25 September 2014
Analysis I
Paul E. Hand
hand@rice.edu

Day 10 — Summary — Norms and Inner Products

1. For finite and infinite sequences x, the ℓ_p norm is $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$. It is a norm for $1 \leq p < \infty$. The ℓ_∞ or sup norm of a sequence x is $\|x\|_\infty = \sup_i |x_i|$.

2. For functions $f : \Omega \to \mathbb{R}$, the L_p norm is $\|f\|_p = \left(\int_\Omega |f|^p\right)^{1/p}$. The L_∞ norm is $\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$.

3. A norm for $C^p[a, b]$ is given by $\|f\| = \sum_{i=0}^p \|f^{(i)}\|_\infty$.

4. Norms can be visualized by their unit ball.

5. An inner product $\langle \cdot, \cdot \rangle$ satisfies the following axioms for all $u, v, w \in V$:

   (a) $\langle v, w\rangle = \langle w, v\rangle$

   (b) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$

   (c) If $c \in \mathbb{R}$, $\langle cv, w\rangle = c\langle v, w\rangle = \langle v, cw\rangle$

   (d) $\langle v, v\rangle \geq 0\ \forall v$ and $\langle v, v\rangle = 0 \Rightarrow v = 0$.

6. Inner products induce a norm $\|v\| = \sqrt{\langle v, v\rangle}$.

7. Inner products satisfy the Cauchy-Schwarz inequality $\langle v, w\rangle \leq \|v\|\|w\|$.

1.) If $X \in \mathbb{R}^n$ or X is infinite sequence,
$X = (X_1, \cdots, X_n)$ $X = (X_1, X_2, \cdots)$

$\|X\|_p = \sqrt[p]{\sum_i |X_i|^p}$ is a norm for $1 \leq p < \infty$

is not a norm for $p < 1$

These are ℓ_p norms.

Claim $\|X\|_1 = \sum_i |X_i|$ is a norm

Pf: $\|X\|_1 \geq 0$ ✓

$\|X\|_1 = 0 \Leftrightarrow X = 0$.

$\Rightarrow$: $\sum |X_i| = 0 \Rightarrow |X_i| = 0 \Rightarrow X_i = 0$

$\Leftarrow$: Trivial

$\|aX\|_1 = \sum_i |aX_i| = \sum_i |a||X_i| = |a| \sum_i |X_i|$

$\|X+y\|_1 = \sum_i |X_i + y_i| \leq \sum_i |X_i| + |y_i| = \sum_i |X_i| + \sum_i |y_i|$

Claim: $\|X\|_\infty = \sup_i |X_i|$ is a norm

Pf: Triangle inequality

$\|X+y\|_\infty = \sup_i |X_i + y_i| \leq \sup_i |X_i| + |y_i| \leq \sup_i |X_i| + \sup_i |y_i|$

Example: Let $X_n = n^{\alpha}$

What values of α are such that $\|X\|_p < \infty$? $\|X\|_\infty < \infty$?

Need $\sum_n n^{p\alpha} < \infty$.

By integral comparison test convergence equivalent to convergence of $\int_1^\infty x^{p\alpha}\, dx$,

This int conv for $p\alpha < -1$, diverges for $p\alpha \geq -1$

$$\alpha < -\frac{1}{p} \Rightarrow \|X\|_p < \infty$$

$$\alpha \leq 0 \Rightarrow \|X\|_\infty < \infty$$

2) L_p norms are analogue of l_p norms for functions.

$$\|f\|_p = \sqrt[p]{\int |f(x)|^p \, dx} \qquad \|f\|_\infty = \sup_x |f(x)|$$

Note: need $\sqrt[p]{}$ to get linear scaling $\|cf\|_p = |c| \, \|f\|_p$

Example: What powers of x^α ~~are~~ have finite L_p norm on $(0,1)$?

Need $\displaystyle\int_0^1 x^{\alpha p} dx < \infty$

$$= \begin{cases} \left.\dfrac{x^{\alpha p+1}}{\alpha p+1}\right|_0^1 \\ \left.\log x\right|_0^1 \end{cases} \qquad \begin{array}{l} \text{finite when } 1+\alpha p > 0 \\ \text{infinite when } 1+\alpha p \le 0 \end{array}$$

$$\alpha > -1/p \implies \|f\|_p < \infty$$

$$\alpha \geq 0 \implies \|f\|_\infty < \infty$$

3) A norm on $C^p[a,b]$ is $\|f\| = \sum_{i=0}^{p} \|f^{(i)}\|_\infty$

Why can it not just be $\|f\| = \|f^{(p)}\|_\infty$?

Because $f(x) = x^{p-1}$ has $f^{(p)} \equiv 0$ yet $f \not\equiv 0$.

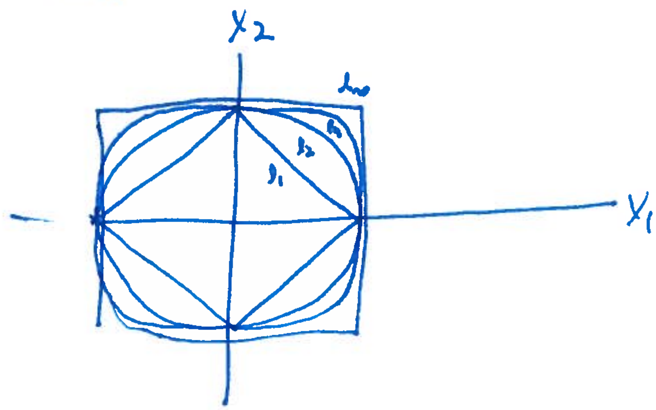
4) open Ball of radius r about x_0 is

$$\{x \mid \|x - x_0\| < r\} = B_r(x_0)$$

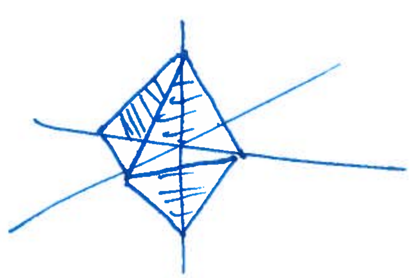
Closed Ball

$$\{x \mid \|x - x_0\| \leq r\} = \overline{B_r(x_0)}$$

Activity: Draw unit ball of $\|\cdot\|_1$ $\|\cdot\|_2$ $\|\cdot\|_4$ $\|\cdot\|_\infty$ in $\mathbb{R}^2$



Draw unit ball of $\|\cdot\|_1$ in $\mathbb{R}^3$



Inner products

$x, y \in \mathbb{R}^n \qquad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$

x, y sequences $\qquad \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$

f, g functions $\qquad \langle f, g \rangle = \int f(x) g(x)\, dx$

Cauchy Schwarz

$$\langle v, w \rangle \leq \|v\| \, \|w\|$$

Compare to: dot product angle formula

$x, y \in \mathbb{R}^n$ $\quad x \cdot y = |x| \, |y| \cos\theta$ $\quad$ so $|x \cdot y| \leq |x| \, |y|$.

Proof: WLOG let $\|v\| = 1$, $\|w\| = 1$.

We show $\langle v, w \rangle \leq 1$.

Consider $0 \leq \langle v - w, v - w \rangle = \|v\|^2 - 2\langle v, w \rangle + \|w\|^2$

$= 2 - 2\langle v, w \rangle$

So $2\langle v, w \rangle \leq 2 \Rightarrow \langle v, w \rangle \leq 1$

Repeat w/ $0 \leq \langle v+w, v-w \rangle$ to get $\langle v, w \rangle \geq -1$

When is inequality ~~strict~~ achieved?

When $\frac{v}{\|v\|} = \pm \frac{w}{\|w\|}$. That is, when $v = cw$

Generalisations: For functions and sequences

$$\langle v, w \rangle \leq \|v\|_1 \, \|w\|_\infty$$

$$\langle v, w \rangle \leq \|v\|_p \, \|w\|_q \qquad \frac{1}{p} + \frac{1}{q} = 1 \quad (\text{Hölder})$$

6) $\|v\| = \sqrt{\langle v, v\rangle}$ is a norm

Triangle inequality

$$
\begin{aligned}
\|v+w\| &= \sqrt{\langle v+w, v+w\rangle} \\
&= \sqrt{\langle v,v\rangle + 2\langle v, w\rangle + \langle w,w\rangle} \\
&\le \sqrt{\|v\|^2 + 2\|v\|\,\|w\| + \|w\|^2} \\
&= \sqrt{(\|v\| + \|w\|)^2} \\
&= \|v\| + \|w\| \quad \blacksquare
\end{aligned}
$$