17 October 2013
Paul E. Hand

Statement on Teaching and Learning

Over the last ten years, I have taught freshmen and masters students from MIT (in *Multivariable Calculus*, *Computational Science and Engineering*), engineering students from The Cooper Union (in *Ordinary Differential Equations*, *Boundary Value Problems*, *Probability and Statistics*), and undergraduates at New York University (in *Quantitative Reasoning*). For all of my classes, I see myself as a coach: an expert, a motivator, and a diagnostician. I try to create an effective learning environment through decisions informed by research in cognition, experience in instruction, and the unique needs of my school and students.

When planning and running a class, I consider four types of questions:

- Knowledge-centered, i.e. What should my students be able to do?
- Learner-centered, i.e. How do I help my students build on and refine their mental models?
- Assessment-centered, i.e. How can I frequently reveal the progress my students have made?
- Community-centered, i.e. How do I capitalize on the community of learners and society at large?

These questions are based on the National Research Council's book *How People Learn* [1]. Below are illustrations of how each has shaped my teaching.

Knowledge-Centered Aspects

At the beginning of a course, I consider what I specifically want my students to be able to do. I try to provide written objectives that are specific enough to guide student thinking and general enough to promote transfer of learning to new contexts. Making these goals explicit led me to realize that my first attempts at teaching were ineffective at causing the transformation I desired in my students.

As an example, consider the attached objectives from the second time I taught *Boundary Value Problems* (BVP). At the end of the prior year, I was bothered that my students were unable to relate the solutions of partial differential equations to the physical situations they describe. The fault, here, lied with me: I implicitly assumed my students would figure out this skill as a byproduct of solving many homework problems. When I taught the course a year later, I decided that a primary goal was for students to pose and interpret these equations. Making this and other objectives explicit led me to overhaul the course's organization, the in-class activities, and the kinds of homework and exam questions I asked. For example, notice that in the attached "Homework from Day 16," the problems are directed squarely at all three objectives from the syllabus.

In addition to mathematical definitions and results, I want my students to explicitly think about the problem solving process. Unfortunately, there are few convenient resources that explicitly delineate thought processes. Lectures can be too ephemeral, textbooks too concise, and online videos too time consuming. To address these concerns in my *Multivariable Calculus* class, I created the website LeadingLesson.com, which has math problems and solutions that explicitly show the thought process. For example, the solution to the problem "Find the plane containing the three points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$" begins with "Recall that to specify a plane, we need a point x_0 and a normal vector n" [2]. This statement is precisely what I want my students to think when they start the problem. While it takes years to master efficient problem solving, I believe resources like this can accelerate the task.

Learner-Centered Aspects

After settling on objectives, I can focus on developing my students' mental models and skills. The main principle here is that we actively build understanding on top of our existing mental structures. In contrast to the belief that information is transmitted from teacher to student, this view requires an awareness of whether students have the knowledge with which to understand the content. As an illustration, Schwartz and Martin [3] present an activity to prepare high school students for learning about the standard deviation of data. Students are given a scatter plot of the outcomes of four baseball pitching machines, and they are asked to devise a single number that represents the reliability of each machine. Students struggled with outliers and the comparison of machines that have different numbers of samples. While they did not guess the formula for standard deviation, the students remembered it better than control subjects once they were told the definition. This result was influential in guiding me to ask the question "How can I prepare students to understand a mathematical abstraction?"

One way to prepare students for abstraction is course organization. Consider the attached syllabus for *Boundary Value Problems*. It does not begin with the mathematically simplest problems; instead, it begins with the physical experience students bring to the class. After developing the interesting qualitative questions about diffusions and waves, we saw how many we could answer with only basic physical assumptions. Then we saw how many we could answer by inspection of a differential equation. Only after that did we deal with the technical analysis of these equations.

Another way I prepare students for abstraction is through homework exercises that create the background knowledge needed for class. In the attached "Homework from Day 16," the second problem concerns separation of variables when solving a diffusion equation. It guides students to contrast the properties of an example from class with two variants for which the technique fails. This problem provides the differentiated knowledge that prepares them to be told a generalization in class, much like the baseball pitching example.

In addition to these high level thought processes, I try not to overlook the necessity for developing fluency in the most basic skills. For example, in my BVP course, I found individual whiteboards to be an excellent medium for such tasks. Each day, we would begin with a 2-3 minute whiteboard warm-up question, such as "Write down a diffusion equation with homogeneous Dirichlet boundary conditions." At first, students would be flipping through their notes for the definition of "homogeneous" and "Dirichlet," but as the semester progressed their recall speed greatly improved. Once my students were fluent with these important definitions, I was comfortable using the terms without any pause or reminder.

The final learner centered aspect I will discuss is motivation. As an instructor, I can transform my students only if they care. A significant part of my job, then, is to convince them that the course's content is worth knowing. The challenge was greatest for my course in *Quantitative Reasoning*, a requirement for non-math majors at NYU. In order to convince my students that the skills of estimation could actually help them, I had to show them that their own wild guesses can be completely wrong. On the second day of class, I had them guess whether they have more brain cells than there are bytes on a typical hard disk. They were unanimous, and wrong. I hope that this activity led my students to be more receptive to the technicalities of estimation.

Assessment-Centered Aspects

As my courses are about meeting objectives, assessments naturally inform my students and me about their progress. I often have a day-to-day ungraded assessment so that my students and I can notice misunderstandings early. For example, five-minute nameless exit quizzes and whiteboard problems fill this purpose.

For some assessments that affect grades, I believe that students should have multiple opportunities to demonstrate competence with the course objectives. If they fail at a task, a revision or two will make them more likely to learn from their mistakes. In my *Probability and Statistics* course, I allowed homework revisions, and the difference between submissions was striking. For example, one problem was to find how many people must be in a room for it to be very likely that at least two share the same birthday. I told my students to provide a reasonable definition of the term 'very likely.' One student's initial submission said "Let very likely $= 0.9$." After revisions, the answer was more like "We call an event 'very likely' if it's probability is at least 0.9." I often saw explanations evolve from being verbose or confusing to being clear and direct.

Even with exams, I have sometimes allowed revisions. In my BVP class, the final was an oral interview in line with the course objectives: I asked them to provide specific examples that illustrate the important qualitative properties of partial differential equations. Many students provided physically ridiculous responses by over-relying on formulas they didn't understand. In these cases, I pointed out how their response contradicts physical observations, and I directed them to relevant course material. After that, I encouraged them to schedule another final. One student said of the revision process:

> I went to the exam expecting to do extremely well, ... [but] I was shocked when I failed the oral final not only once, but twice. During revision, a series of questions was posed to guide me through the process of crystallizing physical phenomena into well defined problems and identifying the appropriate mathematical tools to arrive at meaningful solutions.
>
> The course completely changed the way I approach various problems I encounter. Instead of just memorizing how a particular theorem is derived or how an equation is solved, I always strive to understand why this particular technique is used, when its application becomes inappropriate, and what the solution means.

On the third try at the final, this student's explanations and examples were outstanding.

Community-Centered Aspects

In order to capitalize on the community of learners in the classroom, I sometimes provide in-class activities that allow individual and small group thinking as preparation for a whole-class discussion. For example, in my *Quantitative Reasoning* class, I handed out a faulty calculation of how many bacteria stacked end-to-end it would take to span from Earth to the moon. The purported answer was $3.8 \cdot 10^2$. In this activity, students had 1 minute to individually come up with more than one way to notice the given answer was incorrect. Then, they had 2 minutes to compare and explain their thoughts to their neighbors. After that, we had a short whole-class discussion on the topic. As there were several ways to analyze the calculation, peers can provide new viewpoints to each other. I believe students will be much more likely to remember these multiple ways than if I had simply written them on the board.

What's to Come

I look forward to teaching math classes between the early undergraduate and advanced graduate levels. Each school, course, and set of students will have unique needs, and I enjoy trying to address these needs by asking the types of questions above. Unsurprisingly, answers to these questions prompt more questions. For example, what kinds of experiences best prepare students for learning mathematical abstractions? How can I allow revision without significantly increasing the grading burden? Given the recent interest in online instruction, I am also curious about how to capitalize on online lectures so that I can spend more time in class focusing on skills like problem solving and making precise statements. I hope to explore these and other questions in my future teaching.

References

[1] National Research Council. *How People Learn: Brain, Mind, Experience, and School: Expanded Edition*. Washington, DC: The National Academies Press, 2000.

[2] P. Hand. Problem on finding a plane from three points. *Leading Lesson*. Accessed: 11 October 2013. Available: http://www.leadinglesson.com/problem-on-finding-a-plane-from-three-points

[3] D. Schwartz, T. Martin. Inventing to Prepare for Future Learning: The Hidden Efficiency of Encouraging Original Student Production in Statistics Instruction. *Cognition and Instruction*, 22(2), 129–184, 2004.

17 January 2007
Paul E. Hand
hand@cooper.edu
MA 401

Boundary Value Problems

Objectives

By the end of the course you will be able to:

1. Pose physical situations involving diffusions and wave propagation as boundary value problems and interpret such problems physically.
2. Demonstrate the qualitative behaviors of the diffusion and wave equations through explicit examples.
3. Isolate and explain the main idea and conditions of applicability of several PDE solution techniques.

Syllabus

The course will be broken into the following segments:

I.) Posing physical situations as BVPs

- (i) Posing the major qualitative questions about diffusions and wave propagation.
- (ii) Using first principles (conservation of mass, Fick's Law, Newton's Law, etc.) to assess the reasonability of certain qualitative behaviors.
- (iii) Using diffusion and wave PDEs to assess the reasonability of certain qualitative behaviors.
- (iv) Posing and interpreting boundary value problems.

II.) Solution techniques

- (i) Symmetry and change of coordinates.
- (ii) Travelling wave solutions and similarity solutions.
- (iii) Fourier series and Fourier transform.
- (iv) Eigenfunction expansions and separation of variables.
- (v) Dirac Delta, Green's functions, and method of images.

9 March 2007
Paul E. Hand
hand@cooper.edu
MA 401

Homework from Day 16
Due: 30 Mar

1. Consider a guitar string of tension T, length L, linear density ρ, and ends held fixed. Suppose the string is initially held fixed at a displacement $u(t=0,x) = \sin\frac{\pi x}{L}$ and released.
   (a) Write down the BVP satisfied by the displacement $u(t,x)$.
   (b) Draw a sequence of profiles which you expect to correspond to the behavior of the string. What is a natural question to ask about this system concerning the sound produced?
   (c) Solve the BVP from (a) by separation of variables.
   (d) Use the exact solution from (c) to answer your question from (b).

2. In class, we found that we could solve

$$\begin{aligned} \partial_t u - D\partial_{xx} u &= 0 \\ u|_{t=0} &= \sin\frac{n\pi x}{L} \\ u|_{x=0} &= 0 \\ u|_{x=L} &= 0 \end{aligned}$$

   by separation of variables, where n is an integer.
   (a) Would the same technique work if the initial data were $u|_{t=0} = \sin\frac{\pi x}{2L}$?
   (b) Would the same technique work if the initial data were $u|_{t=0} = x(L-x)$?
   (c) Use (a) and (b) to determine why separation of variables works in the case $u|_{t=0} = \sin\frac{n\pi x}{L}$.