A New Current–Voltage Relation for Duct Precipitators Valid for Low and High Current Densities

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Abstract—**A closed-form analytic current–voltage formula for duct electrostatic precipitators is presented. A short discussion of previous theoretical and numerical solutions is given, followed by an explanation of the theoretical formula derived here. A comparison with experimental data is then given, showing that the present formula is accurate over a wide range of conditions, including wide plate spacing.**

I. INTRODUCTION

PRESENTLY, the most common geometry for electrostatic precipitators is of the wire-plate form. A series of equally spaced vertical wires are placed equidistant between two vertical parallel plates. The plates are grounded and the wire is at high voltage. A corona about the wires causes an ion current to flow from the wires to the plates. It is assumed that a gas is used for which the free electron current is negligible.

Maxwell's equations and the continuity equations provide the following mathematical description:

$$\nabla^2 u = -4\pi\tilde{\rho}$$

$$\nabla \cdot j = 0$$

$$j = -K\tilde{\rho}\nabla u$$

where $u = V_0$ at wire; $u = 0$ at plate; and $\partial u/\partial y = 0$ at points equidistant from adjacent wires.

u is the electrical potential, $\tilde{\rho}$ is the ion space charge density, and y is in a direction parallel to the plates and perpendicular to the wires. It is not necessary to include boundary conditions for $\tilde{\rho}$, since $\tilde{\rho}$ can be derived from u. To see this, note that the three differential eqautions can be replaced by $\nabla u \cdot \nabla(\nabla^2 u) = 0$.

A formula for current density j as a function of the applied voltage V_0 will be given for this geometry. A modification of this formula is found when particles are present in the gas stream. The formula is to be applied using the absolute value of the voltage (i.e., positive sign). However, the formula is valid for positive and negative polarity, provided an account is taken of the differing ion mobilities.

Recently, there has been much increased interest in wide plate spacing (see, for example, [5]). The new formula should be useful because it gives significantly greater accuracy in predicting current density for wide plate spacing, while agreeing with the already accurate formula of P. Cooperman [1] for more conventional spacing.

The added accuracy will be useful because a more accurate model will better predict when a precipitator is operating abnormally. This would be especially important in cases of high resistivity, since the first warning often comes from exceptionally low current densities at lower voltages, and exceptionally high current densities at the higher voltages where the back corona makes itself felt.

II. NUMERICAL METHODS

An important question for the pertinence of any theoretical work is how well a computer may numerically calculate the desired data. So far, most numerical methods have used variations of the following scheme.

The applied voltage is divided into an electrostatic voltage and a space charge voltage where the electrostatic voltage equals the current starting voltage. The space charge voltage is the difference between the total applied voltage and the electrostatic voltage. The electric field strength at the wire is assumed to equal the breakdown field of the gas.

Initially a space charge of zero is assumed. The electrostatic potential equals zero at the plate and the starting voltage at the plate. The potential between the wire and the plate is then computed by a relaxation method.

This electrostatic field is held fixed and an ion distribution is then computed by numerically allowing ions to "flow" in a discrete grid between the wire and the plate. The space charge density is then normalized so that the total potential between the wire and the plate equals the applied voltage.

With the space charge density held fixed and the applied voltage taken as the boundary condition at the wire, the electrostatic field is recomputed. Then the electrostatic field is fixed and the space charge density recomputed. The method is continued until convergence is achieved.

P. Cooperman [1] uses the first two iterations of this procedure to arrive at a theoretical current-voltage formula. The full procedure is also described as a means of obtaining a more accurate formula using further iterations.

Leutert and Bohlen [3] describe and use this numerical procedure and a computer to obtain a number of sample graphs of how field strength and potential vary with position in the precipitator. Unfortunately, they do not normalize their current density in the manner described above. Hence, they are obligated to take current density and voltage as independent parameters. Thus, no current-voltage curves are given.

More recently, McDonald [4] at the Southern Research

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Institute and Paranjpe [6] at Research-Cottrell have constructed computer models using variations of this technique in order to arrive at current-voltage relations. This author has worked on the latter model, and examined the published results of the Stanford Research Institute and Leutert and Bohlen models. The most difficult feature to model seems to be the region near the wire.

Since the wire is typically 0.15 cm in radius, a reasonable grid size might be 0.03 cm or less. When one recalls that the wire-to-plate spacing and the wire-to-wire spacing may be of the order of 15 cm, this means 250 000 grid points. When one considers the many iterations necessary to alternately converge both the ion density and the electric potential, a truly accurate model becomes difficult without some special techniques.

Two tests of any computer model are whether current density and electric flux are conserved in going from the wire to the plate. In this author's experience, one appears to be doing quite well if these quantities change by less than 20 percent. Unfortunately, this author has not had access to detailed printouts from models in the published literature. But it is to be expected that similar numerical methods would yield similar accuracy.

There are several possible alternative numerical techniques. Among these are variable grid size (using a smaller grid size near the wire), variational methods (minimizing a certain integral), and a finite element method. However, it is possible that these methods would contain their own difficulties. In any case, this author has not seen reports on such numerical methods for precipitators in the literature.

Hence, more theoretical techniques are used in hopes of achieving better accuracy. The formula to be here derived is based on two distinct methods created by P. Cooperman and discussed in [9]. For a detailed discussion of one of the methods, see [1].

III. CURRENT-VOLTAGE RELATIONS

A. Description of Previous Results Employed

The first method, discussed in [1], considers the case when the electrostatic field is large compared to the space charge field, as is true for low current densities. The current-voltage formula derived there is

$$j = \frac{\pi \epsilon_0 K}{cb^2 \ln (d/a)} \; V(V - V_0).$$

Note the parabolic form $j = sV(V - V_0)$, which is often observed in experiments. The derivation given here will also numerically approximate a parabolic formula. Perhaps the most important result for the purposes of this paper is the rigorous proof in [1] that the ion density will be uniform under such circumstances.

In the second method [9], one imagines the wires replaced by a uniformly current-emitting plate. Thus, the electrostatic field becomes uniform, and the problem is reduced to one dimension, where it may easily be solved. This approximation will clearly be a good one when the space charge field is large compared to the electrostatic field. The equation derived is

$$j = \frac{9}{8} \frac{\epsilon_0 K}{b^3} (V - V_0)^2.$$

As the voltage or current density becomes very large, the formula given will reduce to the above formula.

B. Relation without Particle Space Charge

Our contribution is to imagine a plane parallel to the plate and between the wire and the plate. On the wire side of the plane, the electrostatic field dominates and the first method is applicable. On the plate side, the electrostatic field is assumed uniform. The first method tells us that the ion density is uniform on the wire side of the plane. Hence, the space charge field is also uniform on the wire side of the plane. Thus, the total field is uniform on the plate side of the plane, and the second method can be used.

The appropriate equation is taken from [9]:

$$\nabla^2 V = \frac{dE}{dx} = \frac{\rho}{\epsilon_0} = -\frac{j}{\epsilon_0 K E}.$$

This yields

$$E_x{}^2 = E_1{}^2 + \frac{2j}{\epsilon_0 K} x.$$

E_1 is the average electrostatic field and can be determined from Gauss' law to be

$$E_1 = \frac{\pi a E_0}{2c} = \frac{\pi V_0}{2c \ln d/a}.$$

It is clear that

$$\int_0^b E_x \, dx = V = (V - V_0) + V_0.$$

However, the formula for E_x was derived under the assumption that we had a uniform electrostatic field E_1. Hence, to maintain the consistency of our approximation, the second term V_0 which represents the electrostatic voltage, must be replaced by bE_1. Strictly, the equation should apply only to the plate side of the plane, but if the plane is close to the wire, the approximation should be a good one:

$$V - V_0 + bE_1 = \int_0^b E_x \, dx = \int_0^b \sqrt{E_1{}^2 + \frac{2j}{\epsilon_0 K} x} \, dx$$

$$V - V_0 + bE_1 = \frac{K\epsilon_0}{2j} \frac{2}{3} \left[\left(E_1{}^2 + \frac{2j}{\epsilon_0 K} b \right)^{3/2} - E_1{}^3 \right]$$

$$j = \frac{\epsilon_0 K}{16b} \left[\alpha + \sqrt{\alpha^2 + 192(V - V_o)(bE_1)^3} \right]$$

where

$$\alpha = 9(V - V_0 + bE_1)^2 - 12(bE_1)^2, \qquad E_1 = \frac{\pi V_0}{2c \ln d/a}.$$

Curiously, in the range of interest for precipitation, this formula numerically approximates one of the form $j = sV(V - V_0)$, where s is a constant. This compares well with experimental data. Further, it exhibits a steeper than usual rise in current at high currents, which is also observed.

C. Relation with Particle Space Charge Present

P. Cooperman [2] has given an extension of his method to dust space charge. Unfortunately, this method is not directly applicable to our equation. However, if one assumes a uniform dust space charge ρ as would be indicated by the high diffusion values of M. Robinson [7], the following modified equation will result:

$$\frac{dE}{dx} = \frac{j}{\epsilon_0 KE} + \frac{\rho}{\epsilon_0}.$$

This has the solution

$$E_x - \frac{j}{\rho K} \ln\left(1 + \frac{\rho K}{j} E_x\right) = \frac{\rho}{\epsilon_0} x + E_1 - \frac{j}{\rho K} \ln\left(1 + \frac{\rho K}{j} E_1\right).$$

This gives E_x implicitly as a function of j and x. By letting

$$V - V_0' + bE_1 + \frac{b^2 \rho}{2\epsilon_0} = \int_0^b E_x \, dx,$$

it should be possible to numerically calculate j as a function of V, even in the presence of dust space charge. (In fact, this formula will give V explicitly in terms of j.) It should be noted here that V_0', the corona starting voltage in the presence of dust space charge, is given by

$$V_0' = V_0 + \frac{b^2 \rho}{2\epsilon_0}.$$

D. Numerical Use of the Relations

V_0 can be calculated by the formula $V_0 = a \ln (d/a) E_0$. E_0 can then be estimated by means of Peek's semi-empirical formula. However, this route is not recommended due to the variability of E_0 with wire roughness, cleanliness, etc. Hence, it is proposed that where possible, the usual method for estimating V_0 be used, whereby a relation $j = sV(V - V_0)$ is assumed for the experimental data from a test run, and V_0 is taken from a straight line fit of the variables j/V versus V.

Finally, once j is known, the ion space charge field is given by the formula

$$E_x = \sqrt{\left(E_1^2 + \frac{2j}{\epsilon_0 K} x\right)} - E_1.$$

The electrostatic field can be calculated by differentiating P. Cooperman's formula for electric potential in a duct is given in [9, p. 97].

IV. EXPERIMENTAL CONFIRMATION

Clearly, an ideal test of this formula would require exceptionally accurate data. In the case of gas with dust it would be difficult to find good data for uniform, measured particle space charge. However, M. Robinson and J. Shepherd [8] have taken highly accurate data for clean air under a wide variety of conditions. Care was taken to avoid edge effects (by using large flared plates), and misalignment. There is some controversy about the proper value of ion mobility. The commonly used value of 2.2 cm^2/s was assumed throughout.

Two examples with positive polarity are given for both conventional and wide plate spacing. This formula consistently overestimates the current density by about 7 percent. Reducing the ion mobility to about 2.05 cm^2/s would give nearly perfect agreement between experiment and theory. Since some researchers have reported the ion mobility for positive ions to be lower than that for negative ions, the experimental results are considered highly encouraging.

For conventional plate spacing (Fig. 1), the new formula agrees with both the experimental data and P. Cooperman's formula to better than 10 percent. For wide plate spacing and thin wires (Fig. 2), the new formula maintains high accuracy, whereas P. Cooperman's formula is too low by about 30 percent.

V. CONCLUSION

The current-voltage relations given here are expected to have an accuracy at least as good as many of the previously considered relations. While it would be difficult to find an error bound, both the derivation and the experimental evidence indicate the accuracy to be good over most normal (i.e., no back corona, etc.) operating conditions. This includes the case of wide-plate spacing, which was not as important when many of the older formulas were designed and tested.

NOMENCLATURE

- b Wire-plate spacing.
- c Half-wire-wire spacing.
- a Wire radius.
- d Equivalent cylinder radius:

$$d = \frac{4b}{\pi} \sum_{m=1}^{\infty} \frac{\cosh (m\pi c/b) + 1}{\cosh (m\pi c/b) - 1}.$$

- V Applied voltage.
- V_0 Corona starting voltage.

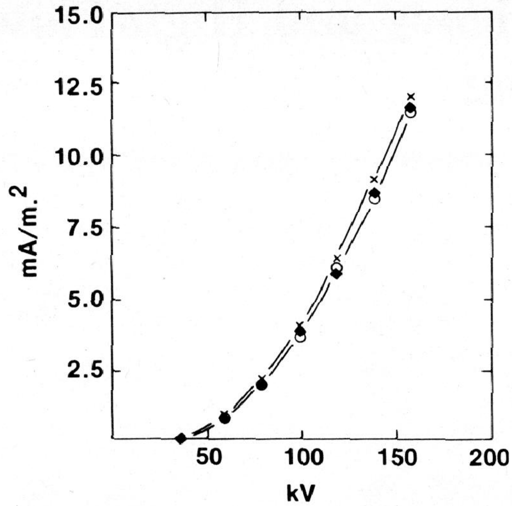

Fig. 1. Current-voltage for conventional plate spacing. Wire diameter 2.768×10^{-3} m (0.109 in). Plate-plate distance: 0.3048 m (12 in). Wire-wire distance: 0.1524 m (6 in). Starting voltage: 37 kV. Diamonds represent results from experiment. Circles represent results from P. Cooperman [1]. Crosses represent results from this paper.

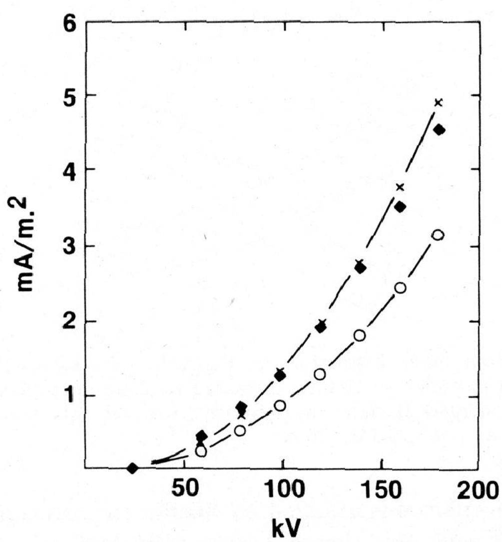

Fig. 2. Current-voltage for wide plate spacing. Wire diameter 17.78×10^{-5} m (0.007 in). Plate-plate distance: 0.4572 m (18 in). Wire-wire distance: 0.2286 m (9 in). Starting voltage: 25 kV. Diamonds represent results from experiment. Circles represent results from P. Cooperman [1]. Crosses represent results from this paper.

V_0' Corona starting voltage in presence of dust space charge.
E_0 Breakdown field of gas at wire.
E_1 Average electostatic field at plate $= \pi a E_0/2c = \pi V_0/2c \ln d/a$.
E_x Field at distance x from centerline.
x Distance from centerline.
j Current density.
ρ Dust space charge density.
K Ion mobility (2.2 cm^2/V-s).
ϵ_0 Permittivity of space (8.85×10^{-12} F/m).

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