A Numerical Model for Time-Dependent Gravity-Driven Flow in a Collapsible Tube

Michelle Borkin & Amanda Peters, and Shreyas Mandre

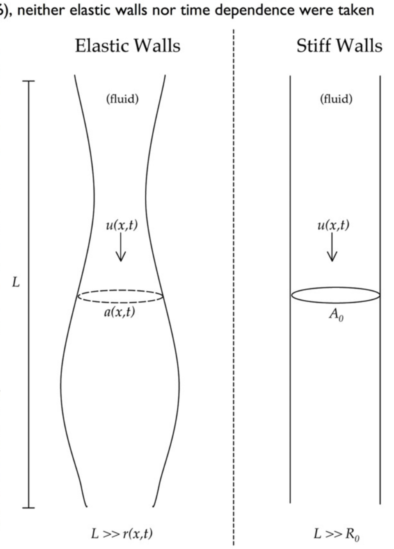
School of Engineering and Applied Sciences, Harvard University

Summary

We present details of a Navier-Stokes solver to address fluid flows through a circular tube with elastic walls. This class of problems, fluid flow through collapsible tubes, is very important to the study of biological systems (respiratory system, circulatory system, etc.) and physical systems (fluid dynamics, engineering, etc.). In contrast to other models, we focus on integrating wall elasticity and time dependence. We successfully model the flow of blood through the jugular vein of a giraffe over time by numerically evaluating a series of hyperbolic PDEs using Lax-Wendroff. We examined the role that elasticity plays at various length scales and determined that it has an impact on the flow velocity over large length scales (i.e. a giraffe) whereas it is negligible over small length scales (i.e. a human) as it is likely overwhelmed by factors such as lateral flow and viscosity. This work presents a strong framework for future CFD studies regarding various human blood flow physiologies including the abdominal aorta.

Background

We were motivated to study the effects of elastic walls relating to blood flow due to our work on the "Multiscale Hemodyanmics Project". We wanted to address whether it would be important to add elasticity into our 3D Lattice-Boltzmann model (MUPHY). The specific approximation addressed in this research is observed in some real life systems including the case of blood flow through the jugular vein in the neck of a giraffe. This problem has been one of interest over the past few decades due to the giraffe's unique anatomy. Due to the giraffe's exceptional height, its circulatory system must cope. A giraffe's heart, for instance, is ~2.5% of its total body mass compared to other mammals (including humans) where the heart is only ~0.5% (Goetz et al., 1960). It is also the case that the blood pressure in the giraffe's head is higher than at the base of its neck (Hargens et al. 1987), which is counter intuitive to the case of standing fluid in a tall glass. This implies that there is a viscous resistance to downward flow in the vein thus the vein must be significantly collapsed (Pedley, 1987). In one classic paper, Pedley et al. (1996), neither elastic walls nor time dependence were taken into account. The paper accounts for wall radius changes modeled as a wall with a fixed change from one large radius to a smaller radius. We by contrast, similar to Brook & Pedley (2002), incorporated dynamic changes in the elastic walls alongside time dependence into our model so as to be able to determine how accurate the various models are and where inaccuracies lie.



Right: Schematic drawing of the system addressed in the paper with the "stiff" case on the right (large K), and the "elastic" case on the left (small K). L is the tube length, u is the fluid velocity, a is the cross sectional area, and A_o is the undistorted cross sectional area.

Mathematical Model & Numerics

Our mathematical model is an approximation of the full time-dependent Navier-Stokes fluid dynamics equations in which we assume the length of the tube is much larger than the radius ($L >> r$) and there are no abrupt changes in the tube's diameter. This allows us to neglect velocity of the fluid perpendicular to the direction of flow. Based on Brook et al. (1999), our non-dimensionalized (i.e. scaled so 1 represents the equilibrium (steady) value) governing equations in our system are:

$$\frac{\partial \alpha}{\partial T} + \frac{\partial}{\partial X}(U\alpha) = 0$$

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + \frac{p_e}{\rho U_c^2}\frac{\partial P}{\partial X} = \frac{gL}{U_c{}^2}\Big(1 - \frac{U}{\alpha^{3/2}}\Big) + D\frac{\partial^2 U}{\partial X^2}$$

where T is time, X is distance along the tube with $X = 0$ representing the top, $\alpha(X,T)$ is the cross sectional area of the tube, $U(X,T)$ is the downstream velocity averaged across the cross sectional area, p_e is the external pressure, ρ is the fluid density, U_c is the critical velocity when the viscous resistance to laminar flow balances the gravitational force, $P(X,T)$ is the pressure, g is gravitational acceleration, L is the tube length, and D is the diffusion constant (discussed below). The first equation, commonly called the "continuity equation", represents conservation of mass and the second equation represents the conservation of momentum in the system. The pressure is defined as:

$$P(X,T) = 1 + \frac{K}{p_e}\left(\alpha^{n_{bio}} - \alpha^{-3/2}\right)$$

where K represents the elasticity (bending stiffness) of the tube, and n_{bio} is chosen to match the biological properties of real blood vessels. We also pre-define the initial geometry of the tube to include one period of a sine wave over the length of the tube:

$$\alpha(X, T=1) = 1 + 0.1 sin\left(\frac{2\pi x}{L}\right)$$

This serves to give an initial perturbation to the system, and can be thought of as small initial pulse of blood through the vein. The initial pressure, based on the above equation for the cross sectional, becomes:

$$P(T=1) = \frac{P(\alpha(T=1))}{P(\alpha(A_o))}$$

where the pressure of the cross sectional area at $T = 1$ is scaled by the pressure of a tube with constant cross sectional area A_o.

To solve our system of equations we applied the finite difference methods Lax-Wenderoff which consists of doing forward difference in time and central difference in space. We chose to use this method due to its simplicity and it is not prone to over-damping at long time scales. In order to ensure conditional stability we needed to both adhere to the Courant-Friedrichs-Lewy condition (CFL condition) and introduce an artificial diffusion constant (D). To illustrate a perpetual wave, we chose to use periodic boundary conditions.

http://hemo.seas.harvard.edu

Results

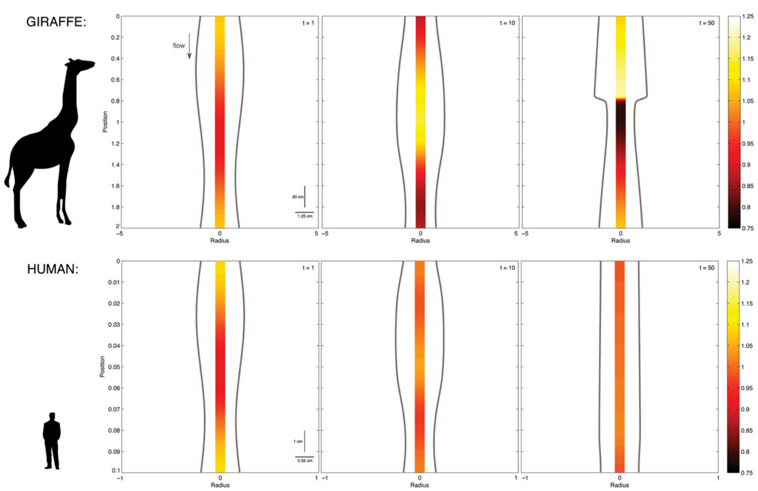
When looking at the model of the blood flow through a giraffe's jugular vein with elasticity taken into account, the significant area of focus was to look at its effect on the flow rate through the tube. We found that:

- Fluctuations in the diameter of the tube will cause the fluid to speed and slow accordingly as it moves through the length of the tube.
- By varying the elasticity of the walls, waves would quickly stop being propagated through the walls of the tube when they were stiff (vs. waves would continue to propagate with elastic walls).
- The wall elasticity determines the change in cross sectional area thus affecting the flow rate.

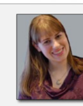
To determine the length scale at which the elasticity would have a significant impact on the flow rate, we investigated the percent change in flow rate created by elastic walls:

- For the giraffe, there is a 31% difference in the height of the peaks of the velocity for stiff vs. elastic walls (elasticity plays a significant role at the giraffe length scale).
- For the human model, the velocity in the elastic wall case is only 2% higher than the velocity calculated with a stiff wall (elasticity less important since other factors begin to overwhelm the velocity calculation when looking at human length scales).

It would also be enlightening to extend this model to higher dimensions to create a more realistic model. We plan to incorporate this knowledge into the Multiscale Hemodyanmics' large-scale 3D blood flow simulations.

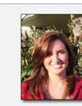


Above: Movie frames from blood flow animations of elastic walls. *Top:* Giraffe jugular vein. Time step 10 shows beginning of elastic walls influencing the velocity, and by time step 50 the elastic walls are semi-collapsing thus slowing down the blood flow (a real anatomically observed effect). *Bottom:* Human jugular vein. Time step 10 shows velocity smoothing out, and by time step 50 the walls and velocity equilibrated.



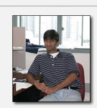
Michelle Borkin

michelle_borkin@harvard.edu



Amanda Peters

apeters@fas.harvard.edu



Shreyas Mandre

shreyas@seas.harvard.edu

Acknowledgements: We would like to thank Prof. Efthimios Kaxiras for supporting, taking the time to brainstorm, and discussing future directions of this project.