is open if for any $x \in S$, there is an open ball (centered at x) contained within S.
2. Definition: A subset S of a normed vector space is closed if its complement is open.
3. The finite intersection of open sets is open.
4. The arbitrary union of open sets is open.
5. The finite union of closed sets is closed.
6. The arbitrary intersection of closed sets is closed.
7. Definition: A point x is a limit point of a set S if there are points in S that are arbitrarily close to x under the provided norm.
8. A set is closed if and only if it contains all its limit points.
9. Definition: The closure of a set is the collection of limit points of that set. Write the closure of S as $\bar{S}$.
10. The closure of a set S is the intersection of all closed sets containing S.
11. Definition: Let $S \subset T$. The set S is dense in the set T if $T \subset \bar{S}$.
12. A function f from one normed vector space to another is continuous if $\lim_{x \to a} f(x) = f(a)$. That is, if $\forall \varepsilon,\ \exists \delta$ such that $\|x - a\| \leq \delta \Rightarrow \|f(x) - f(a)\| < \varepsilon$.
13. A function is continuous if and only if the inverse image of any open set is open.

Activity:

~~Dar~~
Draw / write a sequence in $C^1[0,1]$ that is Cauchy wrt $\|\cdot\|_\infty$ and has a limit not in $C^1[0,1]$.

1.) $B_r(x) = \{ y \mid \|y-x\| < r \}$ $\qquad \overline{B_r(x)} = \{ y \mid \|y-x\| \le r \}$

S is open if $\forall x \ \exists \ \varepsilon > 0$ st $B_\varepsilon(x) \subset S$.

Visually:

Examples:

$\mathbb{R}$: (a, b) is open

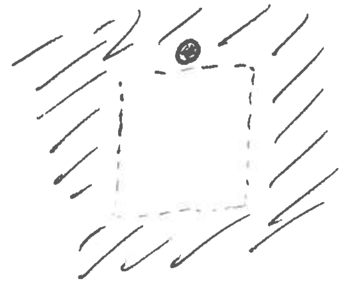
$\mathbb{R}^2$: $\{x \mid \|x\| < 1\}$ is open (under any norm)

ℓ^∞: $\{x \mid \|x\|_\infty < 1\}$ is open under ℓ_∞ norm

2) S is closed if S^c is open

Visually:

has complement



Examples: $\mathbb{R}$: $[a,b]$ closed

$\mathbb{R}^2$: $\{x \mid \|x\| \leq 1\}$ closed

ℓ^∞: $\{x \mid \|x\|_\infty \leq 1\}$ is closed

7) Let V be normal vector space.

Let $S \subset V$.

x is limit point of S if $\forall \varepsilon \; \exists \; y \in V$ st $\|y - x\| < \varepsilon$.

Points you can get arbitrarily close to.

Eg: Any $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

~~The function $f(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| \geq 1 \end{cases}$ is a limit point of ℓ_1,~~

The set $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a limit point of $\underset{S}{\{x \mid \sum |x_i| < \infty\}} \subset \underset{V}{\ell_\infty}$ under the ℓ_∞ norm.

The set of limit points of $B_r(x)$ is $\overline{B_r(x)}$.

Activity: Open, Closed, or Neither, or both

$\mathbb{R}$: $S = \{\frac{1}{i}\}_{i=1}^{\infty}$

$\mathbb{R}$: $S = \{0\} \cup \{\frac{1}{i}\}_{i=1}^{\infty}$

$\mathbb{R}^2$: $S = \emptyset$

$\mathbb{R}^2$: $S = \mathbb{R}^2$

~~ℓ^∞: $S = \{x \mid \|x\|_1 \leq 1\}$ under l_∞ norm closed~~

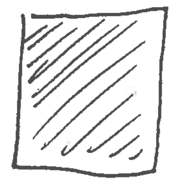
ℓ^∞: $S = \{x \mid \|x\|_1 < 1\}$ under l_∞ norm neither

9) Closure of S is set of all limit points of S.

Eg:

$\{x \in \mathbb{R}^2 \mid \|x\|_\infty < 1\}$

has limit points



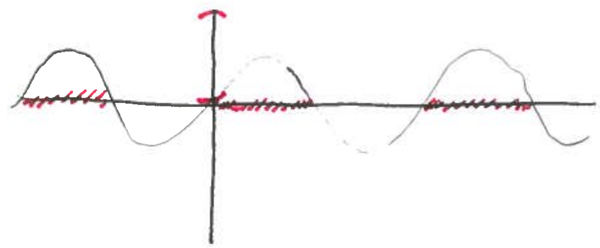
$\{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\}$

The closure of S is closed (requires a proof)

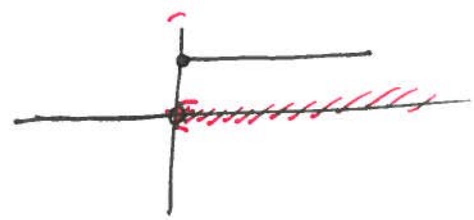
13) Let E, F be normal vectr spaces.

$f: E \to F$ is continuous iff $\forall\, O \subset F$ open, $f^{-1}(O)$ is open.

Example: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin x$. is continuous. $f^{-1}((0,2))$ is open



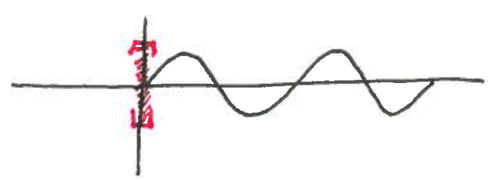
Nonexample $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ is not continuous $f^{-1}((0,2)) = \{x \geq 0\}$ is closed



Note: Image of open sets is not open

$f: \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin x$

$f((0,4\pi)) = [-1,1]$ not open



Proof:

$\Rightarrow$: Let $f: E \to F$ be continuous.

Let $\mathcal{O} \subset F$ be open.

~~If $\mathcal{O}$ empty:~~

~~If $\nexists$ y~~

Let $x \in E$ such that $y = f(x) \in \mathcal{O}$. [If no such x exists, $f^{-1}(\mathcal{O})$ trivially open]

$\exists\, \varepsilon$ st $B_\varepsilon(y) \subset \mathcal{O}$.

As f is continuous, $\exists\, \delta$ st $\|\tilde{x} - x\| < \delta \Rightarrow \|f(\tilde{x}) - f(x)\| < \varepsilon$.

Hence, $f^{-1}(\mathcal{O})$ contains all of $B_\delta(x)$. Hence $f^{-1}(\mathcal{O})$ open.

$\Leftarrow$: Let $f: E \to F$ be such that $\forall$ open $\mathcal{O}$, $f^{-1}(\mathcal{O})$ is open.

~~We will show~~ Let $x \in E$. We will show f cont at x

Need to show: $\forall\, \varepsilon\ \exists\, \delta$ st $\|\tilde{x} - x\| < \delta \Rightarrow \|f(\tilde{x}) - f(x)\| < \varepsilon$

Fix ε. Consider $B_\varepsilon(f(x))$ which is open. Its inverse image is open and contains x. Hence $\exists\, \delta$ st $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$

That is $\|\tilde{x} - x\| < \delta \Rightarrow \|f(\tilde{x}) - f(x)\| < \varepsilon$ ∎