Theorems for Free for Free: Parametricity, With and Without Types

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Contents

1 Polymorphic Blame Calculus

$$
\begin{array}{lrcl}
\textit{Convertibility Labels} & \phi & ::= & +\alpha \mid -\alpha \\
\textit{Compatibility Labels} & p\,,\,q & ::= & +\ell \mid -\ell \\
\textit{Base Types} & \iota & ::= & \mathsf{int} \mid \mathsf{bool} \\
\textit{Types} & A\,,\,B & ::= & \iota \mid A \to B \mid \forall X\,.\,A \mid A \times B \mid X \mid \alpha \mid \star \\
\textit{Ground Types} & G\,,\,H & ::= & \iota \mid \star \to \star \mid \alpha \\
\textit{Operations} & \circledast & ::= & + \mid - \mid * \mid \ldots \\
\textit{Expressions} & e & ::= & n \mid \mathsf{true} \mid \mathsf{false} \mid \text{if } e \text{ then } e \text{ else } e \mid e \circledast e \mid x \mid \lambda(x : A).\, e \mid e\ e \mid \Lambda X\,.v \mid \\
 & & & e\ [B] \mid \langle e , e \rangle \mid \pi_1\ e \mid \pi_2\ e \mid (e : A \overset{\phi}{\Longrightarrow} B) \mid (e : A \overset{p}{\Longrightarrow} B) \mid \text{blame}\, p \\
\textit{Values} & v & ::= & n \mid \mathsf{true} \mid \mathsf{false} \mid \lambda(x : A).\, e \mid \Lambda X\,.v \mid \langle v , v \rangle \mid (v : A \to B \overset{\phi}{\Longrightarrow} A' \to B') \mid \\
 & & & (v : \forall X\,.\,A \overset{\phi}{\Longrightarrow} \forall X\,.\,B) \mid (v : A \overset{-\alpha}{\Longrightarrow} \alpha) \mid (v : A \to B \overset{p}{\Longrightarrow} A' \to B') \mid \\
 & & & (v : A \overset{p}{\Longrightarrow} \forall X\,.\,B) \mid (v : G \overset{p}{\Longrightarrow} \star) \\
\textit{Type-Name Stores} & \Sigma & ::= & \cdot \mid \Sigma, \alpha := A \\
\textit{Type Environments} & \Delta & ::= & \cdot \mid \Delta, X \\
\textit{Environments} & \Gamma & ::= & \cdot \mid \Gamma, x : A \\
\textit{Evaluation Contexts} & E & ::= & [\cdot] \mid E \circledast e \mid v \circledast E \mid \text{if } E \text{ then } e \text{ else } e \mid E\ e \mid v\ E \mid E\ [A] \mid \\
 & & & \langle E , e \rangle \mid \langle v , E \rangle \mid (E : A \overset{\phi}{\Longrightarrow} B) \mid (E : A \overset{p}{\Longrightarrow} B) \\
\textit{Term Variable Closures} & \gamma & ::= & \{x \mapsto (v_1, v_2), \ldots\} \\
\textit{Type Variable Closures} & \rho & ::= & \{X \mapsto \alpha, \ldots\} \\
\textit{Type Name Relations} & \kappa & ::= & \{\alpha \mapsto R, \ldots\} \\
\textit{Worlds} & W & ::= & (j, \Sigma_1, \Sigma_2, \kappa) \\
\textit{Relations} & R & ::= & \{(W, e_1, e_2), \ldots\}
\end{array}
$$

Figure 1: Syntax

Shorthand:

$$
\begin{array}{lcl}
\alpha \notin \phi & \overset{\text{def}}{=} & \phi \neq -\alpha \wedge \phi \neq +\alpha \\
\alpha \notin \Sigma & \overset{\text{def}}{=} & \alpha \notin \text{dom}(\Sigma)
\end{array}
$$

Store Well-Formedness $\boxed{\vdash \Sigma}$

$$\frac{}{\vdash \cdot} \qquad \frac{\alpha \notin \Sigma \qquad \Sigma; \cdot \vdash A}{\vdash \Sigma, \alpha := A}$$

Type Well-Formedness $\boxed{\Sigma; \Delta \vdash A}$ where $\vdash \Sigma$

$$\frac{\vdash \Sigma \qquad X \in \Delta}{\Sigma; \Delta \vdash X} \qquad \frac{\vdash \Sigma \qquad \alpha := A \in \Sigma}{\Sigma; \Delta \vdash \alpha} \qquad \frac{\vdash \Sigma}{\Sigma; \Delta \vdash \mathsf{int}} \qquad \frac{\vdash \Sigma}{\Sigma; \Delta \vdash \mathsf{bool}} \qquad \frac{\vdash \Sigma}{\Sigma; \Delta \vdash \star} \qquad \frac{\Sigma; \Delta \vdash A \qquad \Sigma; \Delta \vdash B}{\Sigma; \Delta \vdash A \to B}$$

$$\frac{\Sigma; \Delta, X \vdash A}{\Sigma; \Delta \vdash \forall X \,.\, A} \qquad \frac{\Sigma; \Delta \vdash A \qquad \Sigma; \Delta \vdash B}{\Sigma; \Delta \vdash A \times B}$$

Convertibility $\boxed{\Sigma; \Delta \vdash A \prec^{\phi} B}$ where $\Sigma; \Delta \vdash A$, $\Sigma; \Delta \vdash B$, and $FTN(\phi) \in \Sigma$

$$\frac{\vdash \Sigma}{\Sigma; \Delta \vdash \mathsf{int} \prec^{\phi} \mathsf{int}} \qquad \frac{\vdash \Sigma}{\Sigma; \Delta \vdash \mathsf{bool} \prec^{\phi} \mathsf{bool}} \qquad \frac{\Sigma; \Delta \vdash A' \prec^{-\phi} A \qquad \Sigma; \Delta \vdash B \prec^{\phi} B'}{\Sigma; \Delta \vdash A \to B \prec^{\phi} A' \to B'} \qquad \frac{\Sigma; \Delta, X \vdash A \prec^{\phi} B}{\Sigma; \Delta \vdash \forall X \,.\, A \prec^{\phi} \forall X \,.\, B}$$

$$\frac{\Sigma; \Delta \vdash A \prec^{\phi} A' \qquad \Sigma; \Delta \vdash B \prec^{\phi} B'}{\Sigma; \Delta \vdash A \times B \prec^{\phi} A' \times B'} \qquad \frac{\vdash \Sigma \qquad \alpha := A \in \Sigma}{\Sigma; \Delta \vdash \alpha \prec^{+\alpha} A} \qquad \frac{\vdash \Sigma \qquad \alpha := A \in \Sigma}{\Sigma; \Delta \vdash A \prec^{-\alpha} \alpha}$$

$$\frac{\vdash \Sigma \qquad \alpha := A \in \Sigma \qquad \alpha \notin \phi}{\Sigma; \Delta \vdash \alpha \prec^{\phi} \alpha} \qquad \frac{\vdash \Sigma \qquad X \in \Delta}{\Sigma; \Delta \vdash X \prec^{\phi} X} \qquad \frac{\vdash \Sigma}{\Sigma; \Delta \vdash \star \prec^{\phi} \star}$$

Label Negation

$$\begin{aligned} -(+\alpha) &\stackrel{\text{def}}{=} -\alpha \\ -(-\alpha) &\stackrel{\text{def}}{=} +\alpha \end{aligned}$$

Compatibility $\boxed{\Sigma; \Delta \vdash A \prec B}$ where $\Sigma; \Delta \vdash A$ and $\Sigma; \Delta \vdash B$

$$\frac{\vdash \Sigma}{\Sigma; \Delta \vdash \mathsf{int} \prec \mathsf{int}} \qquad \frac{\vdash \Sigma}{\Sigma; \Delta \vdash \mathsf{bool} \prec \mathsf{bool}} \qquad \frac{\Sigma; \Delta \vdash A' \prec A \qquad \Sigma; \Delta \vdash B \prec B'}{\Sigma; \Delta \vdash A \to B \prec A' \to B'} \qquad \frac{\Sigma; \Delta, X \vdash A \prec B \qquad X \notin A}{\Sigma; \Delta \vdash A \prec \forall X \,.\, B}$$

$$\frac{\Sigma; \Delta \vdash A[\star/X] \prec B}{\Sigma; \Delta \vdash \forall X \,.\, A \prec B} \qquad \frac{\Sigma; \Delta \vdash A \prec A' \qquad \Sigma; \Delta \vdash B \prec B'}{\Sigma; \Delta \vdash A \times B \prec A' \times B'} \qquad \frac{\vdash \Sigma \qquad \alpha \in \Sigma}{\Sigma; \Delta \vdash \alpha \prec \alpha} \qquad \frac{\vdash \Sigma \qquad X \in \Delta}{\Sigma; \Delta \vdash X \prec X}$$

$$\frac{\Sigma; \Delta \vdash A}{\Sigma; \Delta \vdash A \prec \star} \qquad \frac{\Sigma; \Delta \vdash A}{\Sigma; \Delta \vdash \star \prec A}$$

Figure 2: Type-Level Static Semantics

Environment Well-Formedness $\boxed{\Sigma ; \Delta \vdash \Gamma}$ where $\vdash \Sigma$

$$\frac{\vdash \Sigma}{\Sigma ; \Delta \vdash \cdot} \qquad \frac{\Sigma ; \Delta \vdash \Gamma \qquad \Sigma ; \Delta \vdash A}{\Sigma ; \Delta \vdash \Gamma , x : A}$$

Expression Well-Formedness $\boxed{\Sigma ; \Delta ; \Gamma \vdash e : A}$ where $\Sigma ; \Delta \vdash \Gamma$ and $\Sigma ; \Delta \vdash A$

$$\frac{\Sigma ; \Delta \vdash \Gamma}{\Sigma ; \Delta ; \Gamma \vdash \mathsf{true} : \mathsf{bool}} \qquad \frac{\Sigma ; \Delta \vdash \Gamma}{\Sigma ; \Delta ; \Gamma \vdash \mathsf{false} : \mathsf{bool}} \qquad \frac{\Sigma ; \Delta ; \Gamma \vdash e : \mathsf{bool} \qquad \Sigma ; \Delta ; \Gamma \vdash e_1 : A \qquad \Sigma ; \Delta ; \Gamma \vdash e_2 : A}{\Sigma ; \Delta ; \Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : A}$$

$$\frac{\Sigma ; \Delta \vdash \Gamma}{\Sigma ; \Delta ; \Gamma \vdash n : \mathsf{int}} \qquad \frac{\Sigma ; \Delta ; \Gamma \vdash e : \mathsf{int} \qquad \Sigma ; \Delta ; \Gamma \vdash e' : \mathsf{int}}{\Sigma ; \Delta ; \Gamma \vdash e \circledast e' : \mathsf{int}}$$

$$\frac{\Sigma ; \Delta \vdash \Gamma \qquad \Gamma(x) = A}{\Sigma ; \Delta ; \Gamma \vdash x : A} \qquad \frac{\Sigma ; \Delta ; \Gamma , x : A \vdash e : B}{\Sigma ; \Delta ; \Gamma \vdash \lambda(x : A).\, e : A \to B} \qquad \frac{\Sigma ; \Delta ; \Gamma \vdash e : B \to A \qquad \Sigma ; \Delta ; \Gamma \vdash e' : B}{\Sigma ; \Delta ; \Gamma \vdash e\ e' : A}$$

$$\frac{\Sigma ; \Delta , X ; \Gamma \vdash v : A \qquad \Sigma ; \Delta \vdash \Gamma}{\Sigma ; \Delta ; \Gamma \vdash \Lambda X . v : \forall X . A} \qquad \frac{\Sigma ; \Delta ; \Gamma \vdash e : \forall X . A \qquad \Sigma ; \Delta \vdash B}{\Sigma ; \Delta ; \Gamma \vdash e\ [B] : A[B/X]} \qquad \frac{\Sigma ; \Delta ; \Gamma \vdash e_1 : A \qquad \Sigma ; \Delta ; \Gamma \vdash e_2 : B}{\Sigma ; \Delta ; \Gamma \vdash \langle e_1, e_2 \rangle : A \times B}$$

$$\frac{\Sigma ; \Delta ; \Gamma \vdash e : A \times B}{\Sigma ; \Delta ; \Gamma \vdash \pi_1\, e : A} \qquad \frac{\Sigma ; \Delta ; \Gamma \vdash e : A \times B}{\Sigma ; \Delta ; \Gamma \vdash \pi_2\, e : B} \qquad \frac{\Sigma ; \Delta ; \Gamma \vdash e : A \qquad \Sigma ; \Delta \vdash A \prec^{\phi} B}{\Sigma ; \Delta ; \Gamma \vdash (e : A \overset{\phi}{\Longrightarrow} B) : B}$$

$$\frac{\Sigma ; \Delta ; \Gamma \vdash e : A \qquad \Sigma ; \Delta \vdash A \prec B}{\Sigma ; \Delta ; \Gamma \vdash (e : A \overset{p}{\Longrightarrow} B) : B} \qquad \frac{\Sigma ; \Delta \vdash \Gamma \qquad \Sigma ; \Delta \vdash A}{\Sigma ; \Delta ; \Gamma \vdash \text{blame}\, p : A}$$

Figure 3: Expression-Level Static Semantics

$\boxed{e \longmapsto e'}$

$$
\begin{array}{rcl}
n \circledast n' & \longmapsto & [\![\circledast]\!]\,(n,n') \\
\text{if } \mathsf{true} \text{ then } e_1 \text{ else } e_2 & \longmapsto & e_1 \\
\text{if } \mathsf{false} \text{ then } e_1 \text{ else } e_2 & \longmapsto & e_2 \\
(\lambda(x:A).\,e)\ v & \longmapsto & e\,[v/x] \\
\pi_1\ \langle v_1,v_2\rangle & \longmapsto & v_1 \\
\pi_2\ \langle v_1,v_2\rangle & \longmapsto & v_2 \\
(v:\mathsf{int} \overset{\phi}{\Longrightarrow} \mathsf{int}) & \longmapsto & v \\
(v:\mathsf{bool} \overset{\phi}{\Longrightarrow} \mathsf{bool}) & \longmapsto & v \\
(\langle v_1,v_2\rangle : A \times B \overset{\phi}{\Longrightarrow} A' \times B') & \longmapsto & \langle (v_1 : A \overset{\phi}{\Longrightarrow} B),(v_2 : A' \overset{\phi}{\Longrightarrow} B')\rangle \\
(v : A \to B \overset{\phi}{\Longrightarrow} A' \to B')\ v' & \longmapsto & (v\ \ (v' : A' \overset{-\phi}{\Longrightarrow} A) : B \overset{\phi}{\Longrightarrow} B') \\
(v : \alpha \overset{\phi}{\Longrightarrow} \alpha) & \longmapsto & v \quad \text{if } \alpha \notin \phi \\
((v : A \overset{-\alpha}{\Longrightarrow} \alpha) : \alpha \overset{+\alpha}{\Longrightarrow} A) & \longmapsto & v \\
(v : \star \overset{\phi}{\Longrightarrow} \star) & \longmapsto & v \\
(v:\mathsf{int} \overset{p}{\Longrightarrow} \mathsf{int}) & \longmapsto & v \\
(v:\mathsf{bool} \overset{p}{\Longrightarrow} \mathsf{bool}) & \longmapsto & v \\
(\langle v_1,v_2\rangle : A \times B \overset{p}{\Longrightarrow} A' \times B') & \longmapsto & \langle (v_1 : A \overset{p}{\Longrightarrow} B),(v_2 : A' \overset{p}{\Longrightarrow} B')\rangle \\
(v : A \to B \overset{p}{\Longrightarrow} A' \to B')\ v' & \longmapsto & (v\ \ (v' : A' \overset{-p}{\Longrightarrow} A) : B \overset{p}{\Longrightarrow} B') \\
(v : \forall X\,.\,A \overset{p}{\Longrightarrow} B) & \longmapsto & (v\ [\star] : A[\star/X] \overset{p}{\Longrightarrow} B) \quad \text{if } B \neq \forall Y.\,B' \text{ for any } Y, B' \\
(v : \alpha \overset{p}{\Longrightarrow} \alpha) & \longmapsto & v \\
(v : \star \overset{p}{\Longrightarrow} \star) & \longmapsto & v \\
((v : G \overset{p}{\Longrightarrow} \star) : \star \overset{q}{\Longrightarrow} G) & \longmapsto & v \\
((v : G \overset{p}{\Longrightarrow} \star) : \star \overset{q}{\Longrightarrow} H) & \longmapsto & \text{blame}\ q \quad \text{if } G \neq H \\
(v : A \to B \overset{p}{\Longrightarrow} \star) & \longmapsto & ((v : A \to B \overset{p}{\Longrightarrow} \star \to \star) : \star \to \star \overset{p}{\Longrightarrow} \star) \quad \text{if } A \to B \neq \star \to \star \\
(v : \star \overset{p}{\Longrightarrow} A \to B) & \longmapsto & ((v : \star \overset{p}{\Longrightarrow} \star \to \star) : \star \to \star \overset{p}{\Longrightarrow} A \to B) \quad \text{if } A \to B \neq \star \to \star
\end{array}
$$

$\boxed{\Sigma \triangleright e \ \longmapsto\ \Sigma' \triangleright e'}$

$$
\frac{e \longmapsto e'}{\Sigma \triangleright E\,[e] \ \longmapsto\ \Sigma \triangleright E\,[e']}
\qquad
\frac{\Sigma \triangleright e \ \longmapsto\ \Sigma' \triangleright e'}{\Sigma \triangleright E\,[e] \ \longmapsto\ \Sigma' \triangleright E\,[e']}
\qquad
\frac{}{\Sigma \triangleright E\,[\text{blame}\,p\,] \ \longmapsto\ \Sigma \triangleright \text{blame}\,p}
$$

$$
\frac{\Sigma; (\cdot, X);\ \cdot \vdash v : A \qquad \alpha \notin \text{dom}(\Sigma)}{\Sigma \triangleright (\Lambda X\,.\,v)\ [B] \ \longmapsto\ \Sigma, \alpha{:=}B \triangleright (v\,[\alpha/X] : A[\alpha/X] \overset{+\alpha}{\Longrightarrow} A[B/X])}
$$

$$
\frac{\alpha \notin \text{dom}(\Sigma)}{\Sigma \triangleright (v : A \overset{p}{\Longrightarrow} \forall X.\,A')\ [B] \ \longmapsto\ \Sigma, \alpha{:=}B \triangleright ((v : A \overset{p}{\Longrightarrow} A'[\alpha/X]) : A'[\alpha/X] \overset{+\alpha}{\Longrightarrow} A'[B/X])}
$$

$$
\frac{\alpha \notin \text{dom}(\Sigma)}{\Sigma \triangleright (v : \forall X.\,A \overset{\phi}{\Longrightarrow} \forall X.\,A')\ [B] \ \longmapsto\ \Sigma, \alpha{:=}B \triangleright ((v\ [\alpha] : A[\alpha/X] \overset{\phi}{\Longrightarrow} A'[\alpha/X]) : A'[\alpha/X] \overset{+\alpha}{\Longrightarrow} A'[B/X])}
$$

Figure 4: Dynamic Semantics

2 Context and Contextual Equivalence

$$
\begin{array}{llll}
\textit{Expression Contexts} & C & ::= & [\cdot] \mid C \circledast e \mid e \circledast C \mid \text{if } C \text{ then } e \text{ else } e \mid \text{if } e \text{ then } C \text{ else } e \mid \text{if } e \text{ then } e \text{ else } C \mid \\
& & & \lambda(x : A).\, C \mid C\ e \mid e\ C \mid \Lambda X . C_{\mathrm{v}} \mid C\,[X] \mid \langle C, e\rangle \mid \langle e, C\rangle \mid \pi_1\, C \mid \pi_2\, C \mid \\
& & & (C : A \overset{\phi}{\Longrightarrow} B) \mid (C : A \overset{p}{\Longrightarrow} B) \\
\textit{Value Contexts} & C_{\mathrm{v}} & ::= & [\cdot]_{\mathrm{v}} \mid \lambda(x : A).\, C \mid \Lambda X . C_{\mathrm{v}} \mid \langle C_{\mathrm{v}}, v\rangle \mid \langle v, C_{\mathrm{v}}\rangle \mid (C_{\mathrm{v}} : A \to B \overset{\phi}{\Longrightarrow} A' \to B') \mid \\
& & & (C_{\mathrm{v}} : \forall X . A \overset{\phi}{\Longrightarrow} \forall X . B) \mid (C_{\mathrm{v}} : A \overset{-\alpha}{\Longrightarrow} \alpha) \mid (C_{\mathrm{v}} : A \to B \overset{p}{\Longrightarrow} A' \to B') \mid \\
& & & (C_{\mathrm{v}} : A \overset{p}{\Longrightarrow} \forall X . B) \mid (C_{\mathrm{v}} : G \overset{p}{\Longrightarrow} \star)
\end{array}
$$

Figure 5: Context Syntax

Context Well-Formedness $\boxed{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow (\Sigma';\Delta';\Gamma' \vdash A)}$

$$\frac{\Sigma \subseteq \Sigma' \qquad \Delta \subseteq \Delta' \qquad \Gamma \subseteq \Gamma'}{\vdash [\cdot] : (\Sigma;\Delta;\Gamma \vdash A) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A\big)}$$

$$\frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash \mathsf{bool}\big) \qquad \Sigma';\Delta';\Gamma' \vdash e_1 : A \qquad \Sigma';\Delta';\Gamma' \vdash e_2 : A}{\vdash \text{if } C \text{ then } e_1 \text{ else } e_2 : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A\big)}$$

$$\frac{\Sigma';\Delta';\Gamma' \vdash e : \mathsf{bool} \qquad \vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A\big) \qquad \Sigma';\Delta';\Gamma' \vdash e_2 : A}{\vdash \text{if } e \text{ then } C \text{ else } e_2 : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A\big)}$$

$$\frac{\Sigma';\Delta';\Gamma' \vdash e : \mathsf{bool} \qquad \Sigma';\Delta';\Gamma' \vdash e_1 : A \qquad \vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A\big)}{\vdash \text{if } e \text{ then } e_1 \text{ else } C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A\big)}$$

$$\frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash \mathsf{int}\big) \qquad \Sigma';\Delta';\Gamma' \vdash e : \mathsf{int}}{\vdash C \circledast e : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash \mathsf{int}\big)}$$

$$\frac{\Sigma';\Delta';\Gamma' \vdash e : \mathsf{int} \qquad \vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash \mathsf{int}\big)}{\vdash e \circledast C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash \mathsf{int}\big)}$$

$$\frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma', x : A_1 \vdash A_2\big)}{\vdash \lambda(x : A_1).\, C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_1 \rightarrow A_2\big)}$$

$$\frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_1 \rightarrow A_2\big) \qquad \Sigma';\Delta';\Gamma' \vdash e' : A_1}{\vdash C\ \ e' : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_2\big)}$$

$$\frac{\Sigma';\Delta';\Gamma' \vdash e : A_1 \rightarrow A_2 \qquad \vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_1\big)}{\vdash e\ \ C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_2\big)}$$

$$\frac{\vdash C_{\mathrm{v}} : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta', X;\Gamma' \vdash A\big)}{\vdash \Lambda X.\, C_{\mathrm{v}} : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash \forall X.\, A\big)} \qquad \frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash \forall X.\, A\big) \qquad \Sigma';\Delta' \vdash A'}{\vdash C\ [A'] : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A[A'/X]\big)}$$

$$\frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A\big) \qquad \Sigma';\Delta';\Gamma' \vdash e : A'}{\vdash \langle C, e\rangle : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A \times A'\big)} \qquad \frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A \times A'\big)}{\vdash \pi_1\, C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A\big)}$$

$$\frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A'\big) \qquad \Sigma';\Delta';\Gamma' \vdash e : A}{\vdash \langle e, C\rangle : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A \times A'\big)} \qquad \frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A \times A'\big)}{\vdash \pi_2\, C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A'\big)}$$

$$\frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_1\big) \qquad \Sigma';\Delta' \vdash A_1 \prec^{\phi} A_2}{\vdash (C : A_1 \xRightarrow{\phi} A_2) : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_2\big)}$$

$$\frac{\vdash C : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_1\big) \qquad \Sigma';\Delta' \vdash A_1 \prec A_2}{\vdash (C : A_1 \xRightarrow{p} A_2) : (\Sigma;\Delta;\Gamma \vdash B) \rightsquigarrow \big(\Sigma';\Delta';\Gamma' \vdash A_2\big)}$$

Figure 6: Context Static Semantics

Contextual Equivalence:

$$
\begin{array}{lcl}
\Sigma;\Delta;\Gamma \vdash e_1 \preceq^{ctx} e_2 : A & \stackrel{\text{def}}{=} & \Sigma;\Delta;\Gamma \vdash e_1 : A \;\wedge\; \Sigma;\Delta;\Gamma \vdash e_2 : A \;\wedge \\
 & & \forall C, \Sigma', B.\ \vdash C : (\Sigma;\Delta;\Gamma \vdash A) \leadsto (\Sigma';\ \cdot\ ;\ \cdot \vdash B) \implies \\
 & & \quad (\Sigma' \triangleright C[e_1]\Downarrow \implies \Sigma' \triangleright C[e_2]\Downarrow) \;\wedge \\
 & & \quad (\exists \Sigma_1.\ \Sigma' \triangleright C[e_1] \longmapsto^* \Sigma_1 \triangleright \mathrm{blame}\, p \implies \exists \Sigma_2.\ \Sigma' \triangleright C[e_2] \longmapsto^* \Sigma_2 \triangleright \mathrm{blame}\, p\ ) \\
\Sigma;\Delta;\Gamma \vdash e_1 \approx^{ctx} e_2 : A & \stackrel{\text{def}}{=} & \Sigma;\Delta;\Gamma \vdash e_1 \preceq^{ctx} e_2 : A \;\wedge\; \Sigma;\Delta;\Gamma \vdash e_2 \preceq^{ctx} e_1 : A
\end{array}
$$

Figure 7: Contextual Equivalence

3 Logical Relation

Shorthand:

$$\begin{array}{lcll}
(j, \Sigma_1, \Sigma_2, \kappa).j & = & j & \\
(j, \Sigma_1, \Sigma_2, \kappa).\Sigma_1 & = & \Sigma_1 & \\
(j, \Sigma_1, \Sigma_2, \kappa).\Sigma_2 & = & \Sigma_2 & \\
(j, \Sigma_1, \Sigma_2, \kappa).\kappa & = & \kappa & \\
\gamma_1(x) & = & v_1 & \text{where } \gamma(x) = (v_1, v_2) \\
\gamma_2(x) & = & v_2 & \text{where } \gamma(x) = (v_1, v_2)
\end{array}$$

$$\begin{array}{lcl}
\text{Atom}_n\,[A_1, A_2] & = & \{(W, e_1, e_2) \mid W.j < n \ \wedge\ W \in \text{World}_n \ \wedge\ W.\Sigma_1; \cdot; \cdot \vdash e_1 : A_1 \ \wedge\ W.\Sigma_2; \cdot; \cdot \vdash e_2 : A_2\} \\
\text{Atom}^{\text{val}}_n\,[A_1, A_2] & = & \{(W, v_1, v_2) \in \text{Atom}_n\,[A_1, A_2]\} \\
\text{Rel}_n\,[A_1, A_2] & = & \{R \subseteq \text{Atom}^{\text{val}}_n\,[A_1, A_2] \mid \forall (W, v_1, v_2) \in R.\ \forall W' \sqsupseteq W.\ (W', v_1, v_2) \in R\} \\
\text{World}_n & = & \{(j, \Sigma_1, \Sigma_2, \kappa) \in \text{Nat} \times \text{TNStore} \times \text{TNStore} \times (\text{TName} \overset{\text{fin}}{\to} \text{Rel}_j) \mid \\
& & \quad j < n \ \wedge\ \vdash \Sigma_1 \ \wedge\ \vdash \Sigma_2 \ \wedge \\
& & \quad \forall \alpha \in \text{dom}(\kappa).\ \kappa(\alpha) \in \text{Rel}_j\,[\Sigma_1(\alpha), \Sigma_2(\alpha)]\} \\
\\
\text{Atom}\,[A]\,\rho & = & \bigcup_{n \geq 0} \{(W, e_1, e_2) \in \text{Atom}_n\,[\rho(A), \rho(A)]\} \\
\\
\text{World} & = & \bigcup_{n \geq 0} \text{World}_n
\end{array}$$

$$\begin{array}{lcl}
\Sigma \triangleright e \Downarrow & \overset{\text{def}}{=} & \exists \Sigma', v.\ \Sigma \triangleright e \ \longmapsto^* \ \Sigma' \triangleright v \\
\\
W' \sqsupseteq W & \overset{\text{def}}{=} & W'.j \leq W.j \ \wedge\ W'.\Sigma_1 \supseteq W.\Sigma_1 \ \wedge\ W'.\Sigma_2 \supseteq W.\Sigma_2 \ \wedge\ W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.j} \ \wedge \\
& & \quad W, W' \in \text{World} \\
W' \sqsupseteq_n W & \overset{\text{def}}{=} & W'.j + n = W.j \ \wedge\ W' \sqsupseteq W \\
\kappa' \sqsupseteq \kappa & \overset{\text{def}}{=} & \forall \alpha \in \text{dom}(\kappa).\ \kappa'(\alpha) = \kappa(\alpha) \\
\\
\lfloor R \rfloor_n & = & \{(W, e_1, e_2) \in R \mid W.j < n\} \\
\lfloor \kappa \rfloor_n & = & \{\alpha \mapsto \lfloor R \rfloor_n \mid \kappa(\alpha) = R\} \\
\\
\blacktriangleright R & = & \{(W, e_1, e_2) \mid W.j > 0 \implies (\blacktriangleright W, e_1, e_2) \in R\} \\
\blacktriangleright (j + 1, \Sigma_1, \Sigma_2, \kappa) & \overset{\text{def}}{=} & (j, \Sigma_1, \Sigma_2, \lfloor \kappa \rfloor_j) \\
\\
W \boxplus (\alpha, B_1, B_2, R) & \overset{\text{def}}{=} & (W.j, W.\Sigma_1, \alpha{:=}B_1, W.\Sigma_2, \alpha{:=}B_2, W.\kappa[\alpha \mapsto R])
\end{array}$$

Figure 8: Auxiliary Definitions

κ is a relational interpretation for type names. However, rather than relate terms at the type names themselves, we define the interpretation of a type name α as a relation on terms at their bound types $W.\Sigma_1(\alpha)$ and $W.\Sigma_2(\alpha)$.

$$
\begin{array}{lcl}
\mathcal{V}\llbracket \mathsf{int} \rrbracket \rho & = & \{(W, n, n) \in \mathrm{Atom}\,[\mathsf{int}]\,\rho\} \\
\mathcal{V}\llbracket \mathsf{bool} \rrbracket \rho & = & \{(W, b, b) \in \mathrm{Atom}\,[\mathsf{bool}]\,\rho\} \\
\mathcal{V}\llbracket A \to B \rrbracket \rho & = & \{(W, v_{f1}, v_{f2}) \in \mathrm{Atom}\,[A \to B]\,\rho \mid \\
 & & \quad \forall W' \sqsupseteq W.\ \forall v_1, v_2.\ (W', v_1, v_2) \in \mathcal{V}\llbracket A \rrbracket \rho \implies (W', v_{f1}\ v_1, v_{f2}\ v_2) \in \mathcal{E}\llbracket B \rrbracket \rho\} \\
\mathcal{V}\llbracket X \rrbracket \rho & = & \{(W, (v_1 : A_1 \overset{-\alpha}{\Longrightarrow} \alpha), (v_2 : A_2 \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathrm{Atom}\,[X]\,\rho \mid \\
 & & \quad (W, v_1, v_2) \in \blacktriangleright W.\kappa(\alpha)\} \\
\mathcal{V}\llbracket \alpha \rrbracket \rho & = & \{(W, (v_1 : A_1 \overset{-\alpha}{\Longrightarrow} \alpha), (v_2 : A_2 \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathrm{Atom}\,[\alpha]\,\emptyset \mid \\
 & & \quad (W, v_1, v_2) \in \blacktriangleright W.\kappa(\alpha)\} \\
\mathcal{V}\llbracket \star \rrbracket \rho & = & \{(W, (v : \iota \overset{p}{\Longrightarrow} \star), (v : \iota \overset{p}{\Longrightarrow} \star)) \in \mathrm{Atom}\,[\star]\,\emptyset\} \\
 & \cup & \{(W, (v_1 : \star \!\to\! \star \overset{p}{\Longrightarrow} \star), (v_2 : \star \!\to\! \star \overset{p}{\Longrightarrow} \star)) \in \mathrm{Atom}\,[\star]\,\emptyset \mid (W, v_1, v_2) \in \blacktriangleright \mathcal{V}\llbracket \star \to \star \rrbracket \rho\} \\
 & \cup & \{(W, (v_1 : \alpha \overset{p}{\Longrightarrow} \star), (v_2 : \alpha \overset{p}{\Longrightarrow} \star)) \in \mathrm{Atom}\,[\star]\,\emptyset \mid \\
 & & \quad v_1 = (v_1' : A_1 \overset{-\alpha}{\Longrightarrow} \alpha) \ \wedge\ v_2 = (v_2' : A_2 \overset{-\alpha}{\Longrightarrow} \alpha) \ \wedge\ (W, v_1', v_2') \in \blacktriangleright W.\kappa(\alpha)\} \\
\mathcal{V}\llbracket \forall X.\, A \rrbracket \rho & = & \{(W, v_{f1}, v_{f2}) \in \mathrm{Atom}\,[\forall X.\, A]\,\rho \mid \\
 & & \quad \forall W' \sqsupseteq W.\ \forall B_1, B_2, R.\ \forall e_1, e_2.\ \forall \alpha. \\
 & & \qquad W'.\Sigma_1; \cdot \vdash B_1 \ \wedge\ W'.\Sigma_2; \cdot \vdash B_2 \ \wedge\ R \in \mathrm{Rel}_{W'.j}\,[B_1, B_2] \ \wedge \\
 & & \qquad W'.\Sigma_1 \triangleright v_{f1}\,[B_1] \longmapsto W'.\Sigma_1, \alpha{:=}B_1 \triangleright (e_1 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[B_1/X]) \ \wedge \\
 & & \qquad W'.\Sigma_2 \triangleright v_{f2}\,[B_2] \longmapsto W'.\Sigma_2, \alpha{:=}B_2 \triangleright (e_2 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[B_2/X]) \implies \\
 & & \qquad\quad (W' \boxplus (\alpha, B_1, B_2, R), e_1, e_2) \in \blacktriangleright \mathcal{E}\llbracket A \rrbracket \rho[X \mapsto \alpha]\} \\
\mathcal{V}\llbracket A \times B \rrbracket \rho & = & \{(W, \langle v_1, v_2 \rangle, \langle v_1', v_2' \rangle) \in \mathrm{Atom}\,[A \times B]\,\rho \mid (W, v_1, v_2) \in \mathcal{V}\llbracket A \rrbracket \rho \ \wedge\ (W, v_1', v_2') \in \mathcal{V}\llbracket B \rrbracket \rho\} \\
 & & \\
\mathcal{E}\llbracket A \rrbracket \rho & = & \{(W, e_1, e_2) \in \mathrm{Atom}\,[A]\,\rho \mid \forall j < W.j. \\
 & & \quad (\forall \Sigma_1, v_1.\ W.\Sigma_1 \triangleright e_1 \longrightarrow^{j} \Sigma_1 \triangleright v_1 \implies \exists W', \Sigma_2, v_2.\ W.\Sigma_2 \triangleright e_2 \longrightarrow^{*} \Sigma_2 \triangleright v_2 \ \wedge \\
 & & \qquad W' \sqsupseteq_j W \ \wedge\ W'.\Sigma_1 = \Sigma_1 \ \wedge\ W'.\Sigma_2 = \Sigma_2 \ \wedge\ (W', v_1, v_2) \in \mathcal{V}\llbracket A \rrbracket \rho) \ \wedge \\
 & & \quad (\forall \Sigma_1, p.\ W.\Sigma_1 \triangleright e_1 \longrightarrow^{j} \Sigma_1 \triangleright \mathrm{blame}\, p \implies \exists \Sigma_2.\ W.\Sigma_2 \triangleright e_2 \longrightarrow^{*} \Sigma_2 \triangleright \mathrm{blame}\, p)\} \\
 & & \\
\mathcal{S}\llbracket \cdot \rrbracket & = & \mathrm{World} \\
\mathcal{S}\llbracket \Sigma, \alpha{:=}A \rrbracket & = & \mathcal{S}\llbracket \Sigma \rrbracket \cap \{W \in \mathrm{World} \mid W.\Sigma_1(\alpha) = A \ \wedge\ W.\Sigma_2(\alpha) = A \ \wedge\ \vdash W.\Sigma_1 \ \wedge\ \vdash W.\Sigma_2 \ \wedge \\
 & & \qquad\qquad W.\kappa(\alpha) = \lfloor \mathcal{V}\llbracket A \rrbracket \emptyset \rfloor_{W.j}\} \\
 & & \\
\mathcal{D}\llbracket \cdot \rrbracket & = & \{(W, \emptyset) \mid W \in \mathrm{World}\} \\
\mathcal{D}\llbracket \Delta, X \rrbracket & = & \{(W, \rho[X \mapsto \alpha]) \mid (W, \rho) \in \mathcal{D}\llbracket \Delta \rrbracket \ \wedge\ \alpha \in \mathrm{dom}(W.\kappa)\} \\
\mathcal{G}\llbracket \cdot \rrbracket \rho & = & \{(W, \emptyset) \mid W \in \mathrm{World}\} \\
\mathcal{G}\llbracket \Gamma, x : A \rrbracket \rho & = & \{(W, \gamma[x \mapsto (v_1, v_2)]) \mid (W, \gamma) \in \mathcal{G}\llbracket \Gamma \rrbracket \rho \ \wedge\ (W, v_1, v_2) \in \mathcal{V}\llbracket A \rrbracket \rho\} \\
 & & \\
\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A & \overset{\mathrm{def}}{=} & \Sigma; \Delta; \Gamma \vdash e_1 : A \ \wedge\ \Sigma; \Delta; \Gamma \vdash e_2 : A \ \wedge \\
 & & \forall W, \rho, \gamma.\ (W \in \mathcal{S}\llbracket \Sigma \rrbracket \ \wedge\ (W, \rho) \in \mathcal{D}\llbracket \Delta \rrbracket \ \wedge\ (W, \gamma) \in \mathcal{G}\llbracket \Gamma \rrbracket \rho) \implies \\
 & & \quad (W, \rho(\gamma_1(e_1)), \rho(\gamma_2(e_2))) \in \mathcal{E}\llbracket A \rrbracket \rho \\
 & & \\
\Sigma; \Delta; \Gamma \vdash e_1 \approx e_2 : A & \overset{\mathrm{def}}{=} & \Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A \wedge \Sigma; \Delta; \Gamma \vdash e_2 \preceq e_1 : A
\end{array}
$$

Figure 9: Logical Relation

Note that, in the definition of $\mathcal{V}\llbracket X \rrbracket \rho$, we have that $\rho(X) = \alpha$. This may be observed by expanding the definition of $\mathrm{Atom}\,[X]\,\rho$. The definitions of $\mathcal{V}\llbracket X \rrbracket \rho$ and $\mathcal{V}\llbracket \alpha \rrbracket \rho$ are in fact identical.

Note that the $\mathcal{V}\llbracket A \rrbracket \rho$ relation in Figure 9 could more accurately be written as an interpretation of type well-formedness judgments $\mathcal{V}\llbracket \Sigma; \Delta \vdash A \rrbracket \rho$. In this expanded form, we may observe that $\Delta = \mathrm{dom}(\rho)$. This longhand also serves to clarify the definition of $\mathcal{V}\llbracket \alpha \rrbracket \rho$.

4 Type Safety

Lemma 4.1 (Canonical forms)
If $\Sigma; \Delta; \Gamma \vdash v : A$ *then either*

- $v = n'$ *and* $A = \mathsf{int}$
- *(*$v = \mathsf{true}$ *or* $v = \mathsf{false}$*) and* $A = \mathsf{bool}$
- $v = \lambda(x : A').\, N'$ *and* $A = A' \to B'$
- $v = \Lambda X.\, v'$ *and* $A = \forall X.B'$
- $v = v' : A' {\to} B' \overset{\phi'}{\Longrightarrow} C' {\to} D'$ *and* $A = C' {\to} D'$
- $v = v' : \forall X.A' \overset{\phi'}{\Longrightarrow} \forall X.B'$ *and* $A = \forall X.B'$
- $v = v' : A' \overset{-\alpha'}{\Longrightarrow} \alpha'$ *and* $A = \alpha'$
- $v = v' : A' {\to} B' \overset{p'}{\Longrightarrow} C' {\to} D'$ *and* $A = C' {\to} D'$
- $v = v' : A' \overset{p'}{\Longrightarrow} \forall X.B'$ *and* $A = \forall X.B'$
- $v = v' : G' \overset{p'}{\Longrightarrow} \star$ *and* $A = \star$

where all primed variables are existentially quantified.

Proof

The proof is by cases on v and the last step in the derivation of $\Sigma; \Delta; \Gamma \vdash v : A$. □

Lemma 4.2
If $\Sigma; \Delta \vdash \forall X.A \prec B$*, then* $\Sigma; \Delta \vdash A[X{:=}\star] \prec B$*.*

Proof

We proceed by induction on $\Sigma; \Delta \vdash \forall X.A \prec B$.

- Case $\boxed{\Sigma; \Delta \vdash A' \prec \forall Y.B'}$ (where $A' = \forall X.A$):
  We have $\Sigma; \Delta, Y \vdash \forall X.A \prec B'$. By the induction hypothesis, $\Sigma; \Delta, Y \vdash A[X{:=}\star] \prec B'$. Therefore, $\Sigma; \Delta \vdash A[X{:=}\star] \prec \forall Y.B'[Y]$.
- Case $\boxed{\Sigma; \Delta \vdash \forall X.A \prec B}$:
  We have $\Sigma; \Delta \vdash A[X{:=}\star] \prec B$, which completes this case.
- Case $\boxed{\Sigma; \Delta \vdash A' \prec \star}$ (where $A' = \forall X.A$):
  We have $\Sigma; \Delta \vdash \forall X.A$ and so $\Sigma; \Delta, X \vdash A$. Then by a substitution lemma, we obtain $\Sigma; \Delta \vdash A[X{:=}\star]$. Therefore $\Sigma; \Delta \vdash A[X{:=}\star] \prec \star$.

□

Lemma 4.3 (Subject Reduction)
If $\Sigma;\ \cdot\ ;\ \cdot \vdash M : A$ *and* $M \longmapsto N$*, then* $\Sigma;\ \cdot\ ;\ \cdot \vdash N : A$*.*

Proof

The proof is by cases on $M \longmapsto N$. Many of the cases are trivial or standard. We give the cases that are novel or non-trivial.

- Case $\boxed{\begin{array}{ll} & (V : A' \to B \overset{\phi}{\Longrightarrow} C \to D)\ W \\ \longmapsto & V\ (W : C \overset{-\phi}{\Longrightarrow} A') : B \overset{\phi}{\Longrightarrow} D \end{array}}$:
  We have $\Sigma;\ \cdot \vdash A' \to B \prec^{\phi} C \to D$. So $\Sigma;\ \cdot \vdash C \prec^{-\phi} A'$ and $\Sigma;\ \cdot \vdash B \prec^{\phi} D$. Thus, the RHS also has type $D = A$.
- Case $\boxed{\begin{array}{ll} & (V : A' \to B \overset{p}{\Longrightarrow} C \to D)\ W \\ \longmapsto & V\ (W : C \overset{-p}{\Longrightarrow} A') : B \overset{p}{\Longrightarrow} D \end{array}}$:
  We have $\Sigma;\ \cdot \vdash A' \to B \prec C \to D$. So $\Sigma;\ \cdot \vdash C \prec A'$ and $\Sigma;\ \cdot \vdash B \prec D$. Thus, the RHS also has type $D = A$.
- Case $\boxed{\begin{array}{ll} & (v : \forall X\,.\,A' \overset{p}{\Longrightarrow} B\,) \\ \longmapsto & (v\ [\star] : A'[\star/X\,] \overset{p}{\Longrightarrow} B\,) \end{array}}$: where $B \neq \forall Y.B'$ for any Y, B'.

  We have $\Sigma;\ \cdot\,;\ \cdot \vdash v : \forall X.A'$. So $\Sigma;\ \cdot\,;\ \cdot \vdash v\ [\star] : A'[\star/X]$. We also have $\Sigma;\ \cdot \vdash \forall X.A' \prec B$, so $\Sigma;\ \cdot \vdash A'[\star/X] \prec B$. Therefore $\Sigma;\ \cdot\,;\ \cdot \vdash (v\ [\star] : A'[\star/X] \overset{p}{\Longrightarrow} B) : B$.

□

Definition 4.4
Well-typed contexts, written $\Sigma \vdash E : B \Rightarrow A$, are defined in the usual way.

Lemma 4.5 (Decomposition)
If $\Sigma;\ \cdot\,;\ \cdot \vdash M : A$, then either

1. $M = V'$,
2. $M = E'[\texttt{blame}\, p']$,
3. $M = E'[M']$, $\Sigma \triangleright M' \longmapsto \Sigma' \triangleright N'$, *and* $\Sigma \subseteq \Sigma'$.
4. $M = E'[M']$ *and* $M' \longmapsto N'$.

where all primed variables are existentially quantified.

Proof

The proof is by induction on $\Sigma;\ \cdot\,;\ \cdot \vdash M : A$.

- $\boxed{\Sigma;\ \cdot\,;\ \cdot \vdash n : \mathsf{int}}$ Pick V' to be n.
- $\boxed{\dfrac{\Sigma;\ \cdot\,;\ \cdot \vdash M_i : \mathsf{int} \qquad \forall i \in \{1,2\}}{\Sigma;\ \cdot\,;\ \cdot \vdash M_1 \circledast M_2 : B}}$
  If M_1 and M_2 are all values n_1, n_2, then we have $M_1 \circledast M_2 \longmapsto [\![\circledast]\!](n_1, n_2)$ (We require the primitive operators to be type safe.) Pick $E' = \square$, $M' = n_1 \circledast n_2$, and $N' = [\![op]\!](n_1, n_2)$ to conclude.
  If one of M_i is not a value, let M_i be the first such. Pick $E' = V_1 \circledast E'_1$ if M_1 is a value V_1 and pick $E' = E'_1 \circledast V_2$ if M_2 is a value V_2. By the induction hypothesis, either
  1. $M_i = E'_1[\texttt{blame}\, p']$, or
  2. $M_i = E'_1[M'_i]$ and $\Sigma \triangleright M'_i \longmapsto \Sigma' \triangleright N'_i$, or
  3. $M_i = E'_1[M'_i]$ and $M'_i \longmapsto N'_i$.

  In the first case we have $\Sigma \triangleright E'[\texttt{blame}\, p'] \longmapsto \Sigma \vdash \texttt{blame}\, p'$. In the second case we have $\Sigma \triangleright E'[M'_i] \longmapsto \Sigma' \triangleright E'[N'_i]$. In the third case we have $E'[M'_i] \longmapsto E'[N'_i]$.

- $\boxed{\dfrac{\Sigma;\ \cdot\,;\cdot,x:A \vdash N:B \qquad \Sigma;\cdot \vdash \cdot}{\Sigma;\ \cdot\,;\ \cdot \vdash \lambda(x:A).\,N : A \to B}}$ Pick V' to be $(\lambda(x:A).\,N)$.
- $\boxed{\dfrac{\Sigma;\cdot,X;\ \cdot \vdash V:B \qquad \Sigma;\cdot \vdash \cdot}{\Sigma;\ \cdot\,;\ \cdot \vdash \Lambda X.V : \forall X.B}}$ Pick V' to be $(\Lambda X.\,V)$.
- $\boxed{\dfrac{\Sigma;\ \cdot\,;\ \cdot \vdash L : A \to B \qquad \Sigma;\ \cdot\,;\ \cdot \vdash M_1 : A}{\Sigma;\ \cdot\,;\ \cdot \vdash (L\ M_1) : B}}$
  - If L and M_1 are values, then pick $E' = \square$ and $M' = (L\ M_1)$. By canonical forms (Lemma 4.1), L is in one of the following forms:
    1. $L = \lambda(x:A).\,N_1$, or
    2. $L = V : A' \to B' \overset{\phi}{\Longrightarrow} A \to B$, or
    3. $L = V : A' \to B' \overset{p}{\Longrightarrow} A \to B$.

    In each of these cases, a reduction rule applies, so we have $M' \longrightarrow N'$ for some N'.
  - If L is a value but not M_1, then we apply the induction hypothesis for M_1 to obtain a decomposition E'' of M_1 and then pick $E' = (L\ E'')$.
  - If L is not a value, then we apply the induction hypothesis for L to obtain a decomposition E'' of L and then pick $E' = (E''\ M_1)$.
- $\boxed{\dfrac{\Sigma;\ \cdot\,;\ \cdot \vdash L : \forall X.B \qquad \Sigma;\ \cdot \vdash A'}{\Sigma;\ \cdot\,;\ \cdot \vdash L\,[A'] : B[A'/X]}}$
  - If L is a value, then pick $E' = \square$ and $M' = L\,[A']$. By canonical forms (Lemma 4.1), L is in one of the following forms:
    1. $L = (\Lambda X.\,V')$, or
    2. $L = V' : \forall X.B' \overset{\phi}{\Longrightarrow} \forall X.B$, or
    3. $L = V' : B' \overset{p}{\Longrightarrow} \forall X.B$.

    In each of these cases, a reduction rule applies, so we have $\Sigma \triangleright M' \longmapsto \Sigma' \triangleright N'$ for some Σ', N'. Also, in each case $\Sigma \subseteq \Sigma'$.
  - If L is not a value, we apply the induction hypothesis to obtain a decomposition E'' of L and then pick $E' = E''\,[A']$.
- $\boxed{\dfrac{\Sigma;\ \cdot\,;\ \cdot \vdash M_1 : A \qquad \Sigma;\ \cdot \vdash A \prec^{\phi} B}{\Sigma;\ \cdot\,;\ \cdot \vdash (M_1 : A \overset{\phi}{\Longrightarrow} B) : B}}$
  - If M_1 is a value V, we proceed by cases on $\Sigma;\ \cdot \vdash A \prec^{\phi} B$.
    1. Case $\Sigma;\ \cdot \vdash \mathsf{int} \prec^{\phi} \mathsf{int}$:
       Pick $E' = \square$ and $M' = (V : \mathsf{int} \overset{\phi}{\Longrightarrow} \mathsf{int})$.
       $$V : \mathsf{int} \overset{\phi}{\Longrightarrow} \mathsf{int} \longmapsto V$$
    2. Case $\Sigma;\ \cdot \vdash \mathsf{bool} \prec^{\phi} \mathsf{bool}$:
       Pick $E' = \square$ and $M' = (V : \mathsf{bool} \overset{\phi}{\Longrightarrow} \mathsf{bool})$.
       $$V : \mathsf{bool} \overset{\phi}{\Longrightarrow} \mathsf{bool} \longmapsto V$$
    3. Case $\Sigma;\ \cdot \vdash A_1 {\to} A_2 \prec^{\phi} B_1 {\to} B_2$:
       $(V : A_1 {\to} A_2 \overset{\phi}{\Longrightarrow} B_1 {\to} B_2)$ is a value.

4. Case $\Sigma;\cdot \vdash \forall X.A \prec^{\phi} \forall X.B$:
   $(V : \forall X.A \overset{\phi}{\Longrightarrow} \forall X.B)$ is a value.
5. Case $\Sigma;\cdot \vdash \alpha \prec^{+\alpha} B$:
   So $\Sigma;\cdot;\cdot \vdash V : \alpha$ and by canonical forms (Lemma 4.1), $V = V' : B \overset{-\alpha}{\Longrightarrow} \alpha$. Pick $E' = \square$ and $M' = (V' : B \overset{-\alpha}{\Longrightarrow} \alpha \overset{+\alpha}{\Longrightarrow} B)$.

$$(V' : B \overset{-\alpha}{\Longrightarrow} \alpha \overset{+\alpha}{\Longrightarrow} B) \longmapsto V'$$

6. Case $\Sigma;\cdot \vdash A \prec^{-\alpha} \alpha$:
   $(V : A \overset{-\alpha}{\Longrightarrow} \alpha)$ is a value.
7. Case $\Sigma;\cdot \vdash \alpha \prec^{\phi} \alpha$:
   Pick $E' = \square$ and $M' = (V : \alpha \overset{\phi}{\Longrightarrow} \alpha)$.

$$(V : \alpha \overset{\phi}{\Longrightarrow} \alpha) \longmapsto V$$

8. Case $\Sigma;\cdot \vdash \star \prec^{\phi} \star$:
   Pick $E' = \square$ and $M' = (V : \star \overset{\phi}{\Longrightarrow} \star)$.

$$(V : \star \overset{\phi}{\Longrightarrow} \star) \longmapsto V$$

– If M_1 is not a value, apply the induction hypothesis for M_1 to obtain a decomposition E'' of M_1. Then we pick $E' = (E'' : A \overset{\phi}{\Longrightarrow} B)$.

- $$\boxed{\dfrac{\Sigma;\cdot;\cdot \vdash M_1 : A \qquad \Sigma;\cdot \vdash A \prec B}{\Sigma;\cdot;\cdot \vdash (M_1 : A \overset{p}{\Longrightarrow} B) : B}}$$

– If M_1 is a value V, we proceed by cases on $\Sigma;\cdot \vdash A \prec B$.

1. $\Sigma;\cdot \vdash \mathsf{int} \prec \mathsf{int}$:
   Pick $E' = \square$ and $M' = (V : \mathsf{int} \overset{p}{\Longrightarrow} \mathsf{int})$.

$$V : \mathsf{int} \overset{p}{\Longrightarrow} \mathsf{int} \longmapsto V$$

2. $\Sigma;\cdot \vdash \mathsf{bool} \prec \mathsf{bool}$:
   Pick $E' = \square$ and $M' = (V : \mathsf{bool} \overset{p}{\Longrightarrow} \mathsf{bool})$.

$$V : \mathsf{bool} \overset{p}{\Longrightarrow} \mathsf{bool} \longmapsto V$$

3. $\Sigma;\cdot \vdash A_1 \to A_2 \prec B_1 \to B_2$:
   $(V : A_1 \to A_2 \overset{p}{\Longrightarrow} B_1 \to B_2)$ is a value.
4. $\Sigma;\cdot \vdash A \prec \forall X.B'$:
   $(V : A \overset{p}{\Longrightarrow} \forall X.B')$ is a value.
5. $\Sigma;\cdot \vdash \forall X.A' \prec B$:
   * If $B = \forall X.B'$, then $(V : \forall X.A' \overset{p}{\Longrightarrow} \forall X.B')$ is a value.
   * Otherwise

$$V : \forall X.A' \overset{p}{\Longrightarrow} B \longmapsto (V\ \star) : A'[X{:=}\star] \overset{p}{\Longrightarrow} B$$

   Pick $E' = \square$, $M' = V : \forall X.A' \overset{p}{\Longrightarrow} B$.
6. $\Sigma;\cdot \vdash \alpha \prec \alpha$:
   Pick $E' = \square$ and $M' = (V : \alpha \overset{p}{\Longrightarrow} \alpha)$.

$$(V : \alpha \overset{p}{\Longrightarrow} \alpha) \longmapsto V$$

7. $A \prec \star$:
   * If $A = G$, then $(V : G \overset{p}{\Longrightarrow} \star)$ is a value.
   * If $A = \star$, then
   $$V : \star \overset{p}{\Longrightarrow} \star \longmapsto V$$
   Pick $E' = \square$ and $M' = V : \star \overset{p}{\Longrightarrow} \star$.
   * If A is not ground and not $\star$, then
   $$V : A \overset{p}{\Longrightarrow} \star \longmapsto V : A \overset{p}{\Longrightarrow} G \overset{p}{\Longrightarrow} \star$$
   Pick $E' = \square$ and $M' = V : A \overset{p}{\Longrightarrow} \star$.
8. $\star \prec B$:
   By canonical forms, we have $V = (V' : G \overset{q}{\Longrightarrow} \star)$.
   * If $B = H$ and $G = H$, then
   $$V' : G \overset{q}{\Longrightarrow} \star \overset{p}{\Longrightarrow} G \longmapsto V'$$
   Pick $E' = \square$ and $M' = V : \star \overset{p}{\Longrightarrow} G$.
   * If $B = H$ and $G \neq H$, then
   $$V' : G \overset{q}{\Longrightarrow} \star \overset{p}{\Longrightarrow} H \longmapsto \texttt{blame}\, p$$
   Pick $E' = \square$ and $M' = V : \star \overset{p}{\Longrightarrow} H$.
   * If $B = \star$, then
   $$V : \star \overset{p}{\Longrightarrow} \star \longmapsto V$$
   Pick $E' = \square$ and $M' = V : \star \overset{p}{\Longrightarrow} \star$.
   * If B is not ground and not $\star$, then
   $$V : \star \overset{p}{\Longrightarrow} B \longmapsto V : \star \overset{p}{\Longrightarrow} G \overset{p}{\Longrightarrow} B$$
   where $G \prec B$. Pick $E' = \square$ and $M' = V : \star \overset{p}{\Longrightarrow} B$.

– If M_1 is not a value, the induction hypothesis for M_1 gives us a decomposition E'', so we pick $E' = (E'' : A \overset{p}{\Longrightarrow} B)$.

- $\boxed{\dfrac{\Sigma;\ \cdot \vdash \cdot \qquad \Sigma;\ \cdot \vdash A}{\Sigma;\ \cdot;\ \cdot \vdash \texttt{blame}\, p : A}}$
  We satisfy the fourth option, picking $E' = \square$.

$\square$

Lemma 4.6 (Context Inversion)
If $\Sigma;\ \cdot;\ \cdot \vdash E[M] : A$, then $\Sigma;\ \cdot;\ \cdot \vdash M : B$ and $\Sigma \vdash E : B \Rightarrow A$ for some B.

Lemma 4.7 (Context Weakening)
If $\Sigma \vdash E : B \Rightarrow A$ and $\Sigma \subseteq \Sigma'$, then $\Sigma' \vdash E : B \Rightarrow A$.

Lemma 4.8 (Plug)
If $\Sigma;\ \cdot;\ \cdot \vdash M : B$ and $\Sigma \vdash E : B \Rightarrow A$, then $\Sigma;\ \cdot;\ \cdot \vdash E[M] : A$.

Theorem 4.9 (Type safety)

1. *(Preservation) If $\Sigma;\ \cdot;\ \cdot \vdash e : A$ and $\Sigma \triangleright e \longmapsto \Sigma' \triangleright e'$ then $\Sigma';\ \cdot;\ \cdot \vdash e' : A$ and $\Sigma \subseteq \Sigma'$.*

2. *(Progress) If* $\Sigma;\,\cdot\,;\,\cdot\vdash e : A$ *then either*

   - $e = V'$, *or*
   - $e = \texttt{blame}\,p'$, *or*
   - $\Sigma \triangleright e \longmapsto \Sigma' \triangleright e'$ *and* $\Sigma \subseteq \Sigma'$.

   where all primed variables are existentially quantified.

Proof

1. The proof of preservation is by induction on $\Sigma \triangleright e \longrightarrow \Sigma' \triangleright e'$.
   - $\boxed{\Sigma \triangleright E[e_1] \longrightarrow \Sigma \triangleright E[e_1']}$
     We have $\Sigma;\,\cdot\,;\,\cdot\vdash e_1 : B$ and $\Sigma \vdash E : B \Rightarrow A$ for some B by Lemma 4.6. Then $\Sigma;\,\cdot\,;\,\cdot\vdash e_1' : B$ by Lemma 4.3 (subject reduction). We conclude that $\Sigma \vdash E[e_1'] : A$ by Lemma 4.8.
   - $\boxed{\Sigma \triangleright E\,[e_1] \longmapsto \Sigma' \triangleright E\,[e_1']}$
     We have $\Sigma \triangleright e_1 \longmapsto \Sigma' \triangleright e_1'$. Also, we have $\Sigma;\,\cdot\,;\,\cdot\vdash e_1 : B$ and $\Sigma \vdash E : B \Rightarrow A$ for some B by Lemma 4.6. So by the induction hypothesis, we have $\Sigma';\,\cdot\,;\,\cdot\vdash e_1' : B$ and $\Sigma \subseteq \Sigma'$ for some Σ'. Then we have $\Sigma' \vdash E : B \Rightarrow A$ by Lemma 4.7 and conclude that $\Sigma';\,\cdot\,;\,\cdot\vdash E[e_1'] : : A$ by Lemma 4.8.
   - $\boxed{\Sigma \triangleright (\Lambda X\,.\,v)\,[B] \longmapsto \Sigma, \alpha{:=}B \triangleright (v\,[\alpha/X] : A'[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} A'[B/X])}$
     We have $\Sigma;\cdot, X;\,\cdot\vdash v : A'$ so by a substitution lemma, $\Sigma;\,\cdot\,;\,\cdot\vdash v[\alpha/X] : A'[\alpha/X]$. Also, we have $\Sigma, \alpha{:=}B;\,\cdot\vdash A'[\alpha/X] \prec^{+\alpha} A'[B/X]$. We conclude that $\Sigma, \alpha{:=}B;\,\cdot\,;\,\cdot\vdash (v[\alpha/X] : A'[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} A'[B/X]) : A'[B/X]$.
   - $\boxed{\Sigma \triangleright (v : A_1 \stackrel{p}{\Longrightarrow} \forall X.\, A_1')\,[B] \longmapsto \Sigma, \alpha{:=}B \triangleright ((v : A_1 \stackrel{p}{\Longrightarrow} A_1'[\alpha/X]) : A_1'[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} A_1'[B/X])}$
     We have $\Sigma;\cdot;\cdot\vdash v : A_1$, $\Sigma;\cdot\vdash A_1 \prec \forall X.A_1'$, and $A = A_1'[B/X]$. So we also have $\Sigma;\cdot, X \vdash A_1 \prec A_1'$. Then by a substitution lemma, $\Sigma, \alpha{:=}B;\,\cdot\vdash A_1 \prec A_1'[\alpha/X]$. Also, we have $\Sigma, \alpha{:=}B;\,\cdot\vdash A_1'[\alpha/X] \prec^{+\alpha} A_1'[B/X]$. We conclude that $\Sigma, \alpha{:=}B;\cdot;\cdot\vdash ((v : A_1 \stackrel{p}{\Longrightarrow} A_1'[\alpha/X]) : A_1'[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} A_1'[B/X]) : A_1'[B/X]$.
   - $\boxed{\Sigma \triangleright (v : \forall X.\, A_1 \stackrel{\phi}{\Longrightarrow} \forall X.\, A_1')\,[B] \longmapsto \Sigma, \alpha{:=}B \triangleright ((v\,[\alpha] : A_1[\alpha/X] \stackrel{\phi}{\Longrightarrow} A_1'[\alpha/X]) : A_1'[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} A_1'[B/X])}$
     We have $\Sigma;\,\cdot\,;\,\cdot\vdash v : \forall X.A_1$, $\Sigma;\,\cdot\vdash \forall X.A_1 \prec^{\phi} \forall X.A_1'$, and $A = A_1'[B/X]$. So $\Sigma, \alpha{:=}B;\,\cdot\,;\,\cdot\vdash v\,[\alpha] : A_1[\alpha/X]$. Also, we have $\Sigma;\cdot, X \vdash A_1 \prec^{\phi} A_1'$. By a substitution lemma, we have $\Sigma, \alpha{:=}B;\,\cdot\vdash A_1[\alpha/X] \prec^{\phi} A_1'[\alpha/X]$. Finally, we have $\Sigma, \alpha{:=}B;\,\cdot\vdash A_1'[\alpha/X] \prec^{+\alpha} A_1'[B/X]$. We conclude that $\Sigma, \alpha{:=}B;\cdot;\cdot\vdash ((v[\alpha] : A_1[\alpha/X] \stackrel{\phi}{\Longrightarrow} A_1'[\alpha/X]) : A_1'[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} A_1'[B/X]) : A_1'[B/X]$.
   - $\boxed{\Sigma \triangleright E[\texttt{blame}\,p] \longrightarrow \Sigma \triangleright \texttt{blame}\,p}$
     We immediately have $\Sigma \vdash \texttt{blame}\,p : A$.
2. The proof of progress is a corollary of Decomposition (Lemma 4.5).

□

Lemma 4.10 (Termination Implies Redex Termination)
If $\Sigma \triangleright E[e] \longmapsto^{*} \Sigma_2 \triangleright v$ *then* $\Sigma \triangleright e \longmapsto^{*} \Sigma_2' \triangleright v'$

5 Basic Properties of the Logical Relation

Lemma 5.1 (World Extension is Reflexive and Transitive)
For any $W, W', W'' \in \text{World}$*, we have*

1. $W \sqsupseteq W$
2. *if* $W'' \sqsupseteq W'$ *and* $W' \sqsupseteq W$*, then* $W'' \sqsupseteq W$

Lemma 5.2 (Properties of ▶)
For any $W \in \text{World}$ *or* $R \in \text{Rel}_n$*, we have*

1. $\blacktriangleright W \sqsupseteq W$
2. *If* $R \in \text{Rel}_n\,[A_1, A_2]$ *then* $\blacktriangleright R \in \text{Rel}_n\,[A_1, A_2]$
3. *If* $W' \sqsupseteq W$ *then* $\blacktriangleright W' \sqsupseteq \blacktriangleright W$

Lemma 5.3 (Successive Approximation)
If $j' \leq j$*, then* $\left\lfloor \lfloor R \rfloor_j \right\rfloor_{j'} = \lfloor R \rfloor_{j'}$ *and* $\left\lfloor \lfloor R \rfloor_{j'} \right\rfloor_j = \lfloor R \rfloor_{j'}$.

Lemma 5.4 (Adding to the World Extends It)
If $W.\Sigma_1;\, \cdot \vdash B_1$*,* $W.\Sigma_2;\, \cdot \vdash B_2$*,*
$\alpha \notin W.\Sigma_1$*,* $\alpha \notin W.\Sigma_2$*, and*
$R \in \text{Rel}_{W.j}\,[B_1, B_2]$
then $W \boxplus (\alpha, B_1, B_2, R) \sqsupseteq W$

Lemma 5.5 (Monotonicity of Later Relations in the World)
Let $(W, v_1, v_2) \in \blacktriangleright(W.\kappa(\alpha))$*. If* $W' \sqsupseteq W$ *then* $(W', v_1, v_2) \in \blacktriangleright(W'.\kappa(\alpha))$.

Proof

Suppose that $W'.j > 0$. We need to show that $(\blacktriangleright W', v_1, v_2) \in W'.\kappa(\alpha)$.

Since $W' \sqsupseteq W$, we have that $W.j \geq W'.j > 0$. Hence from the first premise, we have that $(\blacktriangleright W, v_1, v_2) \in W.\kappa(\alpha)$.

Note that $\blacktriangleright W' \sqsupseteq \blacktriangleright W$ by Lemma 5.2. Since $W \in \text{World}$, we have that $W.\kappa(\alpha) \in \text{Rel}_{W.j}\,[W.\Sigma_1(\alpha), W.\Sigma_2(\alpha)]$. Hence, by monotonicity of relations in Rel, we have $(\blacktriangleright W', v_1, v_2) \in W.\kappa(\alpha)$. Moreover, by definition of approximation and ▶, note that $(\blacktriangleright W', v_1, v_2) \in \lfloor W.\kappa(\alpha) \rfloor_{W'.j}$.

Since $W' \sqsupseteq W$ and $\alpha \in \text{dom}(W.\kappa)$, we have that $W'.\kappa(\alpha) = \lfloor W.\kappa(\alpha) \rfloor_{W'.j}$. Hence, we have that $(\blacktriangleright W', v_1, v_2) \in W'.\kappa(\alpha)$ as we were required to show. □

Lemma 5.6 (Monotonicity)
Let $\Sigma; \Delta \vdash A$*. Let* $W \in \mathcal{S}\,[\![\Sigma]\!]$ *and* $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$.
If $W' \sqsupseteq W$ *and* $(W, v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$*, then* $(W', v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$.

Proof

We proceed by induction on A.

Case $A = \mathsf{int}$: Immediate from the definition of $\mathcal{V}\,[\![\mathsf{int}]\!]\,\rho$ and $(W, v_1, v_2) \in \mathcal{V}\,[\![\mathsf{int}]\!]\,\rho$.

Case $A = \mathsf{bool}$: Immediate from the definition of $\mathcal{V}\,[\![\mathsf{bool}]\!]\,\rho$ and $(W, v_1, v_2) \in \mathcal{V}\,[\![\mathsf{bool}]\!]\,\rho$.

Case $A = A' \to B$: Consider arbitrary W'', v_1', v_2' such that

- $W'' \sqsupseteq W'$
- $(W'', v_1', v_2') \in \mathcal{V}[\![A']\!]\,\rho$

It suffices to show that $(W'', v_1\ v_1', v_2\ v_2') \in \mathcal{E}[\![B]\!]\,\rho$.
Instantiate $(W, v_1, v_2) \in \mathcal{V}[\![A' \to B]\!]\,\rho$ with W'', v_1', v_2', noting that $W'' \sqsupseteq W$ by transitivity of world extension (Lemma 5.1) and that $(W'', v_1', v_2') \in \mathcal{V}[\![A']\!]\,\rho$. Hence, we have that $(W'', v_1\ v_1', v_2\ v_2') \in \mathcal{E}[\![B]\!]\,\rho$ as we were required to show.

Case $A = X$: Let $v_1 = (v_1' : B_1 \overset{-\alpha}{\Longrightarrow} \alpha)$ and $v_2 = (v_2' : B_2 \overset{-\alpha}{\Longrightarrow} \alpha)$. We need to show that

$$\begin{aligned}&(W', (v_1' : B_1 \overset{-\alpha}{\Longrightarrow} \alpha), (v_2' : B_2 \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}[\![X]\!]\,\rho\\ &\equiv (W', v_1', v_2') \in \blacktriangleright W'.\kappa(\alpha)\end{aligned}$$

We know that $(W, (v_1' : B_1 \overset{-\alpha}{\Longrightarrow} \alpha), (v_2' : B_2 \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}[\![X]\!]\,\rho$. Therefore, $(W, v_1', v_2') \in \blacktriangleright W.\kappa(\alpha)$. By Lemma 5.5 (monotonicity of later world relations) noting that $W' \sqsupseteq W$, we have that $(W', v_1', v_2') \in \blacktriangleright W'.\kappa(\alpha)$ as we were required to show.

Case $A = \alpha$: The proof is identical to the previous case.

Case $A = \star$: Let $v_1 = (v_1' : G \overset{p}{\Longrightarrow} \star)$ and $v_2 = (v_2' : G \overset{p}{\Longrightarrow} \star)$. We have three cases to consider.

Case $G = \iota$: Since $(W, v_1, v_2) \in \mathcal{V}[\![\star]\!]\,\rho$, we know that $v_1' = v_2'$, so the proof is immediate.

Case $G = \star \to \star$: We are required to show that

$$\begin{aligned}&(W', v_1', v_2') \in \blacktriangleright\mathcal{V}[\![\star \to \star]\!]\,\rho\\ &\equiv W'.j > 0 \implies (\blacktriangleright W', v_1', v_2') \in \mathcal{V}[\![\star \to \star]\!]\,\rho\end{aligned}$$

Assume that $W'.j > 0$. We need to show that $(\blacktriangleright W', v_1', v_2') \in \mathcal{V}[\![\star \to \star]\!]\,\rho$.
Consider arbitrary W'', v_1'', v_2'' such that

- $W'' \sqsupseteq \blacktriangleright W'$
- $(W'', v_1'', v_2'') \in \mathcal{V}[\![\star]\!]\,\rho$

We are required to show that $(W'', v_1'\ v_1'', v_2'\ v_2'') \in \mathcal{E}[\![\star]\!]\,\rho$.
We know that $(W, (v_1' : \star \to \star \overset{p}{\Longrightarrow} \star), (v_2' : \star \to \star \overset{p}{\Longrightarrow} \star)) \in \mathcal{V}[\![\star]\!]\,\rho$. Therefore, we have that

$$\begin{aligned}&(W, v_1', v_2') \in \blacktriangleright\mathcal{V}[\![\star \to \star]\!]\,\rho\\ &\equiv W.j > 0 \implies (\blacktriangleright W, v_1', v_2') \in \mathcal{V}[\![\star \to \star]\!]\,\rho\end{aligned}$$

We know that $W.j > 0$ since $W' \sqsupseteq W$, and $W'.j > 0$, so we have that $(\blacktriangleright W, v_1', v_2') \in \mathcal{V}[\![\star \to \star]\!]\,\rho$. Instantiate this with W'', v_1'', v_2''. Note that $W'' \sqsupseteq \blacktriangleright W$ by transitivity of $\sqsupseteq$ (Lemma 5.1) since $\blacktriangleright W' \sqsupseteq \blacktriangleright W$ by Lemma 5.2 and $W'' \sqsupseteq \blacktriangleright W'$. Also note that $(W'', v_1'', v_2'') \in \mathcal{V}[\![\star]\!]\,\rho$. Hence, we have that $(W'', v_1'\ v_1'', v_2'\ v_2'') \in \mathcal{E}[\![\star]\!]\,\rho$ as we needed to show.

Case $G = \alpha$: We know that $(W, (v_1' : \alpha \overset{p}{\Longrightarrow} \star), (v_2' : \alpha \overset{p}{\Longrightarrow} \star)) \in \mathcal{V}[\![\star]\!]\,\rho$. Therefore, $v_1' = (v_1'' : A_1 \overset{-\alpha}{\Longrightarrow} \alpha)$, $v_2' = (v_2'' : A_2 \overset{-\alpha}{\Longrightarrow} \alpha)$, and $(W, v_1'', v_2'') \in \blacktriangleright W.\kappa(\alpha)$.
It suffices to show that $(W', v_1'', v_2'') \in \blacktriangleright W'.\kappa(\alpha)$.
By Lemma 5.5 (monotonicity of later world relations) noting that $W' \sqsupseteq W$, we have that $(W', v_1'', v_2'') \in \blacktriangleright W'.\kappa(\alpha)$ as we were required to show.

Case $A = \forall X\,.\,B$: Consider arbitrary $W'', B_1, B_2, R, e_1, e_2, \alpha$ such that

- $W'' \sqsupseteq W'$
- $W''.\Sigma_1;\ \cdot \vdash B_1$
- $W''.\Sigma_2;\ \cdot \vdash B_2$

- $R \in \mathrm{Rel}_{W''.j}\,[B_1, B_2]$
- $W'.\Sigma_1 \triangleright v_{f1}\,[B_1] \longmapsto W'.\Sigma_1, \alpha{:=}B_1 \triangleright (e_1 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[B_1/X])$
- $W'.\Sigma_2 \triangleright v_{f2}\,[B_2] \longmapsto W'.\Sigma_2, \alpha{:=}B_2 \triangleright (e_2 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[B_2/X])$

Let $W_2 = W'' \boxplus (\alpha, B_1, B_2, R)$.

It suffices to show that $(W_2, e_1, e_1) \in \blacktriangleright\mathcal{E}\,[\![A]\!]\,\rho[X \mapsto \alpha]$.

Instantiate $(W, v_1, v_2) \in \mathcal{V}\,[\![\forall X\,.\,A]\!]\,\rho$ with $W'', B_1, B_2, R, e_1, e_2,\ \alpha$, noting that $W_2 \sqsupseteq W$ by transitivity (Lemma 5.1) and that all other conditions are immediate. Hence, we have that $(W_2, e_1, e_1) \in \blacktriangleright\mathcal{E}\,[\![A]\!]\,\rho[X \mapsto \alpha]$ as we were required to show.

Case $A = A_1 \times A_2$: The proof of this case is straightforward.

□

Lemma 5.7 (Type Interpretations Valid)
Let $\Sigma; \Delta \vdash A$. If $W \in \mathcal{S}\,[\![\Sigma]\!]$ and $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$, then $\lfloor \mathcal{V}\,[\![A]\!]\,\rho \rfloor_n \in \mathrm{Rel}_n\,[\rho(A), \rho(A)]$.

Proof

The proof follows from two facts: that type interpretations satisfy monotonicity (Lemma 5.6); and that every $(W', v_1, v_2) \in \lfloor \mathcal{V}\,[\![A]\!]\,\rho \rfloor_n$ belongs to $\mathrm{Atom}_n\,[\rho(A), \rho(A)]$, which is immediate from the definition of $\mathcal{V}\,[\![A]\!]\,\rho$. □

Lemma 5.8 (Substitution Monotonicity)
Let $\Sigma; \Delta \vdash \Gamma$. Let $W \in \mathcal{S}\,[\![\Sigma]\!]$ and $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$.
If $W' \sqsupseteq W$ and $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$, then $(W', \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.

Proof

We proceed by induction on Γ.

Case $\Gamma = \cdot$: From $(W, \gamma) \in \mathcal{G}\,[\![\cdot]\!]\,\rho$ we have that $\gamma = \emptyset$. Since $W' \sqsupseteq W$, we have that $W' \in \mathrm{World}$ we completes the proof.

Case $\Gamma = \Gamma', x : A$: From $(W, \gamma) \in \mathcal{G}\,[\![\Gamma', x : A]\!]\,\rho$, we have that $\gamma = \gamma'[x \mapsto (v_1, v_2)]$ and $(W, \gamma') \in \mathcal{G}\,[\![\Gamma']\!]$ and $(W, v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$. By the definition of $\mathcal{G}\,[\![\Gamma', x : A]\!]$, it suffices to show that

1. $(W', \gamma') \in \mathcal{G}\,[\![\Gamma']\!]\,\rho$, which is immediate from the induction hypothesis; and
2. $(W', v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$, which follows from Lemma 5.6 (monotonicity), noting that $W' \sqsupseteq W$ and $(W, v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$.

□

Lemma 5.9 (Store Monotonicity)
Let $\vdash \Sigma$. If $W' \sqsupseteq W$ and $W \in \mathcal{S}\,[\![\Sigma]\!]$, then $W' \in \mathcal{S}\,[\![\Sigma]\!]$.

Proof

We proceed by induction on Σ.

Case $\Sigma = \cdot$: We know that $\mathcal{S}\,[\![\cdot]\!] = \mathrm{World}$. From $W' \sqsupseteq W$, we also know that $W' \in \mathrm{World}$. Therefore, $W' \in \mathcal{S}\,[\![\cdot]\!]$.

Case $\Sigma = \Sigma', \alpha{:=}A$: We need to show that

1. $W' \in \mathcal{S}[\![\Sigma']\!]$, which is immediate from the induction hypothesis, since $W \in \mathcal{S}[\![\Sigma']\!]$.
2. $W'.\Sigma_1(\alpha) = A$ and $W'.\Sigma_2(\alpha) = A$. Since $W \in \mathcal{S}[\![\Sigma]\!]$, we have that $W.\Sigma_1(\alpha) = A$ and $W.\Sigma_2(\alpha) = A$ and, since $W'.\Sigma_1 \supseteq W.\Sigma_1$ and $W'.\Sigma_2 \supseteq W.\Sigma_2$ by the definition of $W' \sqsupseteq W$, we have that $W'.\Sigma_1(\alpha) = A$ and $W'.\Sigma_2(\alpha) = A$ as we needed to show.
3. $\vdash W'.\Sigma_1$ and $\vdash W'.\Sigma_2$, which is immediate from $W' \in \text{World}$, which in turn follows from $W' \sqsupseteq W$.
4. $W'.\kappa(\alpha) = \lfloor \mathcal{V}[\![A]\!]\,\emptyset \rfloor_{W.j}$. From $W \in \mathcal{S}[\![\Sigma', \alpha{:=}A]\!]$ we have that $W.\kappa(\alpha, \alpha) = \lfloor \mathcal{V}[\![A]\!]\,\rho \rfloor_{W.j}$. From $W' \sqsupseteq W$, we have that $W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.j}$. Hence, we have that

$$\begin{aligned}
&W'.\kappa(\alpha, \alpha) \\
&= \lfloor W.\kappa \rfloor_{W'.j}(\alpha, \alpha) \\
&= \lfloor W.\kappa(\alpha, \alpha) \rfloor_{W'.j} \\
&= \left\lfloor \lfloor \mathcal{V}[\![A]\!]\,\rho \rfloor_{W.j} \right\rfloor_{W'.j} \\
&= \lfloor \mathcal{V}[\![A]\!]\,\rho \rfloor_{W'.j}
\end{aligned}$$

   where the last step follows by Lemma 5.3 (nested approximation), noting that $W'.j \leq W.j$ because $W' \sqsupseteq W$.

□

Lemma 5.10 (Monotonicity for Type-Variable Environments)
If $W' \sqsupseteq W$ and $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$, then $(W', \rho) \in \mathcal{D}[\![\Delta]\!]$.

Proof

We proceed by induction on Δ.

Case $\Delta = \cdot$: By the definition of $\mathcal{D}[\![\cdot]\!]$, we have that $\rho = \emptyset$. Hence, to show $(W', \emptyset) \in \mathcal{D}[\![\Delta]\!]$, we need to show that $W' \in \text{World}$, which is immediate from $W' \sqsupseteq W$.

Case $\Delta = \Delta', X$: From $(W, \rho) \in \mathcal{D}[\![\Delta', X]\!]$, we have that $\rho = \rho'[X \mapsto \alpha]$, $(W, \rho') \in \mathcal{D}[\![\Delta']\!]$, and $\alpha \in \text{dom}(W.\kappa)$.

We are required to show that

1. $(W', \rho') \in \mathcal{D}[\![\Delta']\!]$, which follows direcly from the induction hypothesis since $(W, \rho') \in \mathcal{D}[\![\Delta']\!]$.
2. $\alpha \in \text{dom}(W'.\kappa)$. From $W' \sqsupseteq W$, we have that $W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.k}$. Therefore, since $\alpha \in \text{dom}(W.\kappa)$, it follows that $\alpha \in \text{dom}(W'.\kappa)$.

□

Lemma 5.11 (Logical Relation Weakening)
Let $\Sigma; \Delta \vdash A$ and $\Sigma; \Delta \vdash \Gamma$. Let $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \rho[X \mapsto \alpha]) \in \mathcal{D}[\![\Delta, X]\!]$ where $X \notin \Delta$. Then

1. $(W, v_1, v_2) \in \mathcal{V}[\![A]\!]\,\rho$ *iff* $(W, v_1, v_2) \in \mathcal{V}[\![A]\!]\,\rho[X \mapsto \alpha]$.
2. $(W, e_1, e_2) \in \mathcal{E}[\![A]\!]\,\rho$ *iff* $(W, e_1, e_2) \in \mathcal{E}[\![A]\!]\,\rho[X \mapsto \alpha]$.
3. $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\,\rho$ *iff* $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\,\rho[X \mapsto \alpha]$.

Lemma 5.12 (Atom Weakening)
If $\rho \subseteq \rho'$, then $\text{Atom}\,[A]\,\rho \subseteq \text{Atom}\,[A]\,\rho'$.

Lemma 5.13 (Related Values are Related Terms)
If $(W, v_1, v_2) \in \mathcal{V}\llbracket A \rrbracket \rho$ *then* $(W, v_1, v_2) \in \mathcal{E}\llbracket A \rrbracket \rho$.

Proof

Consider arbitrary $i < w.j, \Sigma_1, v_1'$ such that $W.\Sigma_1 \triangleright v_1 \longmapsto^i \Sigma_1 \triangleright v_1'$. It suffices to show that there exist some W', Σ_2, v_2' such that

- $W' \sqsupseteq_i W$
- $W.\Sigma_2 \triangleright v_2 \longmapsto^* \Sigma_2' \triangleright v_2'$
- $W'.\Sigma_1 = \Sigma_1'$
- $W'.\Sigma_2 = \Sigma_2'$
- $(W', v_1', v_2') \in \mathcal{V}\llbracket A \rrbracket \rho$

Since v_1 is a value, we have that $i = 0$, $\Sigma_1 = W.\Sigma_1$, and $v_1' = v_1$.

Choose $W' = W$, $\Sigma_2 = W.\Sigma_2$, and $v_2' = v_2$. We immediately have what we are required to show.

□

Lemma 5.14 ($\mathcal{E}$ Closed Under Anti-Reduction)
Let $(W, e_1, e_2) \in \text{Atom}[A]\rho$. *Given* $W' \sqsupseteq W$, *if* $W.j \leq W'.j + j_1$ *and* $W.\Sigma_1 \triangleright e_1 \longmapsto^{j_1} W'.\Sigma_1 \triangleright e_1'$ *and* $W.\Sigma_2 \triangleright e_2 \longmapsto^* W'.\Sigma_2 \triangleright e_2'$ *then*

$$(W', e_1', e_2') \in \mathcal{E}\llbracket A \rrbracket \rho \implies (W, e_1, e_2) \in \mathcal{E}\llbracket A \rrbracket \rho$$

Proof

We proceed by cases on termination of e_1'.

Case $W'.\Sigma_1 \triangleright e_1' \longmapsto^k \Sigma_1 \triangleright v_1$ where $k < W'.j$: Instantiate the definition of $(W', e_1', e_2') \in \mathcal{E}\llbracket A \rrbracket \rho$ with k, Σ_1, v_1. We have that there exist some W'', Σ_2, v_2 such that

- $W'' \sqsupseteq_k W'$
- $W'.\Sigma_2 \triangleright e_2' \longmapsto^* \Sigma_2 \triangleright v_2$
- $W''.\Sigma_1 = \Sigma_1$
- $W''.\Sigma_2 = \Sigma_2$
- $(W'', v_1, v_2) \in \mathcal{V}\llbracket A \rrbracket \rho$

We are required to show that $(W, e_1, e_2) \in \mathcal{E}\llbracket A \rrbracket \rho$. Consider arbitrary $i < W.j, \Sigma_1', v_1'$ such that $W.\Sigma_1 \triangleright e_1 \longmapsto^i \Sigma_1' \triangleright v_1'$. It suffices to show that there exist W_2, Σ_2', v_2' such that

- $W_2 \sqsupseteq_i W$
- $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2' \triangleright v_2'$
- $W_2.\Sigma_1 = \Sigma_1'$
- $W_2.\Sigma_2 = \Sigma_2'$
- $(W_2, v_1', v_2') \in \mathcal{V}\llbracket A \rrbracket \rho$

We have that

$$W.\Sigma_1 \triangleright e_1 \longmapsto^{j_1} W'.\Sigma_1 \triangleright e_1' \longmapsto^k \Sigma_1 \triangleright v_1$$

and therefore we have $v_1' = v_1$, $\Sigma_1' = \Sigma_1$, and $i = j_1 + k$.
Similarly, we have that

$$W.\Sigma_2 \triangleright e_2 \longmapsto^* W'.\Sigma_2 \triangleright e_2' \longmapsto^* \Sigma_2 \triangleright v_2$$

Choose $W_2 = (W.j - i, \Sigma_1, \Sigma_2, \lfloor W.\kappa \rfloor_{W.j-i})$, $\Sigma_2' = \Sigma_2$, and $v_2' = v_2$.
Note that $W_2 \sqsupseteq W''$ by the definition of $\sqsupseteq$ since $W_2.j = W.j - (j_1 + k)$, $W''.j = W'.j - k$ and $W.j \leq W'.j + j_1$.
We have that

- $W_2 \sqsupseteq_i W$ by the definition of $\sqsupseteq_i$
- $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2' \triangleright v_2'$
- $W_2.\Sigma_1 = \Sigma_1'$
- $W_2.\Sigma_2 = \Sigma_2'$
- $(W_2, v_1', v_2') \in \mathcal{V}[\![A]\!]\rho$ by Lemma 5.6 (monotonicity) since $W_2 \sqsupseteq W''$

as we were required to show.

Case $W'.\Sigma_1 \triangleright e_1' \longmapsto^k \Sigma_1 \triangleright \text{blame}\, p$ where $k < W'.j$: Instantiate the assumption with k, Σ_1, p. We have that there exists some Σ_2 such that

- $W'.\Sigma_2 \triangleright e_2' \longmapsto^* \Sigma_2 \triangleright \text{blame}\, p$

We are required to show that $(W, e_1, e_2) \in \mathcal{E}[\![A]\!]\rho$. Consider arbitrary $i < W.j, \Sigma_1', p'$ such that $W.\Sigma_1 \triangleright e_1 \longmapsto^i \Sigma_1' \triangleright \text{blame}\, p'$. It suffices to show that there exists Σ_2' such that

- $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2' \triangleright \text{blame}\, p'$

By the operational semantics, we have that

$$W.\Sigma_1 \triangleright e_1 \longmapsto^{j_1} W'.\Sigma_1 \triangleright e_1' \longmapsto^k \Sigma_1 \triangleright \text{blame}\, p$$

and therefore $p' = p$, $\Sigma_1' = \Sigma_1$, and $i = j_1 + k$.
Similarly, we have that

$$W.\Sigma_2 \triangleright e_2 \longmapsto^* W'.\Sigma_2 \triangleright e_2' \longmapsto^* \Sigma_2 \triangleright \text{blame}\, p$$

Choose $\Sigma_2' = \Sigma_2$. We have that

- $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2' \triangleright \text{blame}\, p$

as we were required to show.

Case $W'.\Sigma_1 \triangleright e_1' \longmapsto^{W'.j} \Sigma_1 \triangleright e_1''$: By the operational semantics, we have that

$$W.\Sigma_1 \triangleright e_1 \longmapsto^{j_1} W'.\Sigma_1 \triangleright e_1' \longmapsto^{W'.j} \Sigma_1 \triangleright e_1''$$

Since $j_1 + W'.j \geq W.j$, we vacuously have that $(W, e_1, e_2) \in \mathcal{E}[\![A]\!]\rho$ as we were required to show.

□

Lemma 5.15 (Monadic Bind)
If $(W, e_1, e_2) \in \mathcal{E}[\![A]\!]\rho$ *and*

$$\forall W' \sqsupseteq W.\ \forall v_1, v_2.\ (W', v_1, v_2) \in \mathcal{V}[\![A]\!]\rho \implies (W', E_1[v_1], E_2[v_2]) \in \mathcal{E}[\![B]\!]\rho$$

then $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}[\![B]\!]\rho$.

Proof

We proceed by cases on termination of e_1.

Case $W.\Sigma_1 \triangleright e_1 \longmapsto^k \Sigma_1 \triangleright v_1$ where $k < W.j$: Instantiate the first assumption with k, Σ_1, v_1. We have that there exist some W', Σ_2, v_2 such that

- $W' \sqsupseteq_k W$
- $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2 \triangleright v_2'$
- $W'.\Sigma_1 = \Sigma_1$

- $W'.\Sigma_2 = \Sigma_2$
- $(W', v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$

By the operational semantics, we have that

$$\begin{array}{l} W.\Sigma_1 \triangleright E_1[e_1] \longmapsto^k W'.\Sigma_1 \triangleright E_1[v_1] \\ W.\Sigma_2 \triangleright E_2[e_2] \longmapsto^* W'.\Sigma_2 \triangleright E_2[v_2] \end{array}$$

Instantiate the second assumption with W', v_1, v_2, noting that $(W', v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$. We then have that $(W', E_1[v_1], E_2[v_2]) \in \mathcal{E}\,[\![B]\!]\,\rho$.

By Lemma 5.14 (anti-reduction), noting that

- $W' \sqsupseteq W$
- $W.j = W'.j + k$
- $W.\Sigma_1 \triangleright E_1[e_1] \longmapsto^k W'.\Sigma_1 \triangleright E_1[v_1]$
- $W.\Sigma_2 \triangleright E_2[e_2] \longmapsto^* W'.\Sigma_2 \triangleright E_2[v_2]$

We have that $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}\,[\![B]\!]\,\rho$ as we were required to show.

Case $W'.\Sigma_1 \triangleright e_1 \longmapsto^k \Sigma_1 \triangleright \text{blame}\, p$ where $k + 1 < W'.j$: Instantiate the assumption with k, Σ_1, p. We have that there exists some Σ_2 such that

- $W'.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2 \triangleright \text{blame}\, p$

We are required to show that $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}\,[\![B]\!]\,\rho$. Consider arbitrary $i < W.j, \Sigma_1', p'$ such that $W.\Sigma_1 \triangleright E_1[e_1] \longmapsto^i \Sigma_1' \triangleright \text{blame}\, p'$. It suffices to show that there exist Σ_2' such that

- $W.\Sigma_2 \triangleright E_2[e_2] \longmapsto^* \Sigma_2' \triangleright \text{blame}\, p'$

By the operational semantics, we have that

$$\begin{array}{l} W.\Sigma_1 \triangleright E_1[e_1] \longmapsto^{k+1} \Sigma_1 \triangleright \text{blame}\, p \\ W.\Sigma_1 \triangleright E_2[e_2] \longmapsto^* \Sigma_2 \triangleright \text{blame}\, p \end{array}$$

and therefore $p' = p$, $\Sigma_1' = \Sigma_1$, and $i = k + 1$.

Choose $\Sigma_2' = \Sigma_2$. We have that $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2' \triangleright \text{blame}\, p$ as we were required to show.

Case $W'.\Sigma_1 \triangleright e_1 \longmapsto^{W'.j} \Sigma_1 \triangleright e_1'$: By the operational semantics, we have that

$$W.\Sigma_1 \triangleright E_1[e_1] \longmapsto^{W'.j} \Sigma_1 \triangleright E_1[e_1']$$

Since $W'.j \geq W'.j$, we vacuously have that $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}\,[\![B]\!]\,\rho$ as we were required to show.

□

Lemma 5.16 (Atom Compositionality)
$\text{Atom}\,[A]\,\rho[X \mapsto \alpha] = \text{Atom}\,[A\,[\alpha/X]]\,\rho$

Proof

After unfolding the definition, it suffices to show that

$$\rho_i[X \mapsto \alpha](A) = \rho_i(A\,[\alpha/X]) \qquad \text{for } i = 1, 2$$

which is straightforward to prove by induction on A. □

Lemma 5.17 (Compositionality)
If $\Sigma; \Delta, X \vdash A$, *and* $\text{dom}(\rho) = \Delta$ *then*

1. $\mathcal{V}\llbracket A \rrbracket\, \rho[X \mapsto \alpha] = \mathcal{V}\llbracket A[\alpha/X] \rrbracket\, \rho$

2. $\mathcal{E}\llbracket A \rrbracket\, \rho[X \mapsto \alpha] = \mathcal{E}\llbracket A[\alpha/X] \rrbracket\, \rho$

Proof

We prove both claims simultaneously, by induction on the step index and A. We take ρ as universally quantified in the inductive hypothesis. Both cases use Lemma 5.16 (Atom Compositionality), so we omit that reasoning to avoid repetition.

1. We consider the cases for A. In each case, we may equivalently show that

$$(W, v_1, v_2) \in \mathcal{V}\llbracket A \rrbracket\, \rho[X \mapsto \alpha] \Longleftrightarrow (W, v_1, v_2) \in \mathcal{V}\llbracket A[\alpha/X] \rrbracket\, \rho$$

Case $A = \iota$: This case is immediate from Lemma 5.11 (logical relation weakening) since $\iota = \iota[\alpha/X]$.

Case $A = A_1 \to A_2$: We first prove the $\Rightarrow$ direction. Assume that $(W, v_1, v_2) \in \mathcal{V}\llbracket A_1 \to A_2 \rrbracket\, \rho[X \mapsto \alpha]$. We are required to show that $(W, v_1, v_2) \in \mathcal{V}\llbracket (A_1 \to A_2)[\alpha/X] \rrbracket\, \rho$.
Consider arbitrary W', v_1', v_2' such that

- $W' \sqsupseteq W$
- $(W', v_1', v_2') \in \mathcal{V}\llbracket (A_1[\alpha/X] \rrbracket\, \rho$

It suffices to show that

$$(W', v_1\ v_1', v_2\ v_2') \in \mathcal{E}\llbracket A_2[\alpha/X] \rrbracket\, \rho$$

Instantiate the assumption with W', v_1', v_2', noting that $(W', v_1', v_2') \in \mathcal{V}\llbracket A_1 \rrbracket\, \rho[X \mapsto \alpha]$ by the inductive hypothesis of part 1 for A_1. We then have that

$$(W', v_1\ v_1', v_2\ v_2') \in \mathcal{E}\llbracket A_2 \rrbracket\, \rho[X \mapsto \alpha]$$

By the inductive hypothesis of part 2 for A_2, we have that

$$(W', v_1\ v_1', v_2\ v_2') \in \mathcal{E}\llbracket A_2[\alpha/X] \rrbracket\, \rho$$

as we were required to show.
We next prove the $\Leftarrow$ direction. Assume that $(W, v_1, v_2) \in \mathcal{V}\llbracket A_1 \to A_2[\alpha/X] \rrbracket\, \rho$. We are required to show that $(W, v_1, v_2) \in \mathcal{V}\llbracket A_1 \to A_2 \rrbracket\, \rho[X \mapsto \alpha]$.
Consider arbitrary W', v_1', v_2' such that

- $W' \sqsupseteq W$
- $(W', v_1', v_2') \in \mathcal{E}\llbracket A_1 \rrbracket\, \rho[X \mapsto \alpha]$

It suffices to show that

$$(W', v_1\ v_1', v_2\ v_2') \in \mathcal{E}\llbracket A_2 \rrbracket\, \rho[X \mapsto \alpha]$$

Instantiate the assumption with W', v_1', v_2', noting that $(W', v_1', v_2') \in \mathcal{V}\llbracket A_1[\alpha/X] \rrbracket\, \rho$ by the inductive hypothesis of part 1 for A_1. We then have that

$$(W', v_1\ v_1', v_2\ v_2') \in \mathcal{E}\llbracket A_2 \rrbracket\, \rho[X \mapsto \alpha]$$

By the inductive hypothesis of part 2 for A_2, we have that

$$(W', v_1\ v_1', v_2\ v_2') \in \mathcal{E}\llbracket A_2[\alpha/X] \rrbracket\, \rho$$

as we were required to show.

Case $A = \forall Y.\, A'$:
We first prove the $\Rightarrow$ direction. Assume that $(W, v_1, v_2) \in \mathcal{V}\llbracket \forall Y.\, A' \rrbracket\, \rho[X \mapsto \alpha]$. We are required to show that $(W, v_1, v_2) \in \mathcal{V}\llbracket \forall Y.\, A'[\alpha/X] \rrbracket\, \rho$.
Consider arbitrary $W', B_1, B_2, R, e_1, e_2, \alpha'$ such that

- $W' \sqsupseteq W$
- $W'.\Sigma_1;\ \cdot \vdash B_1$ and $W'.\Sigma_2;\ \cdot \vdash B_2$
- $R \in \mathrm{Rel}_{W'.j}\,[B_1, B_2]$
- $W'.\Sigma_1 \triangleright v_1\ [B_1] \longmapsto W'.\Sigma_1, \alpha' {:=} B_1 \triangleright (e_1 : \rho(A)[\alpha'/X] \overset{\pm\alpha'}{\Longrightarrow} \rho(A)[B_1/X])$
- $W'.\Sigma_2 \triangleright v_2\ [B_2] \longmapsto W'.\Sigma_2, \alpha' {:=} B_2 \triangleright (e_2 : \rho(A)[\alpha'/X] \overset{\pm\alpha'}{\Longrightarrow} \rho(A)[B_2/X])$

We need to show that

$$(W' \boxplus (\alpha', B_1, B_2, R), e_1, e_2) \in \mathcal{E}\,[\![A'[\alpha/X]]\!]\,\rho[Y \mapsto \alpha']$$

Instantiate the assumption with $W', B_1, B_2, R, e_1, e_2, \alpha'$, noting that

- $W' \sqsupseteq W$
- $W'.\Sigma_1;\ \cdot \vdash B_1$ and $W'.\Sigma_2;\ \cdot \vdash B_2$
- $R \in \mathrm{Rel}_{W'.j}\,[B_1, B_2]$
- $W'.\Sigma_1 \triangleright v_1\ [B_1] \longmapsto W'.\Sigma_1, \alpha' {:=} B_1 \triangleright (e_1 : \rho(A)[\alpha'/X] \overset{\pm\alpha'}{\Longrightarrow} \rho(A)[B_1/X])$
- $W'.\Sigma_2 \triangleright v_2\ [B_2] \longmapsto W'.\Sigma_2, \alpha' {:=} B_2 \triangleright (e_2 : \rho(A)[\alpha'/X] \overset{\pm\alpha'}{\Longrightarrow} \rho(A)[B_2/X])$

We have that

$$(W' \boxplus (\alpha', B_1, B_2, R), e_1, e_2) \in \mathcal{E}\,[\![A']\!]\,\rho[X \mapsto \alpha][Y \mapsto \alpha']$$

By the inductive hypothesis of part 2 for A', we then have that

$$(W' \boxplus (\alpha', B_1, B_2, R), e_1, e_2) \in \mathcal{E}\,[\![A'[\alpha/X]]\!]\,\rho[Y \mapsto \alpha']$$

as we were required to show.

We next prove the $\Leftarrow$ direction. Assume that $(W, v_1, v_2) \in \mathcal{V}\,[\![\forall Y.\, A'[\alpha/X]]\!]\,\rho$. We are required to show that $(W, v_1, v_2) \in \mathcal{V}\,[\![\forall Y.\, A']\!]\,\rho[X \mapsto \alpha]$.

Consider arbitrary $W', B_1, B_2, R, e_1, e_2, \alpha'$ such that

- $W' \sqsupseteq W$
- $W'.\Sigma_1;\ \cdot \vdash B_1$ and $W'.\Sigma_2;\ \cdot \vdash B_2$
- $R \in \mathrm{Rel}_{W'.j}\,[B_1, B_2]$
- $W'.\Sigma_1 \triangleright v_1\ [B_1] \longmapsto W'.\Sigma_1, \alpha' {:=} B_1 \triangleright (e_1 : \rho(A)[\alpha'/X] \overset{\pm\alpha'}{\Longrightarrow} \rho(A)[B_1/X])$
- $W'.\Sigma_2 \triangleright v_2\ [B_2] \longmapsto W'.\Sigma_2, \alpha' {:=} B_2 \triangleright (e_2 : \rho(A)[\alpha'/X] \overset{\pm\alpha'}{\Longrightarrow} \rho(A)[B_2/X])$

We need to show that

$$(W' \boxplus (\alpha', B_1, B_2, R), e_1, e_2) \in \mathcal{E}\,[\![A']\!]\,\rho[X \mapsto \alpha][Y \mapsto \alpha']$$

Instantiate the assumption with $W', B_1, B_2, R, e_1, e_2, \alpha'$, noting that

- $W' \sqsupseteq W$
- $W'.\Sigma_1;\ \cdot \vdash B_1$ and $W'.\Sigma_2;\ \cdot \vdash B_2$
- $R \in \mathrm{Rel}_{W'.j}\,[B_1, B_2]$
- $W'.\Sigma_1 \triangleright v_1\ [B_1] \longmapsto W'.\Sigma_1, \alpha' {:=} B_1 \triangleright (e_1 : \rho(A)[\alpha'/X] \overset{\pm\alpha'}{\Longrightarrow} \rho(A)[B_1/X])$
- $W'.\Sigma_2 \triangleright v_2\ [B_2] \longmapsto W'.\Sigma_2, \alpha' {:=} B_2 \triangleright (e_2 : \rho(A)[\alpha'/X] \overset{\pm\alpha'}{\Longrightarrow} \rho(A)[B_2/X])$

We have that

$$(W' \boxplus (\alpha', B_1, B_2, R), e_1, e_2) \in \mathcal{E}\,[\![A'[\alpha/X]]\!]\,\rho[Y \mapsto \alpha']$$

By the inductive hypothesis of part 2 for A', we then have that

$$(W' \boxplus (\alpha', B_1, B_2, R), e_1, e_2) \in \mathcal{E}\,[\![A']\!]\,\rho[X \mapsto \alpha][Y \mapsto \alpha']$$

as we were required to show.

Case $A = A_1 \times A_2$: The proof of this case is straightforward.

Case $A = Y$:

Suppose $X = Y$.

$$\mathcal{V}\,[\![Y]\!]\,\rho[X \mapsto \alpha] = \mathcal{V}\,[\![\alpha]\!]\,\emptyset = \mathcal{V}\,[\![Y[\alpha/X]]\!]\,\rho$$

Suppose $X \neq Y$.

$$\mathcal{V}\,[\![Y]\!]\,\rho[X \mapsto \alpha] = \mathcal{V}\,[\![\rho(Y)]\!]\,\emptyset = \mathcal{V}\,[\![Y[\alpha/X]]\!]\,\rho$$

Case $A = \alpha'$: This case is immediate since $\alpha' = \alpha'[\alpha/X]$.

Case $A = \star$: This case is immediate since $\star = \star[\alpha/X]$.

2. We may equivalently show that

$$(W, e_1, e_2) \in \mathcal{E}\,[\![A]\!]\,\rho[X \mapsto \alpha] \Longleftrightarrow (W, e_1, e_2) \in \mathcal{E}\,[\![A\,[\alpha/X]]\!]\,\rho$$

We first prove the $\Rightarrow$ direction. Assume that $(W, e_1, e_2) \in \mathcal{E}\,[\![A]\!]\,\rho[X \mapsto \alpha]$. We are required to show that $(W, e_1, e_2) \in \mathcal{E}\,[\![A\,[\alpha/X]]\!]\,\rho$.

We proceed by cases on termination of e_1.

Case $W.\Sigma_1 \triangleright e_1 \longmapsto^k \Sigma_1 \triangleright v_1$ where $k < W.j$: Instantiate the assumption with k, Σ_1, v_1. We have that there exist some W', Σ_2, v_2 such that

- $W' \sqsupseteq_k W$
- $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2 \triangleright v_2$
- $W'.\Sigma_1 = \Sigma_1$
- $W'.\Sigma_2 = \Sigma_2$
- $(W', v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho[X \mapsto \alpha]$

Choose W', Σ_2, v_2. We have that

- $W' \sqsupseteq_k W$
- $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2 \triangleright v_2$
- $W'.\Sigma_1 = \Sigma_1$
- $W'.\Sigma_2 = \Sigma_2$
- $(W', v_1, v_2) \in \mathcal{V}\,[\![A[\alpha/X]]\!]\,\rho$ by part 1

Therefore, we have that $(W, e_1, e_2) \in \mathcal{E}\,[\![A\,[\alpha/X]]\!]\,\rho$ as we were required to show.

Case $W'.\Sigma_1 \triangleright e_1' \longmapsto^k \Sigma_1 \triangleright \mathrm{blame}\, p$ where $k < W'.j$: Instantiate the assumption with k, Σ_1, p. We have that there exists some Σ_2 such that

- $W'.\Sigma_2 \triangleright e_2' \longmapsto^* \Sigma_2 \triangleright \mathrm{blame}\, p$

We are required to show that $(W, e_1, e_2) \in \mathcal{E}\,[\![A]\!]\,\rho$. Consider arbitrary $i < W.j, \Sigma_1', p'$ such that $W.\Sigma_1 \triangleright e_1 \longmapsto^i \Sigma_1' \triangleright \mathrm{blame}\, p'$. It suffices to show that there exist Σ_2' such that

- $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2' \triangleright \mathrm{blame}\, p'$

Note that $p = p'$.

Choose $\Sigma_2' = \Sigma_2$. We then have that $W.\Sigma_2 \triangleright e_2 \longmapsto^* \Sigma_2' \triangleright \mathrm{blame}\, p'$ as we were required to show.

Case $W'.\Sigma_1 \triangleright e_1' \longmapsto^{W'.j} \Sigma_1 \triangleright e_1''$: By the operational semantics, we have that

$$W.\Sigma_1 \triangleright e_1 \longmapsto^{j_1} W'.\Sigma_1 \triangleright e_1' \longmapsto^{W'.j} \Sigma_1 \triangleright e_1''$$

Since $j_1 + W'.j \geq W'.j$, we vacuously have that $(W, e_1, e_2) \in \mathcal{E}\,[\![A[\alpha/X]]\!]\,\rho$ as we were required to show.

The proof for the the $\Leftarrow$ direction is identical.

$\square$

Lemma 5.18 (Type Application Steps to a Conversion)
If $\Sigma;\ \cdot\,;\ \cdot \vdash v : \forall X.\, A$, $\Sigma;\ \cdot \vdash B$, *and* $\alpha \notin \mathrm{dom}(\Sigma)$
then $\Sigma \triangleright v\ [B] \longmapsto \Sigma, \alpha{:=}B \triangleright (e : A[\alpha/X] \overset{+\alpha}{\Longrightarrow} A[B/X])$ *for some* e.

Proof

We proceed by cases on the canonical forms.

Case $v = \Lambda X.v'$

$$\Sigma \triangleright v\ [B] \longmapsto \Sigma, \alpha{:=}B \triangleright (v'[\alpha/X] : A[\alpha/X] \overset{+\alpha}{\Longrightarrow} A[B/X])$$

Pick $e = v'[\alpha/X]$.

Case $v = (v' : A' \overset{p}{\Longrightarrow} \forall X.\, A)$

$$\Sigma \triangleright v\ [B] \longmapsto \Sigma, \alpha{:=}B \triangleright ((v' : A' \overset{p}{\Longrightarrow} A[\alpha/X]) : A[\alpha/X] \overset{+\alpha}{\Longrightarrow} A[B/X])$$

Pick $e = (v' : A' \overset{p}{\Longrightarrow} A[\alpha/X])$.

Case $v = (v' : \forall X.\, A' \overset{\phi}{\Longrightarrow} \forall X.\, A)$

$$\Sigma \triangleright v\ [B] \longmapsto \Sigma, \alpha{:=}B \triangleright ((v'\ [\alpha] : A'[\alpha/X] \overset{\phi}{\Longrightarrow} A[\alpha/X]) : A[\alpha/X] \overset{+\alpha}{\Longrightarrow} A[B/X])$$

Pick $e = (v'\ [\alpha] : A'[\alpha/X] \overset{\phi}{\Longrightarrow} A[\alpha/X])$.

$\square$

6 Conversion and Cast Lemmas

This section deals with the relatedness of conversions and casts. We treat these terms first for two reasons. First, the proofs about relatedness of conversions must be done via simultaneous induction to have inductive hypotheses for both positive and negative conversions. Second, since in the semantics of casts to polymorphic type we generate a conversion, the cast lemma depends on the conversion lemma.

Lemma 6.1 (Canonical Forms for Conversion)
If $\Sigma; \Delta \vdash A \prec^{+\alpha} B$ or $\Sigma; \Delta \vdash B \prec^{-\alpha} A$, then

- *if* $A = \mathsf{int}$ *then* $B = \mathsf{int}$
- *if* $A = \mathsf{bool}$ *then* $B = \mathsf{bool}$
- *if* $A = X$ *then* $B = X$
- *if* $A = \star$ *then* $B = \star$
- *if* $A = A_1 \to A_2$ *then* $B = B_1 \to B_2$
- *if* $A = \forall X. A'$ *then* $B = \forall X. B'$
- *if* $A = A_1 \times A_2$ *then* $B = B_1 \times B_2$
- *if* $A = \alpha$ *then* $B = \Sigma(\alpha)$
- *if* $A = \alpha'$ *and* $\alpha' \neq \alpha$ *then* $B = \alpha'$

Lemma 6.2 (Convertibility Substitution)
If $\Sigma; \Delta \vdash A$, $\alpha{:=}B \in \Sigma$, and $\alpha \notin FTN(A)$ then

1. $\Sigma; \Delta \vdash A[\alpha/X] \prec^{+\alpha} A[B/X]$
2. $\Sigma; \Delta \vdash A[B/X] \prec^{-\alpha} A[\alpha/X]$

Proof

We prove 1. and 2. simultaneously by induction on the derivation of A. Note that $\vdash \Sigma$ from the assumptions.

Case $A = \mathsf{int}$

1. and 2. are both immediate.

Case $A = \mathsf{bool}$

1. and 2. are both immediate.

Case $A = A_1 \to A_2$

1. By 2. of the inductive hypothesis for A_1, we have that $\Sigma; \Delta \vdash A_1[B/X] \prec^{-\alpha} A_1[\alpha/X]$. By 1. of the inductive hypothesis for A_2, we have that $\Sigma; \Delta \vdash A_2[\alpha/X] \prec^{+\alpha} A_2[B/X]$. Therefore, we have
$$\Sigma; \Delta \vdash (A_1 \to A_2)[\alpha/X] \prec^{+\alpha} (A_1 \to A_2)[B/X]$$
as we were required to show.
2. By 1. of the inductive hypothesis for A_1, we have that $\Sigma; \Delta \vdash A_1[\alpha/X] \prec^{+\alpha} A_1[B/X]$. By 2. of the inductive hypothesis for A_2, we have that $\Sigma; \Delta \vdash A_2[B/X] \prec^{-\alpha} A_2[\alpha/X]$. Therefore, we have
$$\Sigma; \Delta \vdash (A_1 \to A_2)[B/X] \prec^{-\alpha} (A_1 \to A_2)[\alpha/X]$$
as we were required to show.

Case $A = \forall Y.\, A'$

1. By 1. of the inductive hypothesis for A', we have that $\Sigma; \Delta \vdash A'[\alpha/X] \prec^{+\alpha} A'[B/X]$. Therefore, we have

$$\Sigma; \Delta \vdash (\forall Y.\, A')[\alpha/X] \prec^{+\alpha} (\forall Y.\, A')[B/X]$$

as we were required to show.

2. By 2. of the inductive hypothesis for A_2, we have that $\Sigma; \Delta \vdash A'[B/X] \prec^{-\alpha} A'[\alpha/X]$. Therefore, we have

$$\Sigma; \Delta \vdash (\forall Y.\, A')[B/X] \prec^{-\alpha} (\forall Y.\, A')[\alpha/X]$$

as we were required to show.

Case $A = A_1 \times A_2$: The proof of this case is straightforward.

Case $A = Y$

Note that $Y \in \Delta$ because $\Sigma; \Delta \vdash Y$.

If $Y \neq X$ then we have

1. $\Sigma; \Delta \vdash Y[\alpha/X] \prec^{+\alpha} Y[B/X]$
2. $\Sigma; \Delta \vdash Y[\alpha/X] \prec^{-\alpha} Y[B/X]$

as we were required to show.

Otherwise, we need to show

1. $\Sigma; \Delta \vdash X[\alpha/X] \prec^{+\alpha} X[B/X]$
   $= \Sigma; \Delta \vdash \alpha \prec^{+\alpha} B$
2. $\Sigma; \Delta \vdash X[B/X] \prec^{-\alpha} X[\alpha/X]$
   $= \Sigma; \Delta \vdash B \prec^{-\alpha} \alpha$

In both cases, we obtain the result directly from $\alpha{:=}B \in \Sigma$.

Case $A = \alpha'$

1. Note that $\alpha' \notin +\alpha$ because $\alpha \notin A$ and that there exists B' such that $\alpha'{:=}B' \in \Sigma$ because $\Sigma; \Delta \vdash \alpha'$. Therefore, we have

$$\Sigma; \Delta \vdash \alpha'[\alpha/X] \prec^{+\alpha} \alpha'[B/X]$$

as we were required to show.

2. Note that $\alpha' \notin -\alpha$ because $\alpha \notin A$ and that there exists B' such that $\alpha'{:=}B' \in \Sigma$ because $\Sigma; \Delta \vdash \alpha'$. Therefore, we have

$$\Sigma; \Delta \vdash \alpha'[\alpha/X] \prec^{-\alpha} \alpha'[B/X]$$

as we were required to show.

Case $A = \star$

1. and 2. are both immediate.

$\square$

Lemma 6.3 (Conversion)
Let $\alpha{:=}B_b \in \Sigma$, $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(W.\Sigma_1)$, $\alpha{:=}B_b \in W.\Sigma_1$, $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(W.\Sigma_2)$, $\alpha{:=}B_b \in W.\Sigma_2$, $W.\kappa(\alpha) = \lfloor \mathcal{V}[\![B_b]\!]\rho \rfloor_{W.j}$, *and* $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$.

1. *If* $\Sigma; \Delta \vdash A \prec^{+\alpha} B$ *and* $(W, e_1, e_2) \in \mathcal{E}[\![A]\!]\rho$ *then*

$$(W, (e_1 : \rho(A) \overset{+\alpha}{\Longrightarrow} \rho(B)), (e_2 : \rho(A) \overset{+\alpha}{\Longrightarrow} \rho(B))) \in \mathcal{E}[\![B]\!]\rho$$

2. *If* $\Sigma;\Delta \vdash B \prec^{-\alpha} A$ *and* $(W, e_1, e_2) \in \mathcal{E}\,[\![B]\!]\,\rho$ *then*

$$(W, (e_1 : \rho(B) \overset{-\alpha}{\Longrightarrow} \rho(A)), (e_2 : \rho(B) \overset{-\alpha}{\Longrightarrow} \rho(A))) \in \mathcal{E}\,[\![A]\!]\,\rho$$

Proof

By induction on the size of A.

Case $A = \mathsf{int}$

1. We have that $(W, e_1, e_2) \in \mathcal{E}\,[\![\mathsf{int}]\!]\,\rho$ and $\Sigma;\Delta \vdash \mathsf{int} \prec^{+\alpha} \mathsf{int}$ and need to show that

$$(W, (e_1 : \mathsf{int} \overset{+\alpha}{\Longrightarrow} \mathsf{int}), (e_2 : \mathsf{int} \overset{+\alpha}{\Longrightarrow} \mathsf{int})) \in \mathcal{E}\,[\![\mathsf{int}]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15). Consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\,[\![\mathsf{int}]\!]\,\rho$. Note that

$$W'.\Sigma_1 \triangleright (v_1 : \mathsf{int} \overset{+\alpha}{\Longrightarrow} \mathsf{int}) \longmapsto W'.\Sigma_1 \triangleright v_1 \quad \text{and} \quad W'.\Sigma_2 \triangleright (v_2 : \mathsf{int} \overset{+\alpha}{\Longrightarrow} \mathsf{int}) \longmapsto W'.\Sigma_1 \triangleright v_2$$

We apply anti-reduction (Lemma 5.14), so it remains to show that $(\blacktriangleright W', v_1, v_2) \in \mathcal{E}\,[\![\mathsf{int}]\!]\,\rho$. We have $(\blacktriangleright W', v_1, v_2) \in \mathcal{V}\,[\![\mathsf{int}]\!]\,\rho$ by monotonicity (Lemma 5.6) and recall that related values are related terms (Lemma 5.13).

2. The proof of part 2 has the same structure as the proof for part 1 above.

Case $A = \mathsf{bool}$

This proof has the same structure as the proof for the int case above.

Case $A = A_1 \!\rightarrow\! A_2$

In both parts, by Lemma 6.1 (canonical forms for conversion), we have that $B = B_1 \!\rightarrow\! B_2$.

1. We assume that $(W, e_1, e_2) \in \mathcal{E}\,[\![A_1 \!\rightarrow\! A_2]\!]\,\rho$ and $\Sigma;\Delta \vdash A_1 \!\rightarrow\! A_2 \prec^{+\alpha} B_1 \!\rightarrow\! B_2$ and need to show that

$$\left(W,\ \begin{matrix}(e_1 : \rho(A_1)\!\rightarrow\!\rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_1)\!\rightarrow\!\rho(B_2)),\\ (e_2 : \rho(A_1)\!\rightarrow\!\rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_1)\!\rightarrow\!\rho(B_2))\end{matrix}\right) \in \mathcal{E}\,[\![B_1 \!\rightarrow\! B_2]\!]\,\rho$$

Let

$$E_1 = ([\cdot] : \rho(A_1)\!\rightarrow\!\rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_1)\!\rightarrow\!\rho(B_2))$$
$$E_2 = ([\cdot] : \rho(A_1)\!\rightarrow\!\rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_1)\!\rightarrow\!\rho(B_2))$$

We need to show that $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}\,[\![B_1 \!\rightarrow\! B_2]\!]\,\rho$.
We proceed via monadic bind (Lemma 5.15). Consider arbitrary W_1, v_1, v_2 such that $W_1 \sqsupseteq W$ and $(W_1, v_1, v_2) \in \mathcal{V}\,[\![A_1 \!\rightarrow\! A_2]\!]\,\rho$. Related values are related terms (Lemma 5.13), so it suffices to show that $(W_1, E_1[v_1], E_2[v_2]) \in \mathcal{V}\,[\![B_1 \!\rightarrow\! B_2]\!]\,\rho$.
We unfold the definition of $\mathcal{V}\,[\![B_1 \!\rightarrow\! B_2]\!]\,\rho$. Consider arbitrary W_2, v_3, v_4 such that $W_2 \sqsupseteq W_1$ and $(W_2, v_3, v_4) \in \mathcal{V}\,[\![B_1]\!]\,\rho$. We need to show that $(W_2, E_1[v_1]\ v_3, E_2[v_2]\ v_4) \in \mathcal{E}\,[\![B_2]\!]\,\rho$. We have

$$W_2.\Sigma_1 \triangleright E_1[v_1]\ v_3 \longmapsto (v_1\ (v_3 : \rho(B_1) \overset{-\alpha}{\Longrightarrow} \rho(A_1)) : \rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_2))$$
$$W_2.\Sigma_2 \triangleright E_2[v_2]\ v_4 \longmapsto (v_2\ (v_4 : \rho(B_1) \overset{-\alpha}{\Longrightarrow} \rho(A_1)) : \rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_2))$$

So by anti-reduction (Lemma 5.14), it suffices to show

$$\left(\blacktriangleright W_2,\ \begin{matrix}(v_1\ (v_3 : \rho(B_1) \overset{-\alpha}{\Longrightarrow} \rho(A_1)) : \rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_2)),\\ (v_2\ (v_4 : \rho(B_1) \overset{-\alpha}{\Longrightarrow} \rho(A_1)) : \rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_2))\end{matrix}\right) \in \mathcal{E}\,[\![B_2]\!]\,\rho$$

Note that $\blacktriangleright W_2 \sqsupseteq W$.
By 2 of the induction hypothesis for A_1, noting that

- $\alpha{:=}B_b \in \Sigma$
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(\blacktriangleright W_2.\Sigma_1)$ by the definition of world extension
- $\alpha{:=}B_b \in \blacktriangleright W_2.\Sigma_1$ by the definition of world extension
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(\blacktriangleright W_2.\Sigma_2)$ by the definition of world extension
- $\alpha{:=}B_b \in \blacktriangleright W_2.\Sigma_2$ by the definition of world extension
- $\blacktriangleright W_2.\kappa(\alpha) = \Big\lfloor \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{W.j} \Big\rfloor_{\blacktriangleright W_2.j} = \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{\blacktriangleright W_2.j}$ by the definition of world extension and Lemma 5.3 (successive approximation)
- $(\blacktriangleright W_2, \rho) \in \mathcal{D}\,[\![\Delta]\!]$ by Lemma 5.10 (monotonicity)
- $\Sigma; \Delta \vdash B_1 \prec^{-\alpha} A_1$
- $(W_2, v_3, v_4) \in \mathcal{E}\,[\![B_1]\!]\,\rho$ by Lemma 5.13 (related values are related terms)

we have

$$(\blacktriangleright W_2, (v_3 : \rho(B_1) \overset{-\alpha}{\Longrightarrow} \rho(A_1)), (v_4 : \rho(B_1) \overset{-\alpha}{\Longrightarrow} \rho(A_1))) \in \mathcal{E}\,[\![A_1]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15), with the contexts

$$E_3 = (v_1\ [\cdot] : \rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_2)) \quad \text{and } E_4 = (v_2\ [\cdot] : \rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_2))$$

Consider arbitrary W_3, v_3', v_4' such that $W_3 \sqsupseteq \blacktriangleright W_2$ and $(W_3, v_3', v_4') \in \mathcal{V}\,[\![A_2]\!]\,\rho$. We need to show

$$(W_3, E_3[v_3'], E_4[v_4']) \in \mathcal{E}\,[\![B_2]\!]\,\rho$$

Because $(W, v_1, v_2) \in \mathcal{V}\,[\![A_1 \to A_2]\!]\,\rho$, we have $(W_3, v_1\ v_3', v_2\ v_4') \in \mathcal{E}\,[\![A_2]\!]$ by Lemma 5.13 (related values are related terms). We proceed again via monadic bind. Consider arbitrary W_4, v_5, v_6 such that $W_4 \sqsupseteq W_3$ and $(W_4, v_5, v_6) \in \mathcal{V}\,[\![A_2]\!]$. We need to show that

$$(W_4, (v_5 : \rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_2)), (v_6 : \rho(A_2) \overset{+\alpha}{\Longrightarrow} \rho(B_2))) \in \mathcal{E}\,[\![B_2]\!]\,\rho$$

Note that $W_4 \sqsupseteq W$.
By 1 of the induction hypothesis for A_2, noting that

- $\alpha{:=}B_b \in \Sigma$
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(W_4.\Sigma_1)$ by the definition of world extension
- $\alpha{:=}B_b \in W_4.\Sigma_1$ by the definition of world extension
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(W_4.\Sigma_2)$ by the definition of world extension
- $\alpha{:=}B_b \in W_4.\Sigma_2$ by the definition of world extension
- $W_4.\kappa(\alpha) = \Big\lfloor \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{W.j} \Big\rfloor_{W_4.j} = \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{W_4.j}$ by the definition of world extension and Lemma 5.3 (successive approximation)
- $(W_4, \rho) \in \mathcal{D}\,[\![\Delta]\!]$ by Lemma 5.10 (monotonicity)
- $\Sigma; \Delta \vdash A_2 \prec^{+\alpha} B_2$
- $(W_4, v_5, v_6) \in \mathcal{E}\,[\![A_2]\!]$ by Lemma 5.13 (related values are related terms)

we have what we are required to show.

2. We assume that $(W, e_1, e_2) \in \mathcal{E}\,[\![B_1 \to B_2]\!]\,\rho$ and $\Sigma; \Delta \vdash B_1 \to B_2 \prec^{-\alpha} A_1 \to A_2$ and need to show that

$$\left(W, \begin{array}{l} (e_1 : \rho(B_1) \to \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_1) \to \rho(A_2)), \\ (e_2 : \rho(B_1) \to \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_1) \to \rho(A_2)) \end{array} \right) \in \mathcal{E}\,[\![A_1 \to A_2]\!]\,\rho$$

Let

$$E_1 = ([\cdot] : \rho(B_1) \to \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_1) \to \rho(A_2))$$
$$E_2 = ([\cdot] : \rho(B_1) \to \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_1) \to \rho(A_2))$$

We need to show that $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}\,[\![A_1 \to A_2]\!]\,\rho$.
We proceed via monadic bind (Lemma 5.15). Consider arbitrary W_1, v_1, v_2 such that $W_1 \sqsupseteq W$ and $(W_1, v_1, v_2) \in \mathcal{V}\,[\![B_1 \to B_2]\!]\,\rho$. Related values are related terms (Lemma 5.13), so it suffices to show that $(W_1, E_1[v_1], E_2[v_2]) \in \mathcal{V}\,[\![A_1 \to A_2]\!]\,\rho$.
We unfold the definition of $\mathcal{V}\,[\![A_1 \to A_2]\!]\,\rho$. Consider arbitrary W_2, v_3, v_4 such that $W_2 \sqsupseteq W_1$ and $(W_2, v_3, v_4) \in \mathcal{V}\,[\![A_1]\!]\,\rho$. We need to show that $(W_2, E_1[v_1]\ v_3, E_2[v_2]\ v_4) \in \mathcal{E}\,[\![A_2]\!]\,\rho$. We have

$$W_2.\Sigma_1 \triangleright E_1[v_1]\ v_3 \longmapsto (v_1\ (v_3 : \rho(A_1) \overset{+\alpha}{\Longrightarrow} \rho(B_1)) : \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_2))$$
$$W_2.\Sigma_2 \triangleright E_2[v_2]\ v_4 \longmapsto (v_2\ (v_4 : \rho(A_1) \overset{+\alpha}{\Longrightarrow} \rho(B_1)) : \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(B_2))$$

So by anti-reduction (Lemma 5.14), it suffices to show

$$\left(\blacktriangleright W_2,\ \begin{array}{l} (v_1\ (v_3 : \rho(A_1) \overset{+\alpha}{\Longrightarrow} \rho(B_1)) : \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_2)), \\ (v_2\ (v_4 : \rho(A_1) \overset{+\alpha}{\Longrightarrow} \rho(B_1)) : \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_2)) \end{array} \right) \in \mathcal{E}\,[\![A_2]\!]\,\rho$$

Note that $\blacktriangleright W_2 \sqsupseteq W$.
By 1 of the induction hypothesis for A_1, noting that

- $\alpha{:=}B_b \in \Sigma$
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(\blacktriangleright W_2.\Sigma_1)$ by the definition of world extension
- $\alpha{:=}B_b \in \blacktriangleright W_2.\Sigma_1$ by the definition of world extension
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(\blacktriangleright W_2.\Sigma_2)$ by the definition of world extension
- $\alpha{:=}B_b \in \blacktriangleright W_2.\Sigma_2$ by the definition of world extension
- $\blacktriangleright W_2.\kappa(\alpha) = \Big\lfloor \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{W.j} \Big\rfloor_{\blacktriangleright W_2.j} = \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{\blacktriangleright W_2.j}$ by the definition of world extension and Lemma 5.3 (successive approximation)
- $(\blacktriangleright W_2, \rho) \in \mathcal{D}\,[\![\Delta]\!]$ by Lemma 5.10 (monotonicity)
- $\Sigma; \Delta \vdash A_1 \prec^{+\alpha} B_1$
- $(W_2, v_3, v_4) \in \mathcal{E}\,[\![B_1]\!]\,\rho$ by Lemma 5.13 (related values are related terms)

we have

$$(\blacktriangleright W_2, (v_3 : \rho(A_1) \overset{+\alpha}{\Longrightarrow} \rho(B_1)), (v_4 : \rho(A_1) \overset{+\alpha}{\Longrightarrow} \rho(B_1))) \in \mathcal{E}\,[\![B_1]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15), with the contexts

$$E_3 = (v_1\ [\cdot] : \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_2)) \quad \text{and } E_4 = (v_2\ [\cdot] : \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_2))$$

Consider arbitrary W_3, v_3', v_4' such that $W_3 \sqsupseteq \blacktriangleright W_2$ and $(W_3, v_3', v_4') \in \mathcal{V}\,[\![B_2]\!]\,\rho$. We need to show

$$(W_3, E_3[v_3'], E_4[v_4']) \in \mathcal{E}\,[\![B_2]\!]\,\rho$$

Because $(W, v_1, v_2) \in \mathcal{V}\,[\![B_1 \to B_2]\!]\,\rho$, we have $(W_3, v_1\ v_3', v_2\ v_4') \in \mathcal{E}\,[\![B_2]\!]$ by Lemma 5.13 (related values are related terms). We proceed again via monadic bind. Consider arbitrary W_4, v_5, v_6 such that $W_4 \sqsupseteq W_3$ and $(W_4, v_5, v_6) \in \mathcal{V}\,[\![B_2]\!]$. We need to show that

$$(W_4, (v_5 : \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_2)), (v_6 : \rho(B_2) \overset{-\alpha}{\Longrightarrow} \rho(A_2))) \in \mathcal{E}\,[\![A_2]\!]\,\rho$$

Note that $W_4 \sqsupseteq W$.
By 2 of the induction hypothesis for A_2, noting that

- $\alpha{:=}B_b \in \Sigma$
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(W_4.\Sigma_1)$ by the definition of world extension

- $\alpha{:=}B_b \in W_4.\Sigma_1$ by the definition of world extension
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(W_4.\Sigma_2)$ by the definition of world extension
- $\alpha{:=}B_b \in W_4.\Sigma_2$ by the definition of world extension
- $W_4.\kappa(\alpha) = \left\lfloor \lfloor \mathcal{V}\llbracket B_b \rrbracket \rho \rfloor_{W.j} \right\rfloor_{W_4.j} = \lfloor \mathcal{V}\llbracket B_b \rrbracket \rho \rfloor_{W_4.j}$ by the definition of world extension and Lemma 5.3 (successive approximation)
- $(W_4, \rho) \in \mathcal{D}\llbracket \Delta \rrbracket$ by Lemma 5.10 (monotonicity)
- $\Sigma; \Delta \vdash B_2 \prec^{-\alpha} A_2$
- $(W_4, v_5, v_6) \in \mathcal{E}\llbracket A_2 \rrbracket$ by Lemma 5.13 (related values are related terms)

we have what we are required to show.

Case $A = \forall X.\, A'$

In both parts, by Lemma 6.1 (canonical forms for conversion), we have that $B = \forall X.\, B'$.

1. Let $E_1 = ([\cdot] : \forall X.\, \rho(A') \overset{+\alpha}{\Longrightarrow} \forall X.\, \rho(B'))$ and $E_2 = ([\cdot] : \forall X.\, \rho(A') \overset{+\alpha}{\Longrightarrow} \forall X.\, \rho(B'))$. We assume $(W, e_1, e_2) \in \mathcal{E}\llbracket \forall X\,.\, A' \rrbracket \rho$ and need to show $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}\llbracket \forall X.\, B' \rrbracket \rho$. We proceed by monadic bind (Lemma 5.15). Consider arbitrary W_1, v_1, v_2 such that $W_1 \sqsupseteq W$ and $(W_1, v_1, v_2) \in \mathcal{V}\llbracket \forall X\,.\, A' \rrbracket \rho$. Related values are related terms (Lemma 5.13), so it suffices to show
$$(W_1, E_1[v_1], E_2[v_2]) \in \mathcal{V}\llbracket \forall X.\, B' \rrbracket \rho$$
We unfold the definition of $\mathcal{V}\llbracket \forall X.\, B' \rrbracket \rho$. Consider arbitrary $W_2, B_1, B_2, R, e_3, e_4, \alpha'$ such that the following hold:
   - $W_2 \sqsupseteq W_1$
   - $W_2.\Sigma_1; \cdot \vdash B_1$
   - $W_2.\Sigma_2; \cdot \vdash B_2$
   - $R \in \mathrm{Rel}_{W'.j}[B_1, B_2]$
   - $W_2.\Sigma_1 \triangleright E_1[v_1]\,[B_1] \longmapsto W_2.\Sigma_1, \alpha'{:=}B_1 \triangleright (e_3 : \rho(B')[\alpha'/X] \overset{+\alpha'}{\Longrightarrow} \rho(B')[B_1/X])$
   - $W_2.\Sigma_2 \triangleright E_2[v_2]\,[B_2] \longmapsto W_2.\Sigma_2, \alpha'{:=}B_2 \triangleright (e_4 : \rho(B')[\alpha'/X] \overset{+\alpha'}{\Longrightarrow} \rho(B')[B_2/X])$

   Let $W_2' = W_2 \boxplus (\alpha', B_1, B_2, R)$ and $\rho' = \rho[X \mapsto \alpha']$.
   We need to show
$$(W_2', e_3, e_4) \in \blacktriangleright\mathcal{E}\llbracket B' \rrbracket \rho'$$
   Note that from the definition of E_1 and E_2 and the operational semantics, we have that
$$W_2.\Sigma_1 \triangleright E_1[v_1]\,[B_1] \longmapsto ((v_1\,[\alpha'] : \rho_1(A')[\alpha'/X] \overset{+\alpha}{\Longrightarrow} \rho(B')[\alpha'/X]) : \rho(B')[\alpha'/X] \overset{+\alpha'}{\Longrightarrow} \rho(B')[B_1/X])$$
$$= W_2.\Sigma_1 \triangleright E_1[v_1]\,[B_1] \longmapsto ((v_1\,[\alpha'] : \rho'(A') \overset{+\alpha}{\Longrightarrow} \rho'(B')) : \rho'(B') \overset{+\alpha'}{\Longrightarrow} \rho(B')[B_1/X])$$
$$W_2.\Sigma_2 \triangleright E_2[v_2]\,[B_2] \longmapsto ((v_2\,[\alpha'] : \rho(A')[\alpha'/X] \overset{+\alpha}{\Longrightarrow} \rho(B')[\alpha'/X]) : \rho(B')[\alpha'/X] \overset{+\alpha'}{\Longrightarrow} \rho(B')[B_2/X])$$
$$= W_2.\Sigma_2 \triangleright E_2[v_2]\,[B_2] \longmapsto ((v_2\,[\alpha'] : \rho'(A') \overset{+\alpha}{\Longrightarrow} \rho'(B')) : \rho'(B') \overset{+\alpha'}{\Longrightarrow} \rho(B')[B_2/X])$$
   Therefore, it suffices to show that
$$\left(W_2',\ \begin{array}{l} (v_1[\alpha'] : \rho'(A') \overset{+\alpha}{\Longrightarrow} \rho'(B')), \\ (v_2[\alpha'] : \rho'(A') \overset{+\alpha}{\Longrightarrow} \rho'(B')) \end{array} \right) \in \blacktriangleright\mathcal{E}\llbracket B' \rrbracket \rho'$$
   Consider the case where $W_2'.j = 0$. Then, by definition of later relations, we have what we are required to show.
   Otherwise, we are required to show
$$\left(\blacktriangleright W_2',\ \begin{array}{l} (v_1[\alpha'] : \rho'(A') \overset{+\alpha}{\Longrightarrow} \rho'(B')), \\ (v_2[\alpha'] : \rho'(A') \overset{+\alpha}{\Longrightarrow} \rho'(B')) \end{array} \right) \in \mathcal{E}\llbracket B' \rrbracket \rho'$$

Note that $\blacktriangleright W_2' \sqsupseteq W$.
By 1 of the induction hypothesis for A', noting that

- $\alpha{:=}B_b \in \Sigma$
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(\blacktriangleright W_2'.\Sigma_1)$ by the definition of world extension
- $\alpha{:=}B_b \in \blacktriangleright W_2'.\Sigma_1$ by the definition of world extension
- $\mathrm{dom}(\Sigma) \subseteq \mathrm{dom}(\blacktriangleright W_2'.\Sigma_2)$ by the definition of world extension
- $\alpha{:=}B_b \in \blacktriangleright W_2'.\Sigma_2$ by the definition of world extension
- $\blacktriangleright W_2'.\kappa(\alpha) = \left\lfloor \lfloor \mathcal{V}\llbracket B_b \rrbracket \rho' \rfloor_{W.j} \right\rfloor_{\blacktriangleright W_2'.j} = \lfloor \mathcal{V}\llbracket B_b \rrbracket \rho' \rfloor_{\blacktriangleright W_2'.j} = \lfloor \mathcal{V}\llbracket B_b \rrbracket \rho \rfloor_{\blacktriangleright W_2'.j}$ by the definition of world extension, Lemma 5.3 (successive approximation), and since $\Sigma; \cdot \vdash B_b$
- $(\blacktriangleright W_2', \rho') \in \mathcal{D}\llbracket \Delta, X \rrbracket$ by Lemma 5.10 (monotonicity) and the definition of $\mathcal{D}\llbracket \Delta, X \rrbracket$
- $\Sigma; \Delta, X \vdash A' \prec^{+\alpha} B'$ by the structure of the derivation

We have

$$(\blacktriangleright W_2', v_1\,[\alpha'], v_2\,[\alpha']) \in \mathcal{E}\llbracket A' \rrbracket \rho' \implies \left(\blacktriangleright W_2', \begin{array}{l} (v_1[\alpha'] : \rho'(A') \overset{+\alpha}{\Longrightarrow} \rho'(B')), \\ (v_2[\alpha'] : \rho'(A') \overset{+\alpha}{\Longrightarrow} \rho'(B')) \end{array} \right) \in \mathcal{E}\llbracket B' \rrbracket \rho'$$

The conclusion of the above satisfies what we need to show, so it remains to prove the premise.
Since $(W, v_1, v_2) \in \mathcal{V}\llbracket \forall X\,.\,A' \rrbracket \rho$ and $W_2' \sqsupseteq W$, we have that $(W_2', v_1, v_2) \in \mathcal{V}\llbracket \forall X\,.\,A' \rrbracket \rho$ by Lemma 5.6 (monotonicity).
Choose α'' such that $\alpha'' \notin W_2'.\Sigma_1$ and $\alpha'' \notin W_2'.\Sigma_2$.
By Lemma 5.18 (type application steps), there exist e_5 and e_6 such that

$$\begin{array}{l} W_2'.\Sigma_1 \triangleright v_1\,[\alpha'] \longmapsto W_2'.\Sigma_1, \alpha''{:=}\alpha' \triangleright (e_5 : \rho(A')[\alpha''/X] \overset{+\alpha''}{\Longrightarrow} \rho(A')[\alpha'/X]) \\ W_2'.\Sigma_2 \triangleright v_2\,[\alpha'] \longmapsto W_2'.\Sigma_2, \alpha''{:=}\alpha' \triangleright (e_6 : \rho(A')[\alpha''/X] \overset{+\alpha''}{\Longrightarrow} \rho(A')[\alpha'/X]) \end{array}$$

Instantiate $(W_2', v_1, v_2) \in \mathcal{V}\llbracket \forall X\,.\,A' \rrbracket \rho$ with $W_2', \alpha', \lfloor \mathcal{V}\llbracket \alpha' \rrbracket \rho \rfloor_{W_2'.j}, e_5, e_6, \alpha''$. Note the following:

- $W_2' \sqsupseteq W_2'$ by reflexivity.
- $W_2'.\Sigma_1; \cdot \vdash \alpha'$ and $W_2'.\Sigma_2; \cdot \vdash \alpha'$ since $\alpha' \in W_2'.\Sigma_1$ and $\alpha' \in W_2'.\Sigma_2$.
- $\lfloor \mathcal{V}\llbracket \alpha' \rrbracket \rho \rfloor_{W_2'.j} \in \mathrm{Rel}_{W_2'.j}[\alpha', \alpha']$ by Lemma 5.7
- $W_2'.\Sigma_1 \triangleright v_1\,[\alpha'] \longmapsto W_2'.\Sigma_1, \alpha''{:=}\alpha' \triangleright (e_5 : \rho(A')[\alpha''/X] \overset{+\alpha''}{\Longrightarrow} \rho(A')[\alpha'/X])$
- $W_2'.\Sigma_2 \triangleright v_2\,[\alpha'] \longmapsto W_2'.\Sigma_2, \alpha''{:=}\alpha' \triangleright (e_6 : \rho(A')[\alpha''/X] \overset{+\alpha''}{\Longrightarrow} \rho(A')[\alpha'/X])$

Let $W_3 = W_2' \boxplus (\alpha'', \alpha', \alpha', \lfloor \mathcal{V}\llbracket \alpha' \rrbracket \rho \rfloor_{W_2'.j})$.
Hence, we have that $(W_3, e_5, e_6) \in \blacktriangleright\mathcal{E}\llbracket A' \rrbracket \rho[X \mapsto \alpha'']$. Since $W_3.j = W_2'.j > 0$, we have that

$$(\blacktriangleright W_3, e_5, e_6) \in \mathcal{E}\llbracket A' \rrbracket \rho[X \mapsto \alpha'']$$

Let $\Sigma_b = \Sigma, \alpha'{:=}\mathsf{bool}, \alpha''{:=}\alpha'$.
Therefore, by the inductive hypothesis for $A'[\alpha''/X]$, noting that

- $\alpha''{:=}\alpha' \in \Sigma_b$
- $\alpha''{:=}\alpha' \in \blacktriangleright W_3.\Sigma_1$
- $\alpha''{:=}\alpha' \in \blacktriangleright W_3.\Sigma_2$
- $(\blacktriangleright W_3).\kappa(\alpha'') = \lfloor \mathcal{V}\llbracket \alpha' \rrbracket \rho \rfloor_{\blacktriangleright W_3.j}$ by the definition of $\blacktriangleright$
- $(\blacktriangleright W_3, \rho) \in \mathcal{D}\llbracket \Delta \rrbracket$ which follows by Lemma 5.10 (monotonicity) from $(W, \rho) \in \mathcal{D}\llbracket \Delta \rrbracket$
- $(\blacktriangleright W_3, e_5, e_6) \in \mathcal{E}\llbracket A'[\alpha''/X] \rrbracket \rho$
- $\Sigma_b; \Delta \vdash A'[\alpha''/X] \prec^{+\alpha''} A'[\alpha'/X]$ by Lemma 6.2 (convertibility substitution)

- $(\blacktriangleright W_3, e_5, e_6) \in \mathcal{E}\llbracket A'[\alpha''/X]\rrbracket \rho$ by Lemma 5.17 (Compositionality) from $(\blacktriangleright W_3, e_5, e_6) \in \mathcal{E}\llbracket A'\rrbracket \rho[X \mapsto \alpha'']$

we have that

$$(\blacktriangleright W_3, (e_5 : A[\alpha''/X] \overset{\pm\alpha''}{\Longrightarrow} A[\alpha'/X]), (e_6 : A[\alpha''/X] \overset{\pm\alpha''}{\Longrightarrow} A[\alpha'/X])) \in \mathcal{E}\llbracket A'[\alpha'/X]\rrbracket \rho$$

We next apply Lemma 5.14 (anti-reduction) noting the following

- $\blacktriangleright W_3 \sqsupseteq \blacktriangleright W_2'$ by Lemma 5.2 (properties of later relations) since $W_3 \sqsupseteq W_2'$
- $\blacktriangleright W_2'.j \leq \blacktriangleright W_3.j + 1$ since $\blacktriangleright W_3.j = W_2'.j - 1 = \blacktriangleright W_2'.j$
- $\blacktriangleright W_2'.\Sigma_1 \triangleright v_1[\alpha'] \longmapsto \blacktriangleright W_2'.\Sigma_1, \alpha'':=\alpha' \triangleright (e_5 : \rho(A')[\alpha''/X] \overset{\pm\alpha''}{\Longrightarrow} \rho(A')[\alpha'/X])$ since $\blacktriangleright W_2'.\Sigma_1 = W_2'.\Sigma_1$
- $\blacktriangleright W_2'.\Sigma_2 \triangleright v_2[\alpha'] \longmapsto \blacktriangleright W_2'.\Sigma_2, \alpha'':=\alpha' \triangleright (e_6 : \rho(A')[\alpha''/X] \overset{\pm\alpha''}{\Longrightarrow} \rho(A')[\alpha'/X])$ since $\blacktriangleright W_2'.\Sigma_2 = W_2'.\Sigma_2$
- $(\blacktriangleright W_3, (e_5 : A[\alpha''/X] \overset{\pm\alpha''}{\Longrightarrow} A[\alpha'/X]), (e_6 : A[\alpha''/X] \overset{\pm\alpha''}{\Longrightarrow} A[\alpha'/X])) \in \mathcal{E}\llbracket A'[\alpha'/X]\rrbracket \rho$

Hence, we have that $(\blacktriangleright W_2', v_1[\alpha'], v_2[\alpha']) \in \mathcal{E}\llbracket A'[\alpha'/X]\rrbracket \rho$ and therefore $(\blacktriangleright W_2', v_1[\alpha'], v_2[\alpha']) \in \mathcal{E}\llbracket A'[\alpha'/X]\rrbracket \rho'$ by Lemma 5.11 (logical relation weakening) as we were required to show.

2. The proof of part 2 has the same structure as the proof for part 1 above.

Case $A = A_1 \times A_2$: The proof of this case is straightforward.

Case $A = \alpha'$ where $\alpha' \neq \alpha$

1. We have that $(W, e_1, e_2) \in \mathcal{E}\llbracket \alpha'\rrbracket \rho$ and $\Sigma; \Delta \vdash \alpha' \prec^{+\alpha} \alpha'$ and need to show that

$$(W, (e_1 : \alpha' \overset{\pm\alpha}{\Longrightarrow} \alpha'), (e_2 : \alpha' \overset{\pm\alpha}{\Longrightarrow} \alpha')) \in \mathcal{E}\llbracket \alpha'\rrbracket \rho$$

We proceed via monadic bind (Lemma 5.15). Consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\llbracket \alpha'\rrbracket \rho$. Note that

$$W'.\Sigma_1 \triangleright (v_1 : \alpha' \overset{\pm\alpha}{\Longrightarrow} \alpha') \longmapsto v_1 \quad \text{and} \quad W'.\Sigma_2 \triangleright (v_2 : \alpha' \overset{\pm\alpha}{\Longrightarrow} \alpha') \longmapsto v_2$$

We apply anti-reduction, (Lemma 5.14) so it remains to show that $(\blacktriangleright W', v_1, v_2) \in \mathcal{E}\llbracket \alpha'\rrbracket \rho$. We have $(\blacktriangleright W', v_1, v_2) \in \mathcal{V}\llbracket \alpha'\rrbracket \rho$ by monotonicity (Lemma 5.6) and recall that related values are related terms (Lemma 5.13).

2. The proof of part 2 has the same structure as the proof for part 1 above.

Case $A = \alpha$

1. We have that $(W, e_1, e_2) \in \mathcal{E}\llbracket \alpha\rrbracket \rho$ and $\Sigma; \Delta \vdash \alpha \prec^{+\alpha} B_b$ and need to show that

$$(W, (e_1 : \alpha \overset{\pm\alpha}{\Longrightarrow} B_b), (e_2 : \alpha \overset{\pm\alpha}{\Longrightarrow} B_b)) \in \mathcal{E}\llbracket B_b\rrbracket \rho$$

We proceed via monadic bind (Lemma 5.15). Consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\llbracket \alpha\rrbracket \rho$. It suffices to show that

$$(W', (v_1 : \alpha \overset{\pm\alpha}{\Longrightarrow} B_b), (v_2 : \alpha \overset{\pm\alpha}{\Longrightarrow} B_b)) \in \mathcal{E}\llbracket B_b\rrbracket \rho$$

By the definition of $\mathcal{V}\llbracket \alpha\rrbracket \rho$, we have that $v_1 = (v_3 : B_b \overset{-\alpha}{\Longrightarrow} \alpha)$, $v_2 = (v_4 : B_b \overset{-\alpha}{\Longrightarrow} \alpha)$, and $(W', v_3, v_4) \in \blacktriangleright W.\kappa(\alpha) = (W', v_3, v_4) \in \blacktriangleright \lfloor \mathcal{V}\llbracket B_b\rrbracket\rfloor_{W.j}$.
If $W'.j = 0$ then we have what we are required to show. Otherwise, we have that $(\blacktriangleright W', v_3, v_4) \in \lfloor \mathcal{V}\llbracket B_b\rrbracket\rfloor_{W.j}$. Since $\lfloor \mathcal{V}\llbracket B_b\rrbracket\rfloor_{W.j} \subset \mathcal{V}\llbracket B_b\rrbracket$, we have that $(\blacktriangleright W', v_3, v_4) \in \mathcal{V}\llbracket B_b\rrbracket$ and because related values are related terms, we have that $(\blacktriangleright W', v_3, v_4) \in \mathcal{E}\llbracket B_b\rrbracket$.

Note that $\blacktriangleright W' \sqsupseteq W'$ by Lemma 5.2 (properties of later), $W'.j \leq \blacktriangleright W'.j + 1$, and that

$$W'.\Sigma_1 \triangleright ((v_3 : B_b \overset{-\alpha}{\Longrightarrow} \alpha) : \alpha \overset{+\alpha}{\Longrightarrow} B_b) \longmapsto \blacktriangleright W'.\Sigma_1 \triangleright v_3$$
$$W'.\Sigma_2 \triangleright ((v_4 : B_b \overset{-\alpha}{\Longrightarrow} \alpha) : \alpha \overset{+\alpha}{\Longrightarrow} B_b) \longmapsto \blacktriangleright W'.\Sigma_2 \triangleright v_4$$

By anti-reduction, we have that

$$(W', (v_1 : \alpha \overset{+\alpha}{\Longrightarrow} B_b), (v_2 : \alpha \overset{+\alpha}{\Longrightarrow} B_b)) \in \mathcal{E}\,[\![B_b]\!]\,\rho$$

as we were required to show.

2. We have that $(W, e_1, e_2) \in \mathcal{E}\,[\![B_b]\!]\,\rho$ and $\Sigma; \Delta \vdash B_b \prec^{-\alpha} \alpha$ and need to show that

$$(W, (e_1 : B_b \overset{-\alpha}{\Longrightarrow} \alpha), (e_2 : B_b \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{E}\,[\![\alpha]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15). Consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\,[\![B_b]\!]\,\rho$. It suffices to show that

$$(W', (v_1 : B_b \overset{-\alpha}{\Longrightarrow} \alpha), (v_2 : B_b \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\,[\![\alpha]\!]\,\rho$$

Note that $W'.\kappa(\alpha) = \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{W'.j}$ from the premises and the definition of world extension. Assume $W'.j > 0$. Since $W'.j - 1 < W'.j$, we have that $(\blacktriangleright W', v_1, v_2) \in \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{W'.j}$. We then have that $(W', v_1, v_2) \in \blacktriangleright \lfloor \mathcal{V}\,[\![B_b]\!]\,\rho \rfloor_{W'.j}$ and so $(W', v_1, v_2) \in \blacktriangleright W'.\kappa(\alpha)$ by the definition of $\blacktriangleright$. The result is then immediate from the definition of $\mathcal{V}\,[\![\alpha]\!]\,\rho$.

Case $A = X$

1. We have that $(W, e_1, e_2) \in \mathcal{E}\,[\![X]\!]\,\rho$ and $\Sigma; \Delta \vdash X \prec^{+\alpha} X$ and need to show that

$$(W, (e_1 : X \overset{+\alpha}{\Longrightarrow} X), (e_2 : X \overset{+\alpha}{\Longrightarrow} X)) \in \mathcal{E}\,[\![X]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15). Consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\,[\![X]\!]\,\rho$. Note that

$$W'.\Sigma_1 \triangleright (v_1 : X \overset{+\alpha}{\Longrightarrow} X) \longmapsto W'.\Sigma_1 \triangleright v_1 \quad \text{and} \quad W'.\Sigma_2 \triangleright (v_2 : X \overset{+\alpha}{\Longrightarrow} X) \longmapsto W'.\Sigma_2 \triangleright v_2$$

We apply anti-reduction, (Lemma 5.14) so it remains to show that $(\blacktriangleright W', v_1, v_2) \in \mathcal{E}\,[\![X]\!]\,\rho$. We have $(\blacktriangleright W', v_1, v_2) \in \mathcal{V}\,[\![X]\!]\,\rho$ by monotonicity (Lemma 5.6) and recall that related values are related terms (Lemma 5.13).

2. The proof of part 2 has the same structure as the proof for part 1 above.

Case $A = \star$

1. We have that $(W, e_1, e_2) \in \mathcal{E}\,[\![\star]\!]\,\rho$ and $\Sigma; \Delta \vdash \star \prec^{+\alpha} \star$ and need to show that

$$(W, (e_1 : \star \overset{+\alpha}{\Longrightarrow} \star), (e_2 : \star \overset{+\alpha}{\Longrightarrow} \star)) \in \mathcal{E}\,[\![X]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15). Consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\,[\![\star]\!]\,\rho$. Note that

$$W'.\Sigma_1 \triangleright (v_1 : \star \overset{+\alpha}{\Longrightarrow} \star) \longmapsto W'.\Sigma_1 \triangleright v_1 \quad \text{and} \quad W'.\Sigma_2 \triangleright (v_2 : \star \overset{+\alpha}{\Longrightarrow} \star) \longmapsto W'.\Sigma_2 \triangleright v_2$$

We apply anti-reduction, (Lemma 5.14) so it remains to show that $(\blacktriangleright W', v_1, v_2) \in \mathcal{E}\,[\![\star]\!]\,\rho$. We have $(\blacktriangleright W', v_1, v_2) \in \mathcal{V}\,[\![\star]\!]\,\rho$ by monotonicity (Lemma 5.6) and recall that related values are related terms (Lemma 5.13).

2. The proof of part 2 has the same structure as the proof for part 1 above.

$\square$

Lemma 6.4 (Pre-Compatibility: Type Application)
If $(W, v_1, v_2) \in \mathcal{V}\,[\![\forall X.\,A]\!]\,\rho$ *and* $W.\Sigma_1;\,\cdot \vdash \rho(B)$*, then* $(W, v_1\,[\rho(B)], v_1\,[\rho(B)]) \in \mathcal{E}\,[\![A[B/X]]\!]\,\rho$

Proof

Choose α such that $\alpha \notin W.\Sigma_1$ and $\alpha \notin W.\Sigma_2$.

By Lemma 5.18 (type application steps), there exist e_1 and e_2 such that

$$W.\Sigma_1 \triangleright v\,[\rho(B)] \longmapsto W.\Sigma_1, \alpha{:=}\rho(B) \triangleright (e_1 : \rho(A)[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} \rho(A)[\rho(B)/X])$$
$$W.\Sigma_2 \triangleright v\,[\rho(B)] \longmapsto W.\Sigma_2, \alpha{:=}\rho(B) \triangleright (e_2 : \rho(A)[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} \rho(A)[\rho(B)/X])$$

Instantiate $(W, v_1, v_2) \in \mathcal{V}\,[\![\forall X\,.\,A]\!]\,\rho$ with $W, \rho(B), \rho(B), \lfloor \mathcal{V}\,[\![B]\!]\,\rho \rfloor_{W.j}, e_1, e_2, \alpha$. Note the following:

- $W \sqsupseteq W$ by reflexivity.
- $W.\Sigma_1; \cdot \vdash \rho(B)$ from the premises
- $W.\Sigma_2; \cdot \vdash \rho(B)$ from the premises
- $\lfloor \mathcal{V}\,[\![B]\!]\,\rho \rfloor_{W.j} \in \mathrm{Rel}_{W.j}\,[\rho(B), \rho(B)]$ by Lemma 5.7.
- $W.\Sigma_1 \triangleright v\,[\rho(B)] \longmapsto W.\Sigma_1, \alpha{:=}\rho(B) \triangleright (e_1 : \rho(A)[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} \rho(A)[\rho(B)/X])$
- $W.\Sigma_2 \triangleright v\,[\rho(B)] \longmapsto W.\Sigma_2, \alpha{:=}\rho(B) \triangleright (e_2 : \rho(A)[\alpha/X] \stackrel{+\alpha}{\Longrightarrow} \rho(A)[\rho(B)/X])$

Let $W_2 = W \boxplus (\alpha, \rho(B), \rho(B), \lfloor \mathcal{V}\,[\![B]\!]\,\rho \rfloor_{W.j})$.
Hence, we have that $(W_2, e_1, e_2) \in \blacktriangleright\mathcal{E}\,[\![A]\!]\,\rho[X \mapsto \alpha]$.
By Lemma 5.17 (Compositionality), we then have that $(W_2, e_1, e_2) \in \blacktriangleright\mathcal{E}\,[\![A\,[\alpha/X]]\!]\,\rho$.
Assume $W.j = 0$. Then the result is immediate.
Otherwise, $W_2.j = W.j > 0$, and we have that $(\blacktriangleright W_2, e_1, e_2) \in \mathcal{E}\,[\![A\,[\alpha/X]]\!]\,\rho$.
Let $\Sigma_b = \Sigma, \alpha{:=}B$.
Note that, since $\blacktriangleright W_2 \in \mathcal{S}\,[\![\Sigma_b]\!]$ which follows by Lemma 5.9 (monotonicity) from $W \in \mathcal{S}\,[\![\Sigma]\!]$ and from the definition of $\mathcal{S}\,[\![\Sigma_b]\!]$, we have

- $\alpha{:=}B \in \Sigma_b$
- $\mathrm{dom}(\Sigma_b) \subseteq \mathrm{dom}(\blacktriangleright W_2.\Sigma_1)$
- $\alpha{:=}B \in \blacktriangleright W_2.\Sigma_1$
- $\mathrm{dom}(\Sigma_b) \subseteq \mathrm{dom}(\blacktriangleright W_2.\Sigma_2)$
- $\alpha{:=}B \in \blacktriangleright W_2.\Sigma_2$
- $\blacktriangleright W_2.\kappa(\alpha) = \lfloor \mathcal{V}\,[\![B]\!]\,\rho \rfloor_{\blacktriangleright W_2.j}$

By Lemma 6.3 (Conversion), additionally noting that

- $\Sigma_b; \Delta \vdash A\,[\alpha/X\,] \prec^{+\alpha} A\,[B\,/X\,]$ by Lemma 6.2 (convertibility substitution)
- $(\blacktriangleright W_2, \rho) \in \mathcal{D}\,[\![\Delta]\!]$ which follows by Lemma 5.10 (monotonicity) from $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(\blacktriangleright W_2, e_1, e_2) \in \mathcal{E}\,[\![A\,[\alpha/X]]\!]\,\rho$

we have that

$$(\blacktriangleright W_2, (e_1 : A[\alpha/X] \overset{+\alpha}{\Longrightarrow} A[\rho(B)/X]), (e_2 : A[\alpha/X] \overset{+\alpha}{\Longrightarrow} A[\rho(B)/X])) \in \mathcal{E}\,[\![A\,[B\,/\,X\,]]\!]\,\rho$$

We next apply Lemma 5.14 (anti-reduction) noting the following

- $\blacktriangleright W_2 \sqsupseteq W$ by Lemma 5.4 (adding to the world extends it),
- $W.j \leq W_2.j + 1$ since $\blacktriangleright W_2.j + 1 = W.j$
- $W.\Sigma_1 \triangleright v\ [\rho(B)] \longmapsto W.\Sigma_1, \alpha{:=}\rho(B) \triangleright (e_1 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[\rho(B)/X])$ and that $\blacktriangleright W_2.\Sigma_1 = W.\Sigma_1, \alpha{:=}\rho(B)$
- $W.\Sigma_2 \triangleright v\ [\rho(B)] \longmapsto W.\Sigma_2, \alpha{:=}\rho(B) \triangleright (e_2 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[\rho(B)/X])$ and that $\blacktriangleright W_2.\Sigma_2 = W.\Sigma_2, \alpha{:=}\rho(B)$
- $(\blacktriangleright W_2, (e_1 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[\rho(B)/X]), (e_2 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[\rho(B)/X])) \in \mathcal{E}\,[\![A\,[B\,/\,X\,]]\!]\,\rho$

Hence, we have that $(W, v_1\ [\rho(B)], v_2\ [\rho(B)]) \in \mathcal{E}\,[\![A\,[B\,/\,X\,]]\!]\,\rho$ as we were required to show. $\square$

Lemma 6.5 (Cast)
Let $\Sigma; \Delta \vdash A \prec B$, $W \in \mathcal{S}\,[\![\Sigma]\!]$, and $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$.
If $(W, e_1, e_2) \in \mathcal{E}\,[\![A]\!]\,\rho$ then $(W, (e_1 : \rho(A) \overset{p}{\Longrightarrow} \rho(B)), (e_2 : \rho(A) \overset{p}{\Longrightarrow} \rho(B))) \in \mathcal{E}\,[\![B]\!]\,\rho$.

Proof

By induction on the step index and on the derivation of $\Sigma; \Delta \vdash A \prec B$.

Case $\dfrac{\vdash \Sigma}{\Sigma; \Delta \vdash \mathsf{int} \prec \mathsf{int}}$

We assume $(W, e_1, e_2) \in \mathcal{E}\,[\![A]\!]\,\rho$ and need to show that

$$(W, (e_1 : \mathsf{int} \overset{p}{\Longrightarrow} \mathsf{int}), (e_2 : \mathsf{int} \overset{p}{\Longrightarrow} \mathsf{int})) \in \mathcal{E}\,[\![\mathsf{int}]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15). Consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\,[\![\mathsf{int}]\!]\,\rho$. Note that

$$W'.\Sigma_1 \triangleright (v_1 : \mathsf{int} \overset{p}{\Longrightarrow} \mathsf{int}) \longmapsto W'.\Sigma_1 \triangleright v_1 \quad \text{and} \quad W'.\Sigma_2 \triangleright (v_2 : \mathsf{int} \overset{p}{\Longrightarrow} \mathsf{int}) \longmapsto W'.\Sigma_2 \triangleright v_2$$

We apply anti-reduction, (Lemma 5.14) so it remains to show that $(\blacktriangleright W', v_1, v_2) \in \mathcal{E}\,[\![\mathsf{int}]\!]\,\rho$. We have $(\blacktriangleright W', v_1, v_2) \in \mathcal{V}\,[\![\mathsf{int}]\!]\,\rho$ by monotonicity (Lemma 5.6) and recall that related values are related terms (Lemma 5.13).

Case $\dfrac{\vdash \Sigma}{\Sigma; \Delta \vdash \mathsf{bool} \prec \mathsf{bool}}$

This case has the same structure as the case for int.

Case $\dfrac{\Sigma; \Delta \vdash B_1 \prec A_1 \qquad \Sigma; \Delta \vdash A_2 \prec B_2}{\Sigma; \Delta \vdash A_1 \!\to A_2 \prec B_1 \!\to B_2}$

We assume that $(W, e_1, e_2) \in \mathcal{E}\,[\![A_1 \to A_2]\!]\,\rho$. Let

$$E_1 = ([\cdot] : \rho(A_1) \!\to \rho(A_2) \overset{p}{\Longrightarrow} \rho(B_1) \!\to \rho(B_2))$$
$$E_2 = ([\cdot] : \rho(A_1) \!\to \rho(A_2) \overset{p}{\Longrightarrow} \rho(B_1) \!\to \rho(B_2))$$

We need to show that $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}\,[\![B_1 \!\to B_2]\!]\,\rho$.
We proceed via monadic bind (Lemma 5.15). Consider arbitrary W_1, v_1, v_2 such that

- $W_1 \sqsupseteq W$
- $(W_1, v_1, v_2) \in \mathcal{V}\,[\![A_1 \rightarrow A_2]\!]\,\rho$

Related values are related terms (Lemma 5.13), so it suffices to show that $(W_1, E_1[v_1], E_2[v_2]) \in \mathcal{V}\,[\![B_1 \rightarrow B_2]\!]\,\rho$.

We unfold the definition of $\mathcal{V}\,[\![B_1 \rightarrow B_2]\!]\,\rho$. Consider arbitrary W_2, v_3, v_4 such that

- $W_2 \sqsupseteq W_1$
- $(W_2, v_3, v_4) \in \mathcal{V}\,[\![B_1]\!]\,\rho$

We need to show that $(W_2, E_1[v_1]\ v_3, E_2[v_2]\ v_4) \in \mathcal{E}\,[\![B_2]\!]\,\rho$. We have

$$W_2.\Sigma_1 \triangleright E_1[v_1]\ v_3 \longmapsto (v_1\ (v_3 : \rho(B_1) \xRightarrow{-p} \rho(A_1)) : \rho(A_2) \xRightarrow{p} \rho(B_2))$$
$$W_2.\Sigma_2 \triangleright E_2[v_2]\ v_4 \longmapsto (v_2\ (v_4 : \rho(B_1) \xRightarrow{-p} \rho(A_1)) : \rho(A_2) \xRightarrow{p} \rho(B_2))$$

So by anti-reduction (Lemma 5.14), it suffices to show

$$\left(\blacktriangleright W_2,\ \begin{array}{l} (v_1\ (v_3 : \rho(B_1) \xRightarrow{-p} \rho(A_1)) : \rho(A_2) \xRightarrow{p} \rho(B_2)), \\ (v_2\ (v_4 : \rho(B_1) \xRightarrow{-p} \rho(A_1)) : \rho(A_2) \xRightarrow{p} \rho(B_2)) \end{array} \right) \in \mathcal{E}\,[\![B_2]\!]\,\rho$$

By the induction hypothesis for $\Sigma; \Delta \vdash B_1 \prec A_1$, noting that

- $\blacktriangleright W_2 \in \mathcal{S}\,[\![\Sigma]\!]$ by Lemma 5.9 (monotonicity)
- $(\blacktriangleright W_2, \rho) \in \mathcal{D}\,[\![\Delta]\!]$ by Lemma 5.10 (monotonicity)
- $(\blacktriangleright W_2, v_3, v_4) \in \mathcal{V}\,[\![B_1]\!]\,\rho$ by Lemma 5.6 (monotonicity)

we have

$$(\blacktriangleright W_2, (v_3 : \rho(B_1) \xRightarrow{-p} \rho(A_1)), (v_4 : \rho(B_1) \xRightarrow{-p} \rho(A_1))) \in \mathcal{E}\,[\![A_1]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15), with the contexts

$$E_3 = (v_1\ [\cdot] : \rho(A_2) \xRightarrow{p} \rho(B_2)) \quad \text{and } E_4 = (v_2\ [\cdot] : \rho(A_2) \xRightarrow{p} \rho(B_2))$$

Consider arbitrary W_3, v_3', v_4' such that

- $W_3 \sqsupseteq \blacktriangleright W_2$
- $(W_3, v_3', v_4') \in \mathcal{V}\,[\![A_1]\!]\,\rho$

We need to show

$$(W_3, E_3[v_3'], E_4[v_4']) \in \mathcal{E}\,[\![B_2]\!]\,\rho$$

Note that $(W_1, v_1, v_2) \in \mathcal{V}\,[\![A_1 \rightarrow A_2]\!]\,\rho$.

Instantiate the definition of $\mathcal{V}\,[\![A_1 \rightarrow A_2]\!]\,\rho$ with W_3, v_3', v_4', noting that

- $W_3 \sqsupseteq W_1$
- $(W_3, v_3', v_4') \in \mathcal{V}\,[\![A_1]\!]\,\rho$

We have that $(W_3, v_1\ v_3', v_2\ v_4') \in \mathcal{E}\,[\![A_2]\!]\,\rho$.

We proceed again via monadic bind. Consider arbitrary W_4, v_5, v_6 such that

- $W_4 \sqsupseteq W_3$
- $(W_4, v_5, v_6) \in \mathcal{V}\,[\![A_2]\!]$

We need to show

$$(W_4, (v_5 : \rho(A_2) \xRightarrow{p} \rho(B_2)), (v_6 : \rho(A_2) \xRightarrow{p} \rho(B_2))) \in \mathcal{E}\,[\![B_2]\!]\,\rho$$

By the induction hypothesis for $\Sigma; \Delta \vdash A_2 \prec B_2$, noting that

- $W_4 \in \mathcal{S}\llbracket\Sigma\rrbracket$ by Lemma 5.9 (monotonicity)
- $(W_4, \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$ by Lemma 5.10 (monotonicity)
- $(W_4, v_5, v_6) \in \mathcal{E}\llbracket A_2\rrbracket\,\rho$ by Lemma 5.13 (related values are related terms)

we have what we are required to show.

Case $\dfrac{\Sigma; \Delta, X \vdash A \prec B' \qquad X \notin A}{\Sigma; \Delta \vdash A \prec \forall X\,.\,B'}$

We assume $(W, e_1, e_2) \in \mathcal{E}\llbracket A\rrbracket\,\rho$. Let $E_1 = ([\cdot] : \rho(A) \xRightarrow{p} \forall X.\,\rho(B'))$ and $E_2 = ([\cdot] : \rho(A) \xRightarrow{p} \forall X.\,\rho(B'))$. We need to show that

$$(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}\llbracket\forall X.\,B'\rrbracket\,\rho$$

We proceed by monadic bind (Lemma 5.15). Consider arbitrary W_1, v_1, v_2 such that

- $W_1 \sqsupseteq W$
- $(W_1, v_1, v_2) \in \mathcal{V}\llbracket A\rrbracket\,\rho$

We may equivalently show that

$$(W_1, E_1[v_1], E_2[v_2]) \in \mathcal{V}\llbracket\forall X.\,B'\rrbracket\,\rho$$

and then conclude because related values are related terms (Lemma 5.13).

We unfold the definition of $\mathcal{V}\llbracket\forall X.\,B'\rrbracket\,\rho$. Consider arbitrary $W_2, B_1, B_2, R, e_1, e_2, \alpha$ such that

- $W_2 \sqsupseteq W_1$
- $W_2.\Sigma_1; \cdot \vdash B_1$ and $W_2.\Sigma_2; \cdot \vdash B_2$
- $R \in \mathrm{Rel}_{W_1.j}\,[B_1, B_2]$
- $W_2.\Sigma_1 \triangleright E_1[v_1]\,[B_1] \longmapsto W_2.\Sigma_1, \alpha{:=}B_1 \triangleright (e_3 : \rho(B')[\alpha/X] \xRightarrow{+\alpha} \rho(B')[B_1/X])$
- $W_2.\Sigma_2 \triangleright E_2[v_2]\,[B_2] \longmapsto W_2.\Sigma_2, \alpha{:=}B_2 \triangleright (e_4 : \rho(B')[\alpha/X] \xRightarrow{+\alpha} \rho(B')[B_2/X])$

Let $W_3 = W_2 \boxplus (\alpha, B_1, B_2, R)$ and $\rho' = \rho[X \mapsto \alpha]$.

We need to show that

$$\begin{aligned}&(W_3, e_1, e_2) \in \mathcal{E}\llbracket B'[\alpha/X]\rrbracket\,\rho'\\ &= (W_3, e_1, e_2) \in \mathcal{E}\llbracket B'\rrbracket\,\rho'[X \mapsto \alpha]\\ &= (W_3, e_1, e_2) \in \mathcal{E}\llbracket B'\rrbracket\,\rho'\end{aligned}$$

by Lemma 5.17 (Compositionality).

By our operational semantics we have that

$$W_2.\Sigma_1 \triangleright E_1[v_1]\,[B_1] \longmapsto W_2.\Sigma_1, \alpha{:=}B_1 \triangleright ((v_1 : \rho(A) \xRightarrow{p} \rho(B')[\alpha/X]) : \rho(B')[\alpha/X] \xRightarrow{+\alpha} \rho(B')[B_1/X])$$

$$W_2.\Sigma_2 \triangleright E_2[v_2]\,[B_2] \longmapsto W_2.\Sigma_2, \alpha{:=}B_2 \triangleright ((v_2 : \rho(A) \xRightarrow{p} \rho(B')[\alpha/X]) : \rho(B')[\alpha/X] \xRightarrow{+\alpha} \rho(B')[B_2/X])$$

and therefore that

$$e_3 = (v_1 : \rho(A) \xRightarrow{p} \rho(B')[\alpha/X])$$

$$e_4 = (v_2 : \rho(A) \xRightarrow{p} \rho(B')[\alpha/X])$$

Thus, by Lemma 5.13 (related values are related terms), and since $X \notin A$, it suffices to show that

$$\begin{aligned}&(W_3, (v_1 : \rho(A) \xRightarrow{p} \rho(B')[\alpha/X]), (v_2 : \rho(A) \xRightarrow{p} \rho(B')[\alpha/X])) \in \mathcal{V}\llbracket B'\rrbracket\,\rho'\\ &= (W_3, (v_1 : \rho'(A) \xRightarrow{p} \rho'(B')), (v_2 : \rho'(A) \xRightarrow{p} \rho'(B'))) \in \mathcal{V}\llbracket B'\rrbracket\,\rho'\end{aligned}$$

By the inductive hypothesis for $\Sigma; \Delta, X \vdash A \prec B'$, noting that

- $W_3 \in \mathcal{S}\llbracket\Sigma\rrbracket$ by Lemma 5.9 (monotonicity)
- $(W_3, \rho') \in \mathcal{D}\llbracket\Delta, X\rrbracket$ by definition since $(W_3, \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$ by Lemma 5.10 (monotonicity) and $\alpha \in \mathrm{dom}(W_3.\kappa)$
- $(W_3, v_1, v_2) \in \mathcal{V}\llbracket A\rrbracket\,\rho'$ by Lemmas 5.6 (monotonicity) and 5.11 (weakening).

we have that

$$(W_3, (v_1 : \rho'(A) \overset{p}{\Longrightarrow} \rho'(B')), (v_2 : \rho'(A) \overset{p}{\Longrightarrow} \rho'(B'))) \in \mathcal{V}\llbracket B'\rrbracket\,\rho'$$

as we were required to show.

Case $\dfrac{\Sigma; \Delta \vdash A'[\star/X] \prec B}{\Sigma; \Delta \vdash \forall X.\, A' \prec B}$

We assume $(W, e_1, e_2) \in \mathcal{E}\llbracket\forall X.\, A'\rrbracket\,\rho$ and need to show that

$$(W, (e_1 : \forall X.\, \rho(A') \overset{p}{\Longrightarrow} \rho(B)), (e_2 : \forall X.\, \rho(A') \overset{p}{\Longrightarrow} \rho(B))) \in \mathcal{E}\llbracket B\rrbracket\,\rho$$

We proceed by monadic bind (Lemma 5.15). Consider arbitrary W_1, v_1, v_2 such that $W_1 \sqsupseteq W$ and $(W_1, v_1, v_2) \in \mathcal{V}\llbracket\forall X.\, A'\rrbracket\,\rho$. It suffices to show that

$$(W_1, (v_1 : \forall X.\, \rho(A') \overset{p}{\Longrightarrow} \rho(B)), (v_2 : \forall X.\, \rho(A') \overset{p}{\Longrightarrow} \rho(B))) \in \mathcal{E}\llbracket B\rrbracket\,\rho$$

Assume $W_1.j = 0$. Then the result is immediate.
Otherwise, $W_1.j > 0$.
By the operational semantics, we have that

$$\begin{array}{lcl} W_1.\Sigma_1 \triangleright (v_1 : \forall X.\, \rho(A') \overset{p}{\Longrightarrow} \rho(B)) & \longmapsto & W_1.\Sigma_1 \triangleright (v_1\,[\star] : \rho(A')[\star/X] \overset{p}{\Longrightarrow} \rho(B)) \\ W_1.\Sigma_2 \triangleright (v_2 : \forall X.\, \rho(A') \overset{p}{\Longrightarrow} \rho(B)) & \longmapsto & W_1.\Sigma_2 \triangleright (v_2\,[\star] : \rho(A')[\star/X] \overset{p}{\Longrightarrow} \rho(B)) \end{array}$$

So by anti-reduction (Lemma 5.14), noting that $\blacktriangleright W_1 \sqsupseteq W_1$ and $W_1.j = \blacktriangleright W_1.j + 1$, it suffices to show that

$$(\blacktriangleright W_1, (v_1\,[\star] : \rho(A')[\star/X] \overset{p}{\Longrightarrow} \rho(B)), (v_2\,[\star] : \rho(A')[\star/X] \overset{p}{\Longrightarrow} \rho(B))) \in \mathcal{E}\llbracket B\rrbracket\,\rho$$

By the inductive hypothesis for $\Sigma; \Delta \vdash A'[\star/X] \prec B$, noting that

- $\blacktriangleright W_1 \in \mathcal{S}\llbracket\Sigma\rrbracket$ by Lemma 5.9 (monotonicity)
- $(\blacktriangleright W_1, \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$ by Lemma 5.10 (monotonicity)

it suffices to show that

$$(\blacktriangleright W_1, v_1\,[\star], v_2\,[\star]) \in \mathcal{E}\llbracket A'[\star/X]\rrbracket\,\rho$$

Choose α such that $\alpha \notin W_1.\Sigma_1$ and $\alpha \notin W_1.\Sigma_2$.
By lemma 5.18 (type application steps), we have that for some e_3, e_4

$$\begin{array}{lcl} W_1.\Sigma_1 \triangleright v_1\,[\star] & \longmapsto & W_1.\Sigma_1, \alpha{:=}\star \triangleright (e_3 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X])) \\ W_1.\Sigma_2 \triangleright v_2\,[\star] & \longmapsto & W_1.\Sigma_2, \alpha{:=}\star \triangleright (e_4 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X])) \end{array}$$

We instantiate the definition of $(W_1, v_1, v_2) \in \mathcal{V}\llbracket\forall X.\, A'\rrbracket\,\rho$ with $W_1, \star, \star, \lfloor\mathcal{V}\llbracket\star\rrbracket\,\rho\rfloor_{W_1.j}, e_3, e_4, \alpha$, noting that

- $W_1 \sqsupseteq W_1$ by reflexivity
- $W_1.\Sigma_1;\, \cdot \vdash \star$ and $W_1.\Sigma_2;\, \cdot \vdash \star$
- $\lfloor\mathcal{V}\llbracket\star\rrbracket\,\rho\rfloor_{W_1.j} \in \mathrm{Rel}_{W_1.j}\,[\star, \star]$ by Lemma 5.7 (type interpretations are valid)

- $W_1.\Sigma_1 \triangleright v_1\ [\star] \ \longmapsto\ W_1.\Sigma_1, \alpha{:=}\star \triangleright (e_3 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X]))$
- $W_1.\Sigma_2 \triangleright v_2\ [\star] \ \longmapsto\ W_1.\Sigma_2, \alpha{:=}\star \triangleright (e_4 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X]))$

Let $W_2 = W_1 \boxplus (\alpha, \star, \star, \lfloor \mathcal{V}\,[\![\star]\!]\,\rho \rfloor_{W_1.j})$.
Therefore, by Lemma 5.17 (compositionality), we have that

$$\begin{aligned}&(W_2, e_3, e_4) \in \blacktriangleright\mathcal{E}\,[\![A']\!]\,\rho[X \mapsto \alpha]\\ &= (W_2, e_3, e_4) \in \blacktriangleright\mathcal{E}\,[\![A'[\alpha/X]]\!]\,\rho\end{aligned}$$

Note that $\Sigma, \alpha{:=}\star; \Delta \vdash A'[\alpha/X] \prec^{+\alpha} A'[\star/X]$ by Lemma 6.2 (conversion substitution).
Then, by the induction hypothesis for $\Sigma, \alpha{:=}\star; \Delta \vdash A'[\alpha/X] \prec^{+\alpha} A'[\star/X]$, noting that

- $W_2 \in \mathcal{S}\,[\![\Sigma, \alpha{:=}\star]\!]$ by the definition of $\mathcal{S}\,[\![\Sigma, \alpha{:=}\star]\!]$ and from $W_1 \in \mathcal{S}\,[\![\Sigma]\!]$ by Lemma 5.9 (monotonicity)
- $(W_2, \rho) \in \mathcal{D}\,[\![\Delta]\!]$ by Lemma 5.10

since $W_2.j = W_1.j > 0$, we have that

$$\begin{aligned}&(W_2, (e_3 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X])), (e_4 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X]))) \in \blacktriangleright\mathcal{E}\,[\![A'[\star/X]]\!]\,\rho\\ &= (\blacktriangleright W_2, (e_3 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X])), (e_4 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X]))) \in \mathcal{E}\,[\![A'[\star/X]]\!]\,\rho\end{aligned}$$

By Lemma 5.14 (anti-reduction), noting that $\blacktriangleright W_2 \sqsupseteq \blacktriangleright W_1$ by Lemma 5.2 (properties of later relations), $\blacktriangleright W_2.j < \blacktriangleright W_1.j + 1$, and

$$\begin{aligned}\blacktriangleright W_1.\Sigma_1 \triangleright v_1\ [\star] \ &\longmapsto\ \blacktriangleright W_2.\Sigma_1 \triangleright (e_3 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X]))\\ \blacktriangleright W_1.\Sigma_2 \triangleright v_2\ [\star] \ &\longmapsto\ \blacktriangleright W_2.\Sigma_2 \triangleright (e_4 : \rho(A'[\alpha/X]) \overset{+\alpha}{\Longrightarrow} \rho(A'[\star/X]))\end{aligned}$$

since $\blacktriangleright W_1.\Sigma_i = W_1.\Sigma_i$ and $\blacktriangleright W_2.\Sigma_i = W_2.\Sigma_i = W_1.\Sigma_i, \alpha{:=}\star$, we have that

$$(\blacktriangleright W_1, v_1\ [\star], v_2\ [\star]) \in \mathcal{E}\,[\![A'[\star/X]]\!]\,\rho$$

as we were required to show.

Case $\dfrac{\Sigma; \Delta \vdash A_1 \prec B_1 \qquad \Sigma; \Delta \vdash A_2 \prec B_2}{\Sigma; \Delta \vdash A_1 \times A_2 \prec B_1 \times B_2}$: The proof of this case is straightforward.

Case $\dfrac{\vdash \Sigma \qquad \alpha \in \Sigma}{\Sigma; \Delta \vdash \alpha \prec \alpha}$

We assume $(W, e_1, e_2) \in \mathcal{E}\,[\![\alpha]\!]\,\rho$ and need to show

$$(W, (e_1 : \alpha \overset{p}{\Longrightarrow} \alpha), (e_2 : \alpha \overset{p}{\Longrightarrow} \alpha)) \in \mathcal{E}\,[\![\alpha]\!]\,\rho$$

We proceed via monadic bind (Lemma 5.15). Consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\,[\![\alpha]\!]\,\rho$. Note that

$$\begin{aligned}W'.\Sigma_1 \triangleright (v_1 : \alpha \overset{p}{\Longrightarrow} \alpha) \ &\longmapsto\ W'.\Sigma_1 \triangleright v_1\\ W'.\Sigma_2 \triangleright (v_2 : \alpha \overset{p}{\Longrightarrow} \alpha) \ &\longmapsto\ W'.\Sigma_2 \triangleright v_2\end{aligned}$$

We apply anti-reduction (Lemma 5.14), so it suffices to show that $(\blacktriangleright W', v_1, v_2) \in \mathcal{E}\,[\![\alpha]\!]\,\rho$, which we have by monotonicity (Lemma 5.6) since related values are related terms (Lemma 5.13).

Case $\dfrac{\vdash \Sigma \qquad X \in \Delta}{\Sigma; \Delta \vdash X \prec X}$

We assume $(W, e_1, e_2) \in \mathcal{E}\,[\![X]\!]\,\rho$ and need to show that

$$(W, (e_1 : \rho(X) \overset{p}{\Longrightarrow} \rho(X)), (e_2 : \rho(X) \overset{p}{\Longrightarrow} \rho(X))) \in \mathcal{E}\,[\![X]\!]\,\rho$$

Because $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$, we have $\rho(X) = \alpha'$ and $\alpha' \in \mathrm{dom}(W.\kappa)$ for some α'. So it suffices to show

$$(W, (e_1 : \alpha' \overset{p}{\Longrightarrow} \alpha'), (e_2 : \alpha' \overset{p}{\Longrightarrow} \alpha')) \in \mathcal{E}[\![X]\!]\,\rho$$

The rest of this case follows the same structure as the previous case.

Case $\dfrac{\Sigma; \Delta \vdash A}{\Sigma; \Delta \vdash A \prec \star}$

We assume $(W, e_1, e_2) \in \mathcal{E}[\![A]\!]\,\rho$ and need to show that

$$(W, (e_1 : \rho(A) \overset{p}{\Longrightarrow} \star), (e_2 : \rho(A) \overset{p}{\Longrightarrow} \star)) \in \mathcal{E}[\![\star]\!]\,\rho$$

Proceeding by monadic bind (Lemma 5.15), we consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}[\![A]\!]\,\rho$. Next we consider two cases regarding A:

1. $A = G$: Because related values are related terms (Lemma 5.13), it suffices to show

$$(W', (v_1 : G \overset{p}{\Longrightarrow} \star), (v_2 : G \overset{p}{\Longrightarrow} \star)) \in \mathcal{V}[\![\star]\!]\,\rho$$

   We consider the three cases for G.
   - If $G = \iota$, then our goal follows from $(W', v_1, v_2) \in \mathcal{V}[\![\iota]\!]\,\rho$ since $v_1 = v_2$.
   - If $G = \star \to \star$, then by the definition of $\mathcal{V}[\![\star]\!]\,\rho$, our goal follows from $(W', v_1, v_2) \in \mathcal{V}[\![\star \to \star]\!]\,\rho$ by Lemma 5.6 (monotonicity) since $\blacktriangleright W' \sqsupseteq W'$ by Lemma 5.2 (properties of later relations)
   - If $G = \alpha$, by the definition of $\mathcal{V}[\![\alpha]\!]\,\rho$ we have

$$v_i = (v_i' : A_i \overset{-\alpha}{\Longrightarrow} \alpha) \text{ for } i \in \{1, 2\} \text{ and } (W, v_1', v_2') \in \blacktriangleright W'.\kappa(\alpha)$$

     for some v_1', v_2', A_1, A_2. So by the definition of $\mathcal{V}[\![\star]\!]\,\rho$ we conclude

$$(W', (v_1 : \alpha \overset{p}{\Longrightarrow} \star), (v_2 : \alpha \overset{p}{\Longrightarrow} \star)) \in \mathcal{V}[\![\star]\!]\,\rho$$

2. $A = \forall X.\, A'$ we need to show that

$$(W', (v_1 : \forall X.\, A' \overset{p}{\Longrightarrow} \star), (v_2 : \forall X.\, A' \overset{p}{\Longrightarrow} \star)) \in \mathcal{E}[\![\star]\!]\,\rho$$

   By Lemma 5.14 (anti-reduction), noting that
   - $\blacktriangleright W' \sqsupseteq W'$ by Lemma 5.2 (properties of $\blacktriangleright$)
   - $W'.j = W'.j - 1 + 1 = \blacktriangleright W'.j + 1$
   - $W'.\Sigma_1 \triangleright (v_1 : \forall X.\, A' \overset{p}{\Longrightarrow} \star) \longmapsto \blacktriangleright W'.\Sigma_1 \triangleright (v_1\ [\star] : A'[\star/X] \overset{p}{\Longrightarrow} \star)$
   - $W'.\Sigma_2 \triangleright (v_2 : \forall X.\, A' \overset{p}{\Longrightarrow} \star) \longmapsto \blacktriangleright W'.\Sigma_2 \triangleright (v_2\ [\star] : A'[\star/X] \overset{p}{\Longrightarrow} \star)$

   It suffices to show that

$$(\blacktriangleright W', (v_1\ [\star] : A'[\star/X] \overset{p}{\Longrightarrow} \star), (v_2\ [\star] : A'[\star/X] \overset{p}{\Longrightarrow} \star)) \in \mathcal{E}[\![\star]\!]\,\rho$$

   By Lemma 6.4 (pre-compatibility for type application), noting that $(W', v_1, v_2) \in \mathcal{V}[\![\forall X.\, A]\!]\,\rho$, we have that

$$(W', v_1\ [\star], v_2\ [\star]) \in \mathcal{E}[\![A[\star/X]]\!]\,\rho$$

   By induction on the step index, noting that
   - $\Sigma; \Delta \vdash A[\star/X] \prec \star$ by definition since $\Sigma; \Delta \vdash A[\star/X]$
   - $\blacktriangleright W' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma 5.9 (monotonicity)
   - $(\blacktriangleright W', \rho) \in \mathcal{D}[\![\Delta]\!]$ by Lemma 5.10 (monotonicity)

- $(W', v_1\,[\star], v_2\,[\star]) \in \mathcal{E}\,[\![A[\star/X]]\!]\,\rho$

We have that $(\blacktriangleright W', (v_1\,[\star] : A'[\star/X] \overset{p}{\Longrightarrow} \star), (v_2\,[\star] : A'[\star/X] \overset{p}{\Longrightarrow} \star)) \in \mathcal{E}\,[\![\star]\!]\,\rho$ as we were required to show.

3. $A = A_1 \to A_2$ and $A_1 \to A_2 \neq \star \to \star$. We have that

$$W'.\Sigma_i \triangleright (v_i : \rho(A_1 \to A_2) \overset{p}{\Longrightarrow} \star) \longmapsto W'.\Sigma_i \triangleright ((v_i : \rho(A_1 \to A_2) \overset{p}{\Longrightarrow} \star \to \star) : \star \to \star \overset{p}{\Longrightarrow} \star)$$

for $i \in \{1, 2\}$. Let $v_i' = (v_i : \rho(A_1 \to A_2) \overset{p}{\Longrightarrow} \star \to \star)$ and $v_i'' = (v_i' : \star \to \star \overset{p}{\Longrightarrow} \star)$. We apply anti-reduction (Lemma 5.14), noting that

- $\blacktriangleright W' \sqsupseteq W'$ by Lemma 5.2 (properties of $\blacktriangleright$)
- $W'.j = W'.j - 1 + 1 = \blacktriangleright W'.j + 1$
- $W'.\Sigma_1 \triangleright (v_1 : \rho(A_1 \to A_2) \overset{p}{\Longrightarrow} \star) \longmapsto \blacktriangleright W'.\Sigma_1 \triangleright v_1''$
- $W'.\Sigma_2 \triangleright (v_2 : \rho(A_1 \to A_2) \overset{p}{\Longrightarrow} \star) \longmapsto \blacktriangleright W'.\Sigma_2 \triangleright v_2''$

and because related values are related terms (Lemma 5.13), it suffices to show $(\blacktriangleright W', v_1'', v_2'') \in \mathcal{V}\,[\![\star]\!]\,\rho$. So by the definition of $\mathcal{V}\,[\![\star]\!]\,\rho$, we need to show

$$(\blacktriangleright W', v_1', v_2') \in \blacktriangleright \mathcal{V}\,[\![\star \to \star]\!]\,\rho$$

We proceed according to the definition of $\mathcal{V}\,[\![\star \to \star]\!]\,\rho$. Consider arbitrary W'', v_3, v_4 such that

- $W'' \sqsupseteq \blacktriangleright\blacktriangleright W'$
- $(W'', v_3, v_4) \in \mathcal{V}\,[\![\star]\!]\,\rho$

We need to show that $(W'', v_1'\ v_3, v_2'\ v_4) \in \mathcal{E}\,[\![\star]\!]\,\rho$. We have

$$\begin{aligned} W''.\Sigma_1 \triangleright v_1'\ v_3 &\longmapsto W''.\Sigma_1 \triangleright (v_1\ (v_3 : \star \overset{-p}{\Longrightarrow} \rho(A_1)) : \rho(A_2) \overset{p}{\Longrightarrow} \star) \\ W''.\Sigma_1 \triangleright v_2'\ v_4 &\longmapsto W''.\Sigma_2 \triangleright (v_2\ (v_4 : \star \overset{-p}{\Longrightarrow} \rho(A_1)) : \rho(A_2) \overset{p}{\Longrightarrow} \star) \end{aligned}$$

so, by anti-reduction (Lemma 5.14), it suffices to show that

$$(\blacktriangleright W'', (v_1\ (v_3 : \star \overset{-p}{\Longrightarrow} \rho(A_1)) : \rho(A_2) \overset{p}{\Longrightarrow} \star), (v_2\ (v_4 : \star \overset{-p}{\Longrightarrow} \rho(A_1)) : \rho(A_2) \overset{p}{\Longrightarrow} \star)) \in \mathcal{E}\,[\![\star]\!]\,\rho$$

By the inductive hypothesis for the step index, noting that

- $\Sigma; \Delta \vdash \star \prec A_1$ by definition
- $\blacktriangleright W'' \in \mathcal{S}\,[\![\Sigma]\!]$ by Lemma 5.9 (monotonicity)
- $(\blacktriangleright W'', \rho) \in \mathcal{D}\,[\![\Delta]\!]$ by Lemma 5.10 (monotonicity)
- $(\blacktriangleright W'', v_3, v_4) \in \mathcal{E}\,[\![\star]\!]\,\rho$ by Lemma 5.6 (monotonicity) and Lemma 5.13 (related values are related terms)

We have

$$(\blacktriangleright W'', (v_3 : \star \overset{-p}{\Longrightarrow} \rho(A_1)), (v_4 : \star \overset{-p}{\Longrightarrow} \rho(A_1))) \in \mathcal{E}\,[\![A_1]\!]\,\rho$$

We proceed by monadic bind, so we consider arbitrary W_3, v_3', v_4' such that $W_3 \sqsupseteq \blacktriangleright W''$ and $(W_3, v_3', v_4') \in \mathcal{V}\,[\![A_1]\!]\,\rho$. It suffices to show that

$$(W_3, (v_1\ v_3' : \rho(A_2) \overset{p}{\Longrightarrow} \star), (v_2\ v_4' : \rho(A_2) \overset{p}{\Longrightarrow} \star)) \in \mathcal{E}\,[\![\star]\!]\,\rho$$

By the definition of $(W', v_1, v_2) \in \mathcal{V}\,[\![A_1 \to A_2]\!]\,\rho$, we have

$$(W_3, v_1\ v_3', v_2\ v_4') \in \mathcal{E}\,[\![A_2]\!]\,\rho$$

By the inductive hypothesis for the step index, noting that

- $\Sigma; \Delta \vdash A_2 \prec \star$ by definition
- $W_3 \in \mathcal{S}\,[\![\Sigma]\!]$ by Lemma 5.9 (monotonicity)

- $(W_3, \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$ by Lemma 5.10 (monotonicity)
- $(W_3, v_1\ v_3', v_2\ v_4') \in \mathcal{E}\llbracket A_2\rrbracket\rho$

We have what we were required to show.

Case $\dfrac{\Sigma;\Delta \vdash B}{\Sigma;\Delta \vdash \star \prec B}$

We assume $(W, e_1, e_2) \in \mathcal{E}\llbracket\star\rrbracket\rho$ and need to show that

$$(W, (e_1 : \star \overset{p}{\Longrightarrow} \rho(B)), (e_2 : \star \overset{p}{\Longrightarrow} \rho(B))) \in \mathcal{E}\llbracket B\rrbracket\rho$$

We proceed by monadic bind, so we consider arbitrary W', v_1, v_2 such that $W' \sqsupseteq W$ and $(W', v_1, v_2) \in \mathcal{V}\llbracket\star\rrbracket\rho$.

We proceed by cases on B:

1. $B = \iota$. So $\rho(B) = \iota$. Based on $(W', v_1, v_2) \in \mathcal{V}\llbracket\star\rrbracket\rho$, we have three subcases to consider:
   (a) $v_1 = (v : \iota' \overset{q}{\Longrightarrow} \star)$ and $v_2 = (v : \iota' \overset{q}{\Longrightarrow} \star)$
   Suppose $\iota' = \iota$. Then

   $$W'.\Sigma_i \triangleright ((v : \iota \overset{q}{\Longrightarrow} \star) : \star \overset{p}{\Longrightarrow} \iota) \longmapsto W'.\Sigma_i \triangleright v \qquad \text{for } i \in \{1, 2\}.$$

   We apply anti-reduction (Lemma 5.14), so it suffices to prove that

   $$(\blacktriangleright W', v, v) \in \mathcal{E}\llbracket\iota\rrbracket\rho$$

   which is true because related values are related terms (Lemma 5.13).
   Suppose $\iota' \neq \iota$. Then

   $$W'.\Sigma_i \triangleright ((v : \iota' \overset{q}{\Longrightarrow} \star) : \star \overset{p}{\Longrightarrow} \iota) \longmapsto W'.\Sigma_i \triangleright \text{blame}\, p \qquad \text{for } i \in \{1, 2\}.$$

   We apply anti-reduction (Lemma 5.14), so it suffices to prove that

   $$(\blacktriangleright W', \text{blame}\, p\, , \text{blame}\, p\,) \in \mathcal{E}\llbracket\iota\rrbracket\rho$$

   Which is immediate from the definition of $\mathcal{E}\llbracket\iota\rrbracket\rho$.
   (b) $v_1 = (v_1' : \star\!\rightarrow\!\star \overset{q}{\Longrightarrow} \star)$ and $v_2 = (v_2' : \star\!\rightarrow\!\star \overset{q}{\Longrightarrow} \star)$ and $(W', v_1', v_2') \in \mathcal{V}\llbracket\star\!\rightarrow\!\star\rrbracket\rho$
   We have
   $$W'.\Sigma_i \triangleright (v_i : \star \overset{p}{\Longrightarrow} \iota) \longmapsto W'.\Sigma_i \triangleright \text{blame}\, p \qquad \text{for } i \in \{1, 2\}.$$
   so the proof is similar to the above case in which $\iota' \neq \iota$.
   (c) $v_1 = (v_1' : \alpha \overset{q}{\Longrightarrow} \star)$ and $v_2 = (v_2' : \alpha \overset{q}{\Longrightarrow} \star)$ and $(W', v_1', v_2') \in \mathcal{V}\llbracket\alpha\rrbracket\rho$
   We have
   $$W'.\Sigma_i \triangleright (v_i : \star \overset{p}{\Longrightarrow} \iota) \longmapsto W'.\Sigma_i \triangleright \text{blame}\, p \qquad \text{for } i \in \{1, 2\}.$$
   so the proof is similar to the above case.
2. $B = B_1 \rightarrow B_2$.
   Based on $(W', v_1, v_2) \in \mathcal{V}\llbracket\star\rrbracket\rho$, we have three subcases to consider:
   (a) $v_1 = (v : \iota \overset{q}{\Longrightarrow} \star)$ and $v_2 = (v : \iota \overset{q}{\Longrightarrow} \star)$
   We have

   $$W'.\Sigma_i \triangleright (v_i : \star \overset{p}{\Longrightarrow} \rho(B_1)\!\rightarrow\!\rho(B_2)) \longmapsto W'.\Sigma_i \triangleright \text{blame}\, p \qquad \text{for } i \in \{1, 2\}.$$

   and the rest of this case is straightforward.

(b) $v_1 = (v_1' : \star \to \star \xRightarrow{q} \star)$ and $v_2 = (v_2' : \star \to \star \xRightarrow{q} \star)$ and $(W', v_1', v_2') \in \mathcal{V}\llbracket \star \to \star \rrbracket \rho$

Suppose $B = \star \to \star$. Then we have

$$W'.\Sigma_i \triangleright ((v_i' : \star \to \star \xRightarrow{q} \star) : \star \xRightarrow{p} \star \to \star) \longmapsto W'.\Sigma_i \triangleright v_i' \qquad \text{for } i \in \{1, 2\}.$$

We apply anti-reduction (Lemma 5.14). By monotonicity, we have $(\blacktriangleright W', v_1', v_2') \in \mathcal{V}\llbracket \star \to \star \rrbracket \rho$ and we conclude because related values are related terms (Lemma 5.13). Suppose $B \neq \star \to \star$. Then $\rho(B) \neq \star \to \star$.
So

$$\begin{aligned}
& W'.\Sigma_i \triangleright (v_i : \star \xRightarrow{p} \rho(B_1) \to \rho(B_2)) \\
\longrightarrow\ & W'.\Sigma_i \triangleright ((v_i : \star \xRightarrow{p} \star \to \star) : \star \to \star \xRightarrow{p} \rho(B_1) \to \rho(B_2)) \\
\longrightarrow\ & W'.\Sigma_i \triangleright (v_i' : \star \to \star \xRightarrow{p} \rho(B_1) \to \rho(B_2)) \qquad \text{for } i \in \{1, 2\}
\end{aligned}$$

We proceed via anti-reduction (Lemma 5.14), noting that $W'.j = \blacktriangleright\blacktriangleright W'.j + 2$, so we need to show

$$(\blacktriangleright\blacktriangleright W', (v_1' : \star \to \star \xRightarrow{p} \rho(B)), (v_2' : \star \to \star \xRightarrow{p} \rho(B))) \in \mathcal{E}\llbracket B \rrbracket \rho$$

which we obtain by the induction hypothesis for the step index.

(c) $v_1 = (v_1' : \alpha' \xRightarrow{q} \star)$ and $v_2 = (v_2' : \alpha' \xRightarrow{q} \star)$
We have

$$W'.\Sigma_i \triangleright (v_i : \star \xRightarrow{p} \rho(B_1) \to \rho(B_2)) \longmapsto W'.\Sigma_i \triangleright \mathrm{blame}\, p \qquad \text{for } i \in \{1, 2\}.$$

and the rest of this case is straightforward.

3. $B = \forall X.\, B'$: Because related values are related terms (Lemma 5.13) it suffices to show

$$(W', (v_1 : \star \xRightarrow{p} \forall X.\, \rho(B')), (v_2 : \star \xRightarrow{p} \forall X.\, \rho(B'))) \in \mathcal{V}\llbracket \forall X.\, B' \rrbracket \rho$$

Consider arbitrary W'', α' such that $W'' \sqsupseteq W'$ and $\alpha' \in \mathrm{dom}(W''.\kappa)$. Let $v_i' = (v_i : \star \xRightarrow{p} \forall X.\, \rho(B'))$ We need to show that

$$(W'', v_1'\ [\alpha'], v_2'\ [\alpha']) \in \mathcal{E}\llbracket B' \rrbracket \rho[X \mapsto \alpha']$$

We have

$$W''.\Sigma_i \triangleright v_i'\ [\alpha'] \longrightarrow W''.\Sigma_i \triangleright (v_i : \star \xRightarrow{p} \rho(B')[\alpha'/X])$$

We proceed via anti-reduction (Lemma 5.14), so we need to show

$$\left(\blacktriangleright W'', \begin{array}{l} (v_1 : \star \xRightarrow{p} \rho(B')[\alpha'/X]), \\ (v_2 : \star \xRightarrow{p} \rho(B')[\alpha'/X]) \end{array} \right) \in \mathcal{E}\llbracket B' \rrbracket \rho[X \mapsto \alpha']$$

which we obtain by the induction hypothesis for the step index.

4. $B = X$: We have $X \in \Delta$ and $(W', \rho) \in \mathcal{D}\llbracket \Delta \rrbracket$ (by monotonicity), so $\rho(X) = \alpha'$ and $\alpha' \in \mathrm{dom}(W'.\kappa)$ for some α'. We need to show that

$$(W', (v_1 : \star \xRightarrow{p} \alpha'), (v_2 : \star \xRightarrow{p} \alpha')) \in \mathcal{E}\llbracket X \rrbracket \rho$$

The rest of this case follows the same reasoning as for the next case where $B = \alpha$.

5. $B = \alpha$: So $\rho(B) = \alpha$. Based on $(W', v_1, v_2) \in \mathcal{V}\llbracket \star \rrbracket \rho$, we have three subcases to consider:

(a) $v_1 = (v : \iota \overset{q}{\Longrightarrow} \star)$ and $v_2 = (v : \iota \overset{q}{\Longrightarrow} \star)$
We have

$$W'.\Sigma_i \triangleright (v_i : \star \overset{p}{\Longrightarrow} \alpha) \longmapsto W'.\Sigma_i \triangleright \mathrm{blame}\, p \qquad \text{for } i \in \{1, 2\}.$$

so it is straightforward to finish this case.

(b) $v_1 = (v_1' : \star \to \star \overset{q}{\Longrightarrow} \star)$ and $v_2 = (v_2' : \star \to \star \overset{q}{\Longrightarrow} \star)$ and $(W', v_1', v_2') \in \mathcal{V}\,[\![\star \to \star]\!]\,\rho$
We have

$$W'.\Sigma_i \triangleright (v_i : \star \overset{p}{\Longrightarrow} \alpha) \longmapsto W'.\Sigma_i \triangleright \mathrm{blame}\, p \qquad \text{for } i \in \{1, 2\}.$$

so it is straightforward to complete this case.

(c) $v_1 = ((v_1'' : A_1 \overset{-\alpha'}{\Longrightarrow} \alpha') : \alpha' \overset{q}{\Longrightarrow} \star)$ and $v_2 = ((v_2'' : A_2 \overset{-\alpha'}{\Longrightarrow} \alpha') : \alpha' \overset{q}{\Longrightarrow} \star)$
and $(W', v_1'', v_2'') \in \blacktriangleright W'.\kappa(\alpha')$.
Let $v_1' = (v_1'' : A_1 \overset{-\alpha'}{\Longrightarrow} \alpha')$ and $v_2' = (v_2'' : A_2 \overset{-\alpha'}{\Longrightarrow} \alpha')$.

We need to show that

$$(W', ((v_1' : \alpha' \overset{q}{\Longrightarrow} \star) : \star \overset{p}{\Longrightarrow} \alpha), ((v_2' : \alpha' \overset{q}{\Longrightarrow} \star) : \star \overset{p}{\Longrightarrow} \alpha)) \in \mathcal{E}\,[\![\alpha]\!]\,\rho$$

As an aside, recall that $\Sigma; \Delta \vdash B$ and $B = \alpha$ so $\alpha \in \mathrm{dom}(\Sigma)$. Also, because $W' \in \mathcal{S}\,[\![\Sigma]\!]$ (by monotonicity, Lemma 5.6), we have $\alpha \in \mathrm{dom}(W'.\kappa)$.

- Suppose $\alpha = \alpha'$. Then

$$W'.\Sigma_i \triangleright ((v_i' : \alpha' \overset{q}{\Longrightarrow} \star) : \star \overset{p}{\Longrightarrow} \alpha) \longmapsto W'.\Sigma_i \triangleright v_i' \qquad \text{for } i \in \{1, 2\}.$$

  We proceed via anti-reduction (Lemma 5.14). From $(W', v_1'', v_2'') \in \blacktriangleright W'.\kappa(\alpha')$, we have $(W', v_1', v_2') \in \mathcal{V}\,[\![\alpha]\!]\,\rho$ and so we conclude that $(W', v_1', v_2') \in \mathcal{E}\,[\![\alpha]\!]\,\rho$ because related values are related terms (Lemma 5.13).

- Suppose $\alpha \neq \alpha'$. Then

$$W'.\Sigma_i \triangleright ((v_i' : \alpha' \overset{q}{\Longrightarrow} \star) : \star \overset{p}{\Longrightarrow} \alpha) \longmapsto W'.\Sigma_i \triangleright \mathrm{blame}\, p \qquad \text{for } i \in \{1, 2\}.$$

So it is straightforward to complete this case.

6. $B = \star$: We have

$$W'.\Sigma_i \triangleright (v_i : \star \overset{p}{\Longrightarrow} \star) \longrightarrow W'.\Sigma_i \triangleright v_i \qquad \text{for } i \in \{1, 2\}$$

We conclude this case by anti-reduction (Lemma 5.14) and note that related values are related terms (Lemma 5.13).

□

7 Fundamental Property / Parametricity

Lemma 7.1 (Compatibility: True)
$\Sigma; \Delta; \Gamma \vdash \mathsf{true} \preceq \mathsf{true} : \mathsf{bool}$.

Proof

Clearly, $\Sigma; \Delta; \Gamma \vdash \mathsf{true} : \mathsf{bool}$.

Consider arbitrary W,ρ, γ such that:

- $W \in \mathcal{S}\,[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$

We need to show that

$$\begin{aligned}&(W, \rho(\gamma_1(\mathsf{true})), \rho(\gamma_2(\mathsf{true}))) \in \mathcal{E}\,[\![\mathsf{bool}]\!]\,\rho\\ \equiv\ &(W, \mathsf{true}, \mathsf{true}) \in \mathcal{E}\,[\![\mathsf{bool}]\!]\,\rho\end{aligned}$$

Since $(W, \mathsf{true}, \mathsf{true}) \in \mathcal{V}\,[\![\mathsf{bool}]\!]\,\rho$, by Lemma a 5.13 (related values are related terms) we have what we needed to show.

□

Lemma 7.2 (Compatibility: False)
$\Sigma; \Delta; \Gamma \vdash \mathsf{false} \preceq \mathsf{false} : \mathsf{bool}$.

Proof

The proof is analogous to that for true. □

Lemma 7.3 (Compatibility: If)
If $\Sigma; \Delta; \Gamma \vdash e \preceq e' : \mathsf{bool}$, $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_1' : A$, *and* $\Sigma; \Delta; \Gamma \vdash e_2 \preceq e_2' : A$,
then $\Sigma; \Delta; \Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 \preceq \text{if } e' \text{ then } e_1' \text{ else } e_2' : A$.

Proof

Note that $\Sigma; \Delta; \Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : A$ and $\Sigma; \Delta; \Gamma \vdash \text{if } e' \text{ then } e_1' \text{ else } e_2' : A$ are immediate from the premise.

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}\,[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$

We need to show that

$$\begin{aligned}&(W, \rho(\gamma_1(\text{if } e \text{ then } e_1 \text{ else } e_2)), \rho(\gamma_2(\text{if } e' \text{ then } e_1' \text{ else } e_2'))) \in \mathcal{E}\,[\![A]\!]\,\rho\\ \equiv\ &(W, \text{if } \rho(\gamma_1(e)) \text{ then } \rho(\gamma_1(e_1)) \text{ else } \rho(\gamma_1(e_2)), \text{if } \rho(\gamma_2(e')) \text{ then } \rho(\gamma_2(e_1')) \text{ else } \rho(\gamma_2(e_2'))) \in \mathcal{E}\,[\![A]\!]\,\rho\end{aligned}$$

Let

$$E_1 = \text{if } [\cdot] \text{ then } \rho(\gamma_1(e_1)) \text{ else } \rho(\gamma_1(e_2))$$
$$E_2 = \text{if } [\cdot] \text{ then } \rho(\gamma_2(e_1')) \text{ else } \rho(\gamma_2(e_2'))$$

Instantiating the first premise with W, ρ, and γ, we have that $(W, \rho(\gamma_1(e)), \rho(\gamma_2(e'))) \in \mathcal{E}\,[\![\mathsf{bool}]\!]\,\rho$. We will use monadic bind to proceed.

Let $W' \sqsupseteq W$ and let $(W', v_1, v_2) \in \mathcal{V}\,[\![\mathsf{bool}]\!]\,\rho$. By Lemma 5.15 (Monadic Bind), it suffices to show that

$$(W', E_1[v_1], E_2[v_2]) \in \mathcal{E}\,[\![A]\!]\,\rho$$

So $v_1 = v_2 = b$ by the definition of $\mathcal{V}\,[\![\mathsf{bool}]\!]\,\rho$, and depending on whether $b = \mathsf{true}$ or $b = \mathsf{false}$ we have either

$$E_1[\mathsf{true}] \longmapsto \rho(\gamma_1(e_1)) \quad \text{and} \quad E_2[\mathsf{true}] \longmapsto \rho(\gamma_2(e_1'))$$

or

$$E_1[\mathsf{false}] \longmapsto \rho(\gamma_1(e_2)) \quad \text{and} \quad E_2[\mathsf{false}] \longmapsto \rho(\gamma_2(e_2'))$$

Hence, by Lemma 5.14 (anti-reduction), noting that $W' \sqsupseteq W'$ and $W'.j \leq W'.j + 1$, it suffices to show that

$$(W', \rho(\gamma_1(e_1)), \rho(\gamma_2(e_1'))) \in \mathcal{E}\,[\![A]\!]\,\rho \quad \text{and} \quad (W', \rho(\gamma_1(e_2)), \rho(\gamma_2(e_2'))) \in \mathcal{E}\,[\![A]\!]\,\rho$$

We can obtain these by instantiating the second and third premises with W', ρ, and γ, noting that $W' \in \mathcal{S}\,[\![\Sigma]\!]$, $(W', \rho) \in \mathcal{D}\,[\![\Delta]\!]$, and $(W', \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$ by the monotonicity lemmas (5.8,5.9, 5.10).

□

Lemma 7.4 (Compatibility: Int)
$\Sigma; \Delta; \Gamma \vdash n \preceq n : \mathsf{int}$.

Proof

Clearly, $\Sigma; \Delta; \Gamma \vdash n : \mathsf{int}$.

Consider arbitrary W, ρ, γ such that:

- $W \in \mathcal{S}\,[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$

We need to show that

$$(W, \rho(\gamma_1(n)), \rho(\gamma_2(n))) \in \mathcal{E}\,[\![\mathsf{int}]\!]\,\rho$$
$$\equiv (W, n, n) \in \mathcal{E}\,[\![\mathsf{int}]\!]\,\rho$$

Since $(W, n, n) \in \mathcal{V}\,[\![\mathsf{int}]\!]\,\rho$, by Lemma 5.13 (related values are related terms), we have what we needed to show.

□

Lemma 7.5 (Compatibility: Op)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_1' : \mathsf{int}$ *and* $\Sigma; \Delta; \Gamma \vdash e_2 \preceq e_2' : \mathsf{int}$, *then* $\Sigma; \Delta; \Gamma \vdash e_1 \circledast e_2 \preceq e_1' \circledast e_2' : \mathsf{int}$.

Proof

The first and second conjuncts are immediate.

For the third conjunct, consider arbitrary W, ρ, γ such that:

- $W \in \mathcal{S}\llbracket\Sigma\rrbracket$
- $(W, \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$
- $(W, \gamma) \in \mathcal{G}\llbracket\Gamma\rrbracket\,\rho$

We are required to show that:

$$\begin{aligned}&(W, \rho(\gamma_1(e_1 \circledast e_2)), \rho(\gamma_2(e_1' \circledast e_2'))) \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho\\ &= (W, \rho(\gamma_1(e_1) \circledast \rho(\gamma_1(e_2))), \rho(\gamma_2(e_1') \circledast \rho(\gamma_2(e_2')))) \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho\end{aligned}$$

Instantiate the first premise with W, ρ, γ, noting that

- $W \in \mathcal{S}\llbracket\Sigma\rrbracket$
- $(W, \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$
- $(W, \gamma) \in \mathcal{G}\llbracket\Gamma\rrbracket\,\rho$

We have that $(W, \rho(\gamma_1(e_1)), \rho(\gamma_2(e_1'))) \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho$.

We will use monadic bind to proceed. Consider arbitrary W', v_1, v_1' such that

- $W' \sqsupseteq W$
- $(W', v_1, v_1') \in \mathcal{V}\llbracket\mathsf{int}\rrbracket\,\rho$

It suffices to show that $(W', v_1 \circledast \rho(\gamma_1(e_2))), v_1' \circledast \rho(\gamma_2(e_2')))) \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho$.

By the definition of $\mathcal{V}\llbracket\mathsf{int}\rrbracket\,\rho$, we have that $v_1 = v_1'$, so we may equivalently show that

$$(W', v_1 \circledast \rho(\gamma_1(e_2)), v_1 \circledast \rho(\gamma_2(e_2'))) \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho$$

Instantiate the second premise with W', ρ, γ, noting that

- $W' \in \mathcal{S}\llbracket\Sigma\rrbracket$ by Lemma 5.9 (monotonicity)
- $(W', \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$ by Lemma 5.10 (monotonicity)
- $(W', \gamma) \in \mathcal{G}\llbracket\Gamma\rrbracket\,\rho$ by Lemma 5.8 (monotonicity)

We have that $(W', \rho(\gamma_1(e_2)), \rho(\gamma_2(e_2'))) \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho$.

We will use monadic bind to proceed. Consider arbitrary W'', v_2, v_2' such that

- $W'' \sqsupseteq W'$
- $(W'', v_2, v_2') \in \mathcal{V}\llbracket\mathsf{int}\rrbracket\,\rho$

It suffices to show that $(W'', v_1 \circledast v_2, v_1 \circledast v_2') \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho$.

By the definition of $\mathcal{V}\llbracket\mathsf{int}\rrbracket\,\rho$, we have that $v_2 = v_2'$, so we may equivalently show that

$$(W'', v_1 \circledast v_2, v_1 \circledast v_2) \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho$$

If $W''.j = 0$, we have what we are required to show. Otherwise, by Lemma 5.14 (anti-reduction), it suffices to show that $(\blacktriangleright W'', \llbracket\circledast\rrbracket\,(v_1,v_2), \llbracket\circledast\rrbracket\,(v_1,v_2)) \in \mathcal{E}\llbracket\mathsf{int}\rrbracket\,\rho$. Then, since related values are related terms (Lemma 5.13), it suffices to show that $(\blacktriangleright W'', \llbracket\circledast\rrbracket\,(v_1,v_2), \llbracket\circledast\rrbracket\,(v_1,v_2)) \in \mathcal{V}\llbracket\mathsf{int}\rrbracket\,\rho$, which is immediate.

$\square$

Lemma 7.6 (Compatibility: Var)
If $\Sigma;\Delta;\Gamma \vdash x : A$ *then* $\Sigma;\Delta;\Gamma \vdash x \preceq x : A$.

Proof

The first and second conjuncts are exactly equivalent to the premise.

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}\llbracket\Sigma\rrbracket$
- $(W, \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$
- $(W, \gamma) \in \mathcal{G}\llbracket\Gamma\rrbracket\rho$

We are required to show that

$$\begin{aligned}&(W, \rho(\gamma_1(x))), \rho(\gamma_2(x)))) \in \mathcal{E}\llbracket A\rrbracket\rho\\ &\equiv (W, \gamma_1(x), \gamma_2(x)) \in \mathcal{E}\llbracket A\rrbracket\rho\end{aligned}$$

Since $(W, \gamma) \in \mathcal{G}\llbracket\Gamma\rrbracket\rho$, there exist v_1 and v_2 such that:

- $\gamma(x) = (v_1, v_2)$
- $(W, v_1, v_2) \in \mathcal{V}\llbracket A\rrbracket\rho$

By lemma 5.13 (related values are related terms), we then have that

$$\begin{aligned}&(W, v_1, v_2) \in \mathcal{E}\llbracket A\rrbracket\rho\\ &\equiv (W, \gamma_1(x), \gamma_2(x)) \in \mathcal{E}\llbracket A\rrbracket\rho\end{aligned}$$

as we were required to show. □

Lemma 7.7 (Compatibility: Lambda)
If $\Sigma;\Delta;\Gamma, x : A \vdash e_1 \preceq e_2 : B$ *then* $\Sigma;\Delta;\Gamma \vdash \lambda(x : A).\, e_1 \preceq \lambda(x : A).\, e_2 : A \to B$.

Proof

Note that $\Sigma;\Delta;\Gamma \vdash \lambda(x : A).\, e_1 : A \to B$ and $\Sigma;\Delta;\Gamma \vdash \lambda(x : A).\, e_2 : A \to B$ are immediate from the premise.

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}\llbracket\Sigma\rrbracket$
- $(W, \rho) \in \mathcal{D}\llbracket\Delta\rrbracket$
- $(W, \gamma) \in \mathcal{G}\llbracket\Gamma\rrbracket\rho$

We are required to show that:

$$\begin{aligned}&(W, \rho(\gamma_1(\lambda(x : A).\, e_1)), \rho(\gamma_2(\lambda(x : A).\, e_2))) \in \mathcal{E}\llbracket A \to B\rrbracket\rho\\ &\equiv (W, \lambda(x : \rho(A)).\, \rho(\gamma_1(e_1)), \lambda(x : \rho(A)).\, \rho(\gamma_2(e_2))) \in \mathcal{E}\llbracket A \to B\rrbracket\rho\end{aligned}$$

By Lemma 5.13 (related values are related terms) it suffices to show that

$$(W, \lambda(x : \rho(A)).\, \rho(\gamma_1(e_1)), \lambda(x : \rho(A)).\, \rho(\gamma_2(e_2))) \in \mathcal{V}\llbracket A \to B\rrbracket\rho$$

Consider arbitrary W', v_1, v_2 such that

- $W' \sqsupseteq W$
- $(W', v_1, v_2) \in \mathcal{V}\llbracket A \rrbracket \rho$

It suffices to show that

$$(W', (\lambda(x:\rho(A)).\, \rho(\gamma_1(e_1)))\ v_1, (\lambda(x:\rho(A)).\, \rho(\gamma_2(e_2)))\ v_2) \in \mathcal{E}\llbracket B \rrbracket \rho$$

Note that

$$W'.\Sigma_1 \triangleright (\lambda(x:\rho(A)).\, \rho(\gamma_1(e_1)))\ v_1 \longmapsto W'.\Sigma_1 \triangleright \rho(\gamma_1(e_1))[v_1/x]$$

and

$$W'.\Sigma_2 \triangleright (\lambda(x:\rho(A)).\, \rho(\gamma_2(e_2)))\ v_2 \longmapsto W'.\Sigma_2 \triangleright \rho(\gamma_2(e_2))[v_2/x]$$

Hence, by Lemma 5.14 (anti-reduction), noting $W' \sqsupseteq W'$ by reflexivity and $W'.j \leq W'.j + 1$, it further suffices to show that

$$(W', \rho(\gamma_1(e_1))[v_1/x], \rho(\gamma_2(e_2))[v_2/x]) \in \mathcal{E}\llbracket B \rrbracket \rho$$

Instantiate the first premise with W', ρ, and $\gamma[x \mapsto (v_1, v_2)]$. Note that

- $W' \in \mathcal{S}\llbracket \Sigma \rrbracket$ by store monotonicity (Lemma 5.9)
- $(W', \rho) \in \mathcal{D}\llbracket \Delta \rrbracket$ by monotonicity (Lemma 5.10)
- $(W', \gamma[x \mapsto (v_1, v_2)]) \in \mathcal{G}\llbracket \Gamma, x:A \rrbracket \rho$, which follows from $(W', \gamma) \in \mathcal{G}\llbracket \Gamma \rrbracket$ (which we have by monotonocity (Lemma 5.8)) and $(W', v_1, v_2) \in \mathcal{V}\llbracket B \rrbracket \rho$ (which we have from above)

Hence we have

$$(W', \rho(\gamma_1[x \mapsto v_1](e_1)), \rho(\gamma_2[x \mapsto v_2](e_2))) \in \mathcal{E}\llbracket B \rrbracket \rho$$

Since v_1 and v_2 contain no free type or term variables, the above is equivalent to

$$(W', \rho(\gamma_1(e_1))[v_1/x], \rho(\gamma_2(e_2))[v_2/x]) \in \mathcal{E}\llbracket B \rrbracket \rho$$

which is what we needed to show. $\square$

Lemma 7.8 (Compatibility: Application)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : B \to A$ *and* $\Sigma; \Delta; \Gamma \vdash e_1' \preceq e_2' : B$*, then* $\Sigma; \Delta; \Gamma \vdash e_1\ e_1' \preceq e_2\ e_2' : A$*.*

Proof

Note that $\Sigma; \Delta; \Gamma \vdash e_1\ e_1' : A$ and $\Sigma; \Delta; \Gamma \vdash e_2\ e_2' : A$ are immediate from the premises.

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}\llbracket \Sigma \rrbracket$
- $(W, \rho) \in \mathcal{D}\llbracket \Delta \rrbracket$
- $(W, \gamma) \in \mathcal{G}\llbracket \Gamma \rrbracket \rho$

We are required to show that

$$\begin{aligned}&(W, \rho(\gamma_1(e_1\ e_1')), \rho(\gamma_2(e_2\ e_2'))) \in \mathcal{E}\llbracket A \rrbracket \rho \\ &\equiv (W, (\rho(\gamma_1(e_1)))\ (\rho(\gamma_1(e_1'))), (\rho(\gamma_2(e_2)))\ (\rho(\gamma_2(e_2')))) \in \mathcal{E}\llbracket A \rrbracket \rho\end{aligned}$$

Instantiating the first premise with W, ρ, and γ, we have that $(W, \rho(\gamma_1(e_1)), \rho(\gamma_2(e_2))) \in \mathcal{E}\,[\![B \to A]\!]\,\rho$. We will use monadic bind to proceed.

Let $W' \sqsupseteq W$ and let $(W', v_1, v_2) \in \mathcal{V}\,[\![B \to A]\!]\,\rho$. By Lemma 5.15 (monadic bind), it suffices to show

$$(W', v_1\ (\rho(\gamma_1(e_1'))), v_2\ (\rho(\gamma_2(e_2')))) \in \mathcal{E}\,[\![A]\!]\,\rho$$

Instantiating the second premise with W, ρ, and γ, we have that $(W, \rho(\gamma_1(e_1')), \rho(\gamma_2(e_2'))) \in \mathcal{E}\,[\![B]\!]\,\rho$. We will again use monadic bind to proceed.

Let $W'' \sqsupseteq W'$ and let $(W'', v_1', v_2') \in \mathcal{V}\,[\![B]\!]\,\rho$. Applying Lemma 5.15 (monadic bind), it suffices to show

$$(W'', v_1\ v_1', v_2\ v_2') \in \mathcal{E}\,[\![A]\!]\,\rho$$

Instantiate $(W', v_1, v_2) \in \mathcal{V}\,[\![B \to A]\!]\,\rho$ with W'', v_1', and v_2', noting that $W'' \sqsupseteq W'$ and $(W'', v_1', v_2') \in \mathcal{V}\,[\![B]\!]\,\rho$. Hence, we have $(W'', v_1\ v_1', v_2\ v_2') \in \mathcal{E}\,[\![A]\!]\,\rho$ as we needed to show. ☐

Lemma 7.9 (Compatibility: Type Abstraction)
If $\Sigma; \Delta, X; \Gamma \vdash v_1 \preceq v_2 : A$ *and* $\Sigma; \Delta \vdash \Gamma$*, then* $\Sigma; \Delta; \Gamma \vdash \Lambda X\,.v_1 \preceq \Lambda X\,.v_2 : \forall X\,.\,A$*.*

Proof

Note that $\Sigma; \Delta, X; \Gamma \vdash \Lambda X\,.v_1 : \forall X\,.\,A$ and $\Sigma; \Delta, X; \Gamma \vdash \Lambda X\,.v_2 : \forall X\,.\,A$ are immediate from the premise.

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}\,[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$

We are required to show that

$$\begin{aligned}&(W, \rho(\gamma_1(\Lambda X\,.v_1)), \rho(\gamma_2(\Lambda X\,.v_2))) \in \mathcal{E}\,[\![\forall X\,.\,A]\!]\,\rho\\ &\equiv (W, \Lambda X\,.\rho(\gamma_1(v_1)), \Lambda X\,.\rho(\gamma_2(v_2))) \in \mathcal{E}\,[\![\forall X\,.\,A]\!]\,\rho\end{aligned}$$

By Lemma 5.13 (related values are related terms), it suffices to show that

$$(W, \Lambda X\,.\rho(\gamma_1(v_1)), \Lambda X\,.\rho(\gamma_2(v_2))) \in \mathcal{V}\,[\![\Lambda X\,.A]\!]\,\rho$$

Consider arbitrary $W', B_1, B_2, R, e_1, e_2, \alpha$ such that

- $W' \sqsupseteq W$
- $W'.\Sigma_1;\ \cdot \vdash B_1$ and $W'.\Sigma_2;\ \cdot \vdash B_2$
- $R \in \mathrm{Rel}_{W'.j}\,[B_1, B_2]$
- $W'.\Sigma_1 \triangleright \Lambda X\,.\rho(\gamma_1(v_1))\ [B_1] \longmapsto W'.\Sigma_1, \alpha{:=}B_1 \triangleright (e_1 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[B_1/X])$
- $W'.\Sigma_2 \triangleright \Lambda X\,.\rho(\gamma_2(v_2))\ [B_2] \longmapsto W'.\Sigma_2, \alpha{:=}B_2 \triangleright (e_2 : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[B_2/X])$

Let $W_2 = W' \boxplus (\alpha, B_1, B_2, R)$.

It suffices to show that
$$(W_2, e_1, e_2) \in \mathcal{E}\,[\![A]\!]\,\rho[X \mapsto \alpha]$$

Note that by the operational semantics,

$$\begin{array}{lll} W'.\Sigma_1 \triangleright \Lambda X\,.\rho(\gamma_1(v_1))\,[B_1] & \longmapsto & W'.\Sigma_1, \alpha{:=}B_1 \triangleright (\rho(\gamma_1(v_1))[\alpha/X] : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[B_1/X]) \\ W'.\Sigma_2 \triangleright \Lambda X\,.\rho(\gamma_2(v_2))\,[B_2] & \longmapsto & W'.\Sigma_2, \alpha{:=}B_2 \triangleright (\rho(\gamma_2(v_2))[\alpha/X] : \rho(A)[\alpha/X] \overset{+\alpha}{\Longrightarrow} \rho(A)[B_2/X]) \end{array}$$

Therefore, we may equivalently show that

$$\begin{array}{l} (W_2, \rho(\gamma_1(v_1))[\alpha/X], \rho(\gamma_2(v_2))[\alpha/X]) \in \mathcal{E}\,[\![A]\!]\,\rho[X \mapsto \alpha] \\ = (W_2, \rho'(\gamma_1(v_1)), \rho'(\gamma_2(v_2))) \in \mathcal{E}\,[\![A]\!]\,\rho' \end{array}$$

where $\rho' = \rho[X \mapsto \alpha]$.

Instantiate the assumption with W_2, ρ', γ, noting that

- $W_2 \in \mathcal{S}\,[\![\Sigma]\!]$ by Lemma 5.9 (monotonicity) since $W_2 \sqsupseteq W$
- $(W_2, \rho') \in \mathcal{D}\,[\![\Delta, X]\!]$ by the definition of $\mathcal{D}\,[\![\Delta, X]\!]$ and from $(W_2, \rho) \in \mathcal{D}\,[\![\Delta]\!]$, which we obtain from Lemma 5.10 (monotonicity)
- $(W_2, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho'$ by Lemma 5.8 (monotonicity) and Lemma 5.11 (logical relation weakening)

We then have that
$$(W_2, \rho'(\gamma_1(v_1)), \rho'(\gamma_2(v_2))) \in \mathcal{E}\,[\![A]\!]\,\rho'$$
as we were required to show. □

Lemma 7.10 (Compatibility: Type Application)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : \forall X\,.\,A$ *and* $\Sigma; \Delta \vdash B$ *then* $\Sigma; \Delta; \Gamma \vdash e_1\,[B] \preceq e_2\,[B] : A[B/X]$.

Proof

First, note that $\Sigma; \Delta; \Gamma \vdash e_1\,[B] : A[B/X]$ and $\Sigma; \Delta; \Gamma \vdash e_2\,[B] : A[B/X]$ are immediate from the first two premises.

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}\,[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$

We are required to show that

$$\begin{array}{l} (W, \rho(\gamma_1(e_1\,[B])), \rho(\gamma_2(e_2\,[B]))) \in \mathcal{E}\,[\![A[B/X]]\!]\,\rho \\ \equiv (W, \rho(\gamma_1(e_1))\,[\rho(B)], \rho(\gamma_2(e_2))\,[\rho(B)]) \in \mathcal{E}\,[\![A[B/X]]\!]\,\rho \end{array}$$

Instantiating the first premise with W, ρ, γ we have that $(W, \rho(\gamma_1(e_1)), \rho(\gamma_2(e_2))) \in \mathcal{E}\,[\![\forall X\,.\,A]\!]\,\rho$. We will use monadic bind to proceed.

Let $W' \sqsupseteq W$ and let $(W', v_1, v_2) \in \mathcal{V}\,[\![\forall X\,.\,A]\!]\,\rho$. By Lemma 5.15 (monadic bind), it suffices to show that

$$(W', v_1\,[\rho(B)], v_2\,[\rho(B)]) \in \mathcal{E}\,[\![A\,[B/X\,]]\!]\,\rho$$

Note that $W'.\Sigma_i; \cdot \vdash \rho(B)$, which follows from the premise $\Sigma; \Delta \vdash B$ along with $W' \in \mathcal{S}\,[\![\Sigma]\!]$ (which follows by monotonicity from $W \in \mathcal{S}\,[\![\Sigma]\!]$) and $(W', \rho) \in \mathcal{D}\,[\![\Delta]\!]$ (which follows by monotonicity from $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$).

The result is then immediate from Lemma 6.4 (pre-compatibility for type application).

□

Lemma 7.11 (Compatibility: Pair)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A$ *and* $\Sigma; \Delta; \Gamma \vdash e'_1 \preceq e'_2 : B$*, then* $\Sigma; \Delta; \Gamma \vdash \langle e_1, e'_1\rangle \preceq \langle e_2, e'_2\rangle : A \times B$.

Proof

The proof of the lemma is standard. □

Lemma 7.12 (Compatibility: Left Projection)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A \times B$ *then* $\Sigma; \Delta; \Gamma \vdash \pi_1\; e_1 \preceq \pi_1\; e_2 : A$.

Proof

The proof of the lemma is standard. □

Lemma 7.13 (Compatibility: Right Projection)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A \times B$ *then* $\Sigma; \Delta; \Gamma \vdash \pi_2\; e_1 \preceq \pi_2\; e_2 : B$.

Proof

The proof of the lemma is standard. □

Lemma 7.14 (Compatibility: Conversion)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A$ *and* $\Sigma; \Delta \vdash A \prec^{\phi} B$*, then* $\Sigma; \Delta; \Gamma \vdash (e_1 : A \overset{\phi}{\Longrightarrow} B) \preceq (e_2 : A \overset{\phi}{\Longrightarrow} B) : B$.

Proof

Note that $\Sigma; \Delta; \Gamma \vdash (e_1 : A \overset{\phi}{\Longrightarrow} B) : B$ and $\Sigma; \Delta; \Gamma \vdash (e_2 : A \overset{\phi}{\Longrightarrow} B) : B$ follow from the premises.

Consider arbitrary W,ρ, γ such that

- $W \in \mathcal{S}\,[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$

We are required to show that

$$\begin{aligned}
&(W, \rho(\gamma_1(e_1 : A \overset{\phi}{\Longrightarrow} B)), \rho(\gamma_2(e_2 : A \overset{\phi}{\Longrightarrow} B))) \in \mathcal{E}\,[\![B]\!]\,\rho \\
&\equiv (W, (\rho(\gamma_1(e_1)) : A_1 \overset{\phi}{\Longrightarrow} B_2), (\rho(\gamma_2(e_2)) : A_2 \overset{\phi}{\Longrightarrow} B_2)) \in \mathcal{E}\,[\![B]\!]\,\rho
\end{aligned}$$

where $A_1 = \rho(A)$, $A_2 = \rho(A)$, $B_1 = \rho(B)$, and $B_2 = \rho(B)$.

Instantiating the first premise with W, ρ, γ we have that

$$(W, \rho(\gamma_1(e_1))), \rho(\gamma_2(e_2)))) \in \mathcal{E}\,[\![A]\!]\,\rho$$

We will use monadic bind to proceed.

Let $W' \sqsupseteq W$ and let $(W', v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$. By Lemma 5.15 (monadic bind), it suffices to show that

$$(W', (v_1 : A_1 \stackrel{\phi}{\Longrightarrow} B_1), (v_2 : A_2 \stackrel{\phi}{\Longrightarrow} B_2)) \in \mathcal{E}\,[\![B]\!]\,\rho$$

Note that we have $\Sigma; \Delta \vdash A \prec^{\phi} B$ as a premise, and that by monotonicity we have $W' \in \mathcal{S}\,[\![\Sigma]\!]$ (by Lemma 5.9) and $(W', \rho) \in \mathcal{D}\,[\![\Delta]\!]$ (by Lemma 5.10). Also, note that $(W', v_1, v_2) \in \mathcal{E}\,[\![A]\!]\,\rho$ since related values are related terms (by Lemma 5.13).

The desired result now follows by the Conversion Lemma (Lemma 6.3). □

Lemma 7.15 (Compatibility: Cast)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A$, $\Sigma; \Delta \vdash A \prec B$ *then* $\Sigma; \Delta; \Gamma \vdash (e_1 : A \stackrel{p}{\Longrightarrow} B) \preceq (e_2 : A \stackrel{p}{\Longrightarrow} B) : B$.

Proof

Note that $\Sigma; \Delta; \Gamma \vdash (e_1 : A \stackrel{p}{\Longrightarrow} B) : B$ and $\Sigma; \Delta; \Gamma \vdash (e_2 : A \stackrel{p}{\Longrightarrow} B) : B$ follow from the premises.

Consider arbitrary W,ρ, γ such that

- $W \in \mathcal{S}\,[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$

We are required to show that

$$\begin{aligned}&(W, \rho(\gamma_1(e_1 : A \stackrel{p}{\Longrightarrow} B)), \rho(\gamma_2(e_2 : A \stackrel{p}{\Longrightarrow} B))) \in \mathcal{E}\,[\![B]\!]\,\rho \\ &\equiv (W, (\rho(\gamma_1(e_1)) : \rho(A) \stackrel{p}{\Longrightarrow} \rho(B)), (\rho(\gamma_2(e_2)) : \rho(A) \stackrel{p}{\Longrightarrow} \rho(B))) \in \mathcal{E}\,[\![B]\!]\,\rho\end{aligned}$$

Instantiating the first premise with W, ρ, γ we have that

$$(W, \rho_1(\gamma_1(e_1)), \rho_2(\gamma_2(e_2))) \in \mathcal{E}\,[\![A]\!]\,\rho$$

We will use monadic bind to proceed.

Let $W' \sqsupseteq W$ and let $(W', v_1, v_2) \in \mathcal{V}\,[\![A]\!]\,\rho$. By Lemma 5.15 (monadic bind), it suffices to show that

$$(W', (v_1 : \rho(A) \stackrel{p}{\Longrightarrow} \rho(B)), (v_2 : \rho(A) \stackrel{p}{\Longrightarrow} \rho(B))) \in \mathcal{E}\,[\![B]\!]\,\rho$$

Note that we have $\Sigma; \Delta \vdash A \prec B$ as a premise, and that by monotonicity we have $W' \in \mathcal{S}\,[\![\Sigma]\!]$ (by Lemma 5.9) and $(W', \rho) \in \mathcal{D}\,[\![\Delta]\!]$ (by Lemma 5.10). Also, note that $(W', v_1, v_2) \in \mathcal{E}\,[\![A]\!]\,\rho$ since related values are related terms (by Lemma 5.13).

The desired result now follows by the Cast Lemma (Lemma 6.5). □

Lemma 7.16 (Compatibility: Blame)
$\Sigma; \Delta; \Gamma \vdash \mathrm{blame}\,p \preceq \mathrm{blame}\,p : A$.

Proof

Clearly, $\Sigma ; \Delta ; \Gamma \vdash \text{blame}\, p\ : A\,.$

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}\,[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}\,[\![\Delta]\!]$
- $(W, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$

We are required to show that

$$
\begin{aligned}
&(W, \rho(\gamma_1(\text{blame}\, p\ )), \rho(\gamma_2(\text{blame}\, p\ ))) \in \mathcal{E}\,[\![A]\!]\,\rho \\
&\equiv (W, \text{blame}\, p\ , \text{blame}\, p\ ) \in \mathcal{E}\,[\![A]\!]\,\rho
\end{aligned}
$$

This is immediate from the definition of $\mathcal{E}\,[\![A]\!]\,\rho$.

□

Theorem 7.17 (Fundamental Property)
If $\Sigma ; \Delta ; \Gamma \vdash e : A\,$, *then* $\Sigma ; \Delta ; \Gamma \vdash e \preceq e : A\,.$

Proof

By induction on the derivation of $\Sigma ; \Delta ; \Gamma \vdash e : A\,.$ Each case follows from the appropriate compatibility lemma. □

8 Soundness W.r.t. Contextual Equivalence

Lemma 8.1 (Weakening)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A$, $\Sigma' \supseteq \Sigma$, $\Delta' \supseteq \Delta$, *and* $\Gamma' \supseteq \Gamma$ *then* $\Sigma'; \Delta'; \Gamma' \vdash e_1 \preceq e_2 : A$.

Proof

Consider arbitrary W, ρ', γ' such that

- $W \in \mathcal{S}[\![\Sigma']\!]$
- $(W, \rho') \in \mathcal{D}[\![\Delta']\!]$
- $(W, \gamma') \in \mathcal{G}[\![\Gamma']\!]\,\rho'$

We need to show that $(W, \rho'(\gamma'_1(e_1)), \rho'(\gamma'_2(e_2))) \in \mathcal{E}[\![A]\!]\,\rho'$.

Let $\rho \sqsupseteq \rho'$ such that $\mathrm{dom}(\rho) = \Delta$. Let $\gamma \sqsupseteq \gamma'$ such that $\mathrm{dom}(\gamma) = \mathrm{dom}(\Gamma)$.

Since $\Sigma; \Delta; \Gamma \vdash e_1 : A$, for any x in e_1, $x \in \mathrm{dom}(\Gamma)$. Similarly, for any X in e_1, $X \in \Delta$. Therefore, $\rho'(\gamma'_1(e_1)) = \rho(\gamma_1(e_1))$.

The same reasoning holds for e_2.

Hence, it suffices to show that $(W, \rho(\gamma_1(e_1)), \rho(\gamma_2(e_2))) \in \mathcal{E}[\![A]\!]\,\rho'$.

Further, by Lemma 5.11, it suffices to show that $(W, \rho(\gamma_1(e_1)), \rho(\gamma_2(e_2))) \in \mathcal{E}[\![A]\!]\,\rho$.

Instantiate the premise with W, ρ, γ. Note that $W \in \mathcal{S}[\![\Sigma]\!]$ since $\mathcal{S}[\![\Sigma]\!] \supseteq \mathcal{S}[\![\Sigma']\!]$. Further note that $(W, \rho) \in \mathcal{D}[\![\Delta]\!]$ because for all $X \in \Delta$, we have that $\rho(X) = \rho'(X) = \alpha$ where $\alpha \in \mathrm{dom}(W.\kappa)$.

We claim that $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\,\rho$. For all $x \in \mathrm{dom}(\Gamma)$, we have that $\gamma(x) = \gamma'(x) = (v_1, v_2)$ where $(W, v_1, v_2) \in \mathcal{V}[\![\Gamma(x)]\!]\,\rho'$. By Lemma 5.11 (logical relation weakening), we have that $(W, v_1, v_2) \in \mathcal{V}[\![\Gamma(x)]\!]\,\rho$. This gives us $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\,\rho$.

Hence, we have that $(W, \rho(\gamma_1(e_1)), \rho(\gamma_2(e_2))) \in \mathcal{E}[\![A]\!]\,\rho$ as we were required to show.

□

Lemma 8.2 (Congruence)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A$ *and* $\vdash C : (\Sigma; \Delta; \Gamma \vdash A) \rightsquigarrow (\Sigma'; \Delta'; \Gamma' \vdash B)$ *then* $\Sigma'; \Delta'; \Gamma' \vdash C[e_1] \preceq C[e_2] : B$.

Proof

By induction on the type derivation for C, using Lemma 8.1 (weakening) for the cases where C is empty, and the compatibility lemmas for all other cases.

□

Lemma 8.3 (Adequacy)
If $\Sigma;\, \cdot\,;\, \cdot \vdash e_1 \preceq e_2 : A$ *and* $\Sigma \triangleright e_1 \Downarrow$ *then* $\Sigma \triangleright e_2 \Downarrow$.

Proof

Assume that $\Sigma \triangleright e_1 \Downarrow^j$.

Let $W = (j, \Sigma, \Sigma, \{\alpha \mapsto \lfloor \mathcal{V}[\![A]\!]\,\emptyset \rfloor_j \mid \alpha{:=}A \in \Sigma\})$.

Instantiate the premise with W, $\emptyset$, $\emptyset$, noting that $W \in \mathcal{S}[\![\Sigma]\!]$, $(W, \emptyset) \in \mathcal{D}[\![\cdot]\!]$, and $(W, \emptyset) \in \mathcal{G}[\![\cdot]\!]\,\emptyset$. Hence, we have that $(W, e_1, e_2) \in \mathcal{E}[\![A]\!]\,\emptyset$. Thus, we also have that $\Sigma \triangleright e_2 \Downarrow$ as we were required to show.

□

Theorem 8.4 (Soundness: Logical Approx. implies Contextual Approx.)
If $\Sigma; \Delta; \Gamma \vdash e_1 \preceq e_2 : A$ *then* $\Sigma; \Delta; \Gamma \vdash e_1 \preceq^{ctx} e_2 : A$.

Proof

Clearly, $\Sigma; \Delta; \Gamma \vdash e_1 : A$ and $\Sigma; \Delta; \Gamma \vdash e_2 : A$.

Consider arbitrary C, Σ', B such that

- $\vdash C : (\Sigma; \Delta; \Gamma \vdash A) \rightsquigarrow (\Sigma'; \cdot; \cdot \vdash B)$

We are required to show that $\Sigma' \triangleright C[e_1]\Downarrow \implies \Sigma' \triangleright C[e_2]\Downarrow$.

By Lemma 8.2 (congruence), we have that $\Sigma'; \cdot; \cdot \vdash C[e_1] \preceq C[e_2] : B$.

Additionally, by Lemma 8.3 (adequacy), we have that $\Sigma' \triangleright C[e_1]\Downarrow \implies \Sigma' \triangleright C[e_2]\Downarrow$ as we were required to show. ∎

9 Examples

Lemma 9.1 (Identity Conversions are Contextually Equivalent)
If $\Sigma; \Delta; \Gamma \vdash (e : A \overset{\phi}{\Longrightarrow} A) : A$ *then* $\Sigma; \Delta; \Gamma \vdash (e : A \overset{\phi}{\Longrightarrow} A) \approx^{ctx} e : A$

Theorem 9.2 (Free Theorem: K-Combinator)
Suppose $\Sigma; \cdot\,; \cdot \vdash v : \forall X.\, \forall Y.\, X \to Y \to X$, $\Sigma; \cdot\,; \cdot \vdash v_1 : A$, *and* $\Sigma; \cdot\,; \cdot \vdash v_2 : B$. *Then either*

1. $\Sigma \triangleright v\,[A]\,[B]\; v_1\; v_2 \longmapsto^* \Sigma' \triangleright v_e \;\wedge\; \Sigma'; \cdot\,; \cdot \vdash v_e \approx^{ctx} v_1 : A$ *for some* Σ', v_e, *or*
2. $\Sigma \triangleright v\,[A]\,[B]\; v_1\; v_2 \Uparrow$, *or*
3. $\Sigma \triangleright v\,[A]\,[B]\; v_1\; v_2 \longmapsto^* \Sigma' \triangleright \mathrm{blame}\, p$ *for some* Σ', p

Proof

Let $e = v\,[A]\,[B]\; v_1\; v_2$.

If $\Sigma \triangleright e \Uparrow$ or $\Sigma \triangleright e \longmapsto^* \Sigma' \triangleright \mathrm{blame}\, p$ then we have what we are required to show.

Otherwise, it must be that $\Sigma \triangleright e \longmapsto^* \Sigma' \triangleright v_{res}$ and we are required to show that $\Sigma'; \cdot; \cdot \vdash v_{res} \approx^{ctx} v_1 : A$.

We use Lemma 4.10 (redex termination) to guarantee that all subexpressions terminate in values. This reasoning is omitted in the rest of the proof for brevity.

Let $\kappa = \{\alpha \mapsto \lfloor \mathcal{V}\,[\![\Sigma(\alpha)]\!]\,\emptyset \rfloor_{n+1} \mid \alpha \in \Sigma\}$ and let $W_0 = (n+1, \Sigma, \Sigma, \kappa)$.

By the Fundamental Property (Theorem 7.17), $\cdot; \cdot\,; \cdot \vdash v \preceq v : \forall X.\, \forall Y.\, X \to Y \to X$. We instantiate this with $W_0, \emptyset, \emptyset$ so we have $(W_0, v, v) \in \mathcal{E}\,[\![\forall X.\, \forall Y.\, X \to Y \to X]\!]\,\emptyset$ and therefore

$$(W_0, v, v) \in \mathcal{V}\,[\![\forall X.\, \forall Y.\, X \to Y \to X]\!]\,\emptyset$$

Choose some α. Define

$$\begin{aligned} R_X &= \{(W, v_r, v_r) \in \mathrm{Atom}^{\mathrm{val}}_{W_0.j}\,[A, A] \mid v_r = v_1 \;\vee\; v_r = (v_1 : A \overset{\phi'}{\Longrightarrow} A)\} \\ W_1 &= W_0 \boxplus (\alpha, A, A, R_X) \end{aligned}$$

By Lemma 5.18 (instantiation steps), we have

$$W_0.\Sigma_i \triangleright v\,[A] \longmapsto W_1.\Sigma_i \triangleright (e_1 : \forall Y.\, \alpha \to Y \to \alpha \overset{+\alpha}{\Longrightarrow} \forall Y.\, A \to Y \to A)$$

for some e_1.

Instantiate $(W_0, v, v) \in \mathcal{V}\,[\![\forall X.\, \forall Y.\, X \to Y \to X]\!]\,\emptyset$ with $W_0, A, A, R_X, e_1, e_1, \alpha$, noting that

- $W_0 \sqsupseteq W_0$ by reflexivity
- $\cdot; \cdot \vdash A$
- $R_X \in \mathrm{Rel}_{W_0.j}\,[A, A]$
- $W_0.\Sigma_i \triangleright v\,[A] \longmapsto W_1.\Sigma_i \triangleright (e_1 : \forall Y.\, \alpha \to Y \to \alpha \overset{+\alpha}{\Longrightarrow} \forall Y.\, A \to Y \to A)$

We have that $(W_1, e_1, e_1) \in \blacktriangleright\mathcal{E}\,[\![\forall Y.\, X \to Y \to X]\!]\,\emptyset[X \mapsto \alpha]$ or equivalently, by Lemma 5.17 (compositionality) and since $W_1.j > 0$, that $(\blacktriangleright W_1, e_1, e_1) \in \mathcal{E}\,[\![\forall Y.\, \alpha \to Y \to \alpha]\!]\,\emptyset$.

We have that $\blacktriangleright W_1.\Sigma_i \triangleright e_1 \longmapsto^m \Sigma_1 \triangleright v'$ where $m < n$, $\Sigma_1 \supseteq \blacktriangleright W_1.\Sigma_i$, and $W_2 = (\blacktriangleright W_1.j - m, \Sigma_1, \Sigma_1, \lfloor \blacktriangleright W_1.\kappa \rfloor_{\blacktriangleright W_1.j-m})$. Therefore, we have $(W_2, v', v') \in \mathcal{V}\,[\![\forall Y.\, \alpha \to Y \to \alpha]\!]\,\emptyset$.

Choose α' such that $\alpha' \notin \Sigma_1$.

By the operational semantics, we have that

$$
\begin{aligned}
&\cdot \triangleright e \longmapsto^* W_1.\Sigma_i \triangleright e_1' \; [B] \; v_1 \; v_2 \\
&\longmapsto^* \Sigma_1, \alpha' {:=} B \triangleright ((v' \; [\alpha'] : \alpha \to \alpha' \to \alpha \overset{+\alpha}{\Longrightarrow} A \to \alpha' \to A) : A \to \alpha' \to A \overset{+\alpha'}{\Longrightarrow} A \to B \to A) \; v_1 \; v_2
\end{aligned}
$$

Choose some α'' such that $\alpha'' \notin (\Sigma_1, \alpha' {:=} B)$. Define

$$
\begin{aligned}
W_3 &= W_2 \boxplus (\alpha', B, B, \lfloor \mathcal{V} [\![B]\!] \rho \rfloor_{W_2.j}) \\
W_4 &= W_3 \boxplus (\alpha'', \alpha', \alpha', \lfloor \mathcal{V} [\![\alpha']\!] \rho \rfloor_{W_2.j})
\end{aligned}
$$

By Lemma 5.18 (instantiation steps), we have

$$W_2.\Sigma_i, \alpha' {:=} B \triangleright v' \; [\alpha'] \longmapsto W_4.\Sigma_i \triangleright (e_2 : \alpha \to \alpha' \to \alpha \overset{+\alpha'}{\Longrightarrow} \alpha \to B \to \alpha)$$

for some e_2.

Instantiate $(W_2, v', v') \in \mathcal{V} [\![\forall Y. A \to Y \to A]\!] \emptyset$ with $W_3, \alpha', \alpha', \lfloor \mathcal{V} [\![\alpha']\!] \rho \rfloor_{W_2.j}, e_2, e_2, \alpha''$, noting that

- $W_3 \sqsupseteq W_2$ by the definition of world extension
- $W_3.\Sigma_i; \cdot \vdash \alpha'$
- $\lfloor \mathcal{V} [\![\alpha']\!] \rho \rfloor_{W_2.j} \in \mathrm{Rel}_{W_3.j} [\alpha', \alpha']$
- $W_3.\Sigma_i \triangleright v' \; [\alpha'] \longmapsto W_4.\Sigma_i \triangleright (e_2 : \alpha \to \alpha'' \to \alpha \overset{+\alpha''}{\Longrightarrow} \alpha \to \alpha' \to \alpha)$

We have that $(W_4, e_2, e_2) \in \blacktriangleright \mathcal{E} [\![\alpha \to Y \to \alpha]\!] \emptyset [Y \mapsto \alpha'']$ or, by Lemma 5.17 (compositionality) and since $W_4.j > 0$, equivalently that $(\blacktriangleright W_4, e_2, e_2) \in \mathcal{E} [\![\alpha \to \alpha'' \to \alpha]\!] \emptyset$.

We have that $\blacktriangleright W_4.\Sigma_i \triangleright e_2 \longmapsto^l \Sigma_2 \triangleright v''$ where $l < \blacktriangleright W_4.j$, $\Sigma_2 \supseteq \blacktriangleright W_4.\Sigma_i$, and $W_5 = (\blacktriangleright W_4.j - l, \Sigma_2, \Sigma_2, \lfloor \blacktriangleright W_4.\kappa \rfloor_{\blacktriangleright W_4.j-l})$. Therefore, we have $(W_5, v'', v'') \in \mathcal{V} [\![\alpha \to \alpha'' \to \alpha]\!] \emptyset$.

By the operational semantics we have that

$$
\begin{aligned}
&\cdot \triangleright e \longmapsto^* W_2.\Sigma_i \triangleright ((v' \; [\alpha'] : \alpha \to \alpha' \to \alpha \overset{+\alpha}{\Longrightarrow} A \to \alpha' \to A) : A \to \alpha' \to A \overset{+\alpha'}{\Longrightarrow} A \to B \to A) \; v_1 \; v_2 \\
&\longmapsto^* \Sigma_2 \triangleright (((v'' : \alpha \to \alpha'' \to \alpha \overset{+\alpha''}{\Longrightarrow} \alpha \to \alpha' \to \alpha) : \alpha \to \alpha' \to \alpha \overset{+\alpha}{\Longrightarrow} A \to \alpha' \to A) : A \to \alpha' \to A \overset{+\alpha'}{\Longrightarrow} A \to B \to A) \; v_1 \; v_2 \\
&\longmapsto^* \Sigma_2 \triangleright (((v'' \; v_1' : \alpha'' \to \alpha \overset{+\alpha''}{\Longrightarrow} \alpha' \to \alpha) : \alpha' \to \alpha \overset{+\alpha}{\Longrightarrow} \alpha' \to A) : \alpha' \to A \overset{+\alpha'}{\Longrightarrow} B \to A) \; v_2
\end{aligned}
$$

where either $v_1' = ((v_1 : A \overset{-\alpha'}{\Longrightarrow} A) : A \overset{-\alpha}{\Longrightarrow} \alpha)$ or $v_1' = (v_1 : A \overset{-\alpha}{\Longrightarrow} \alpha)$. Then by preservation, $\Sigma_2; \cdot; \cdot \vdash v_1' : \alpha$.

We have that $(W_5, v_1', v_1') \in \mathcal{V} [\![\alpha]\!] \emptyset$ by definition since $(W_5, (v_1 : A \overset{-\alpha'}{\Longrightarrow} A), (v_1 : A \overset{-\alpha'}{\Longrightarrow} A)) \in R_X$ and $(W_5, v_1, v_1) \in R_X$.

Instantiate the definition of $(W_5, v'', v'') \in \mathcal{V} [\![\alpha \to \alpha'' \to \alpha]\!] \emptyset$ with W_5, v_1', v_1', noting that

- $W_5 \sqsupseteq W_5$ by reflexivity
- $(W_5, v_1', v_1') \in \mathcal{V} [\![\alpha]\!] \emptyset$

We have that $(W_5, v'' \; v_1', v'' \; v_1') \in \mathcal{E} [\![\alpha'' \to \alpha]\!] \emptyset$ and that $\Sigma_2 \triangleright v'' \; v_1' \longmapsto^k \Sigma_3 \triangleright v_3$ for some $k < W_5.j$, v_3, and $\Sigma_3 \supseteq \Sigma_2$.

Let $W_6 = (W_5.j - k, \Sigma_3, \Sigma_3, \lfloor W_5.\kappa \rfloor_{W_5.j-k})$.

We then have that $(W_6, v_3, v_3) \in \mathcal{V}' [\![\alpha'' \to \alpha]\!] \emptyset$.

By the operational semantics, we have that

$$\begin{aligned}
\cdot \triangleright e &\longmapsto^* W_4.\Sigma_i \triangleright (((v''\ v_1' : \alpha'' \to \alpha \overset{+\alpha''}{\Longrightarrow} \alpha' \to \alpha) : \alpha' \to \alpha \overset{+\alpha}{\Longrightarrow} \alpha' \to A) : \alpha' \to A \overset{+\alpha'}{\Longrightarrow} B \to A)\ v_2 \\
&\longmapsto^* W_5.\Sigma_i \triangleright (((v_3 : \alpha'' \to \alpha \overset{+\alpha''}{\Longrightarrow} \alpha' \to \alpha) : \alpha' \to \alpha \overset{+\alpha}{\Longrightarrow} \alpha' \to A) : \alpha' \to A \overset{+\alpha'}{\Longrightarrow} B \to A)\ v_2 \\
&\longmapsto^* W_5.\Sigma_i \triangleright (((v_3\ v_2' : \alpha \overset{+\alpha''}{\Longrightarrow} \alpha) : \alpha \overset{+\alpha}{\Longrightarrow} A) : A \overset{+\alpha'}{\Longrightarrow} A)
\end{aligned}$$

for some v_2' such that, by preservation, $\Sigma_3;\ \cdot\,;\ \cdot \vdash v_2' : \alpha''$.

By the Fundamental Property (Theorem 7.17), $\Sigma, \alpha' {:=} B, \alpha'' {:=} \alpha';\ \cdot\,;\ \cdot \vdash v_2' \preceq v_2' : \alpha''$. Instantiate this with $W_6, \emptyset, \emptyset$. We have that $(W_6, v_2', v_2') \in \mathcal{E}\,[\![\alpha'']\!]\,\emptyset$ and therefore

$$(W_6, v_2', v_2') \in \mathcal{V}\,[\![\alpha'']\!]\,\emptyset$$

Instantiate the definition of $(W_6, v_3, v_3) \in \mathcal{V}\,[\![\alpha'' \to \alpha]\!]\,\emptyset$ with W_6, v_2', v_2', noting that

- $W_6 \sqsupseteq W_6$ by reflexivity
- $(W_6, v_2', v_2') \in \mathcal{V}\,[\![\alpha'']\!]\,\emptyset$

We have that $(W_6, v_3\ v_2', v_3\ v_2') \in \mathcal{E}\,[\![\alpha]\!]\,\emptyset$ and that $\Sigma_3 \triangleright v_3\ v_2' \longmapsto^{n'} \Sigma_4 \triangleright v_4$ for some $n' < W_6.j$, v_4, and $\Sigma_4 \supseteq \Sigma_3$.

Let $W_7 = (W_6.j - n', \Sigma_4, \Sigma_4, \lfloor W_6.\kappa \rfloor_{W_6.j-n'})$.

We then have that $(W_7, v_4, v_4) \in \mathcal{V}\,[\![\alpha]\!]\,\emptyset$ and by the definition of $\mathcal{V}\,[\![\alpha]\!]\,\emptyset$, we have that $v_4 = (v_4' : A \overset{-\alpha}{\Longrightarrow} \alpha)$ and $(W_7, v_4', v_4') \in \blacktriangleright W_7.\kappa(\alpha)$.

By the operational semantics, we have that

$$\begin{aligned}
\cdot \triangleright e &\longmapsto^* \Sigma_2 \triangleright (((v_3\ v_2' : \alpha \overset{+\alpha''}{\Longrightarrow} \alpha) : \alpha \overset{+\alpha}{\Longrightarrow} A) : A \overset{+\alpha'}{\Longrightarrow} A) \\
&\longmapsto^* \Sigma_4 \triangleright (((v_4 : \alpha \overset{+\alpha''}{\Longrightarrow} \alpha) : \alpha \overset{+\alpha}{\Longrightarrow} A) : A \overset{+\alpha'}{\Longrightarrow} A) \\
&\longmapsto \Sigma_4 \triangleright (((v_4' : A \overset{-\alpha}{\Longrightarrow} \alpha) : \alpha \overset{+\alpha}{\Longrightarrow} A) : A \overset{+\alpha'}{\Longrightarrow} A) \\
&\longmapsto \Sigma_4 \triangleright (v_4' : A \overset{+\alpha'}{\Longrightarrow} A)
\end{aligned}$$

Since $W_7.j > 0$, we have that $(\blacktriangleright W_7, v_4', v_4') \in \lfloor R_X \rfloor_{W_7.j}$. Thus, we have that either $v_4' = v_1$ or $v_4' = (v_1 : A \overset{-\alpha'}{\Longrightarrow} A)$.

Consider the case where $v_4' = v_1$. By Lemma 9.1 (identity conversion equivalence), we have $\Sigma_4;\ \cdot\,;\ \cdot \vdash (v_4' : A \overset{+\alpha'}{\Longrightarrow} A) \approx^{ctx} v_1 : A$ as we were required to show.

Otherwise, $v_4' = (v_1 : A \overset{-\alpha'}{\Longrightarrow} A)$. By two applications of Lemma 9.1 (identity conversion equivalence), we have $\Sigma_4;\ \cdot\,;\ \cdot \vdash (v_4' : A \overset{+\alpha'}{\Longrightarrow} A) \approx^{ctx} v_1 : A$ as we were required to show.

$\square$

Theorem 9.3 (Free Theorem: Swap)
Let

$$\begin{array}{lcl}
\Sigma;\ \cdot\,;\ \cdot \vdash r & : & \forall X.\, X \times X \to X \times X \\
\Sigma;\ \cdot\,;\ \cdot \vdash f & : & A \to B \\
\Sigma;\ \cdot\,;\ \cdot \vdash v & : & A \times A \\
f^{\times} & = & \lambda(x : A).\, \lambda(y : A).\, \langle f\ x, f\ y \rangle
\end{array}$$

If $\Sigma \triangleright f\ (\pi_1\ v){\Downarrow}$ *and* $\Sigma \triangleright f\ (\pi_2\ v){\Downarrow}$ *then* $\Sigma;\ \cdot\,;\ \cdot \vdash f^{\times}\ (r\ [A]\ v) \approx^{ctx} r\ [B]\ (f^{\times}\ v) : B \times B$.

Proof

By Lemma 4.1 (canonical forms), we have that $v = \langle v_1, v_2\rangle$.

From the assumptions, we have that $\Sigma \triangleright f\ (\pi_1\ v) \longmapsto^* \Sigma, \Sigma_1 \triangleright v_1'$ and $\Sigma \triangleright f\ (\pi_2\ v) \longmapsto^* \Sigma, \Sigma_2 \triangleright v_2'$ for some $\Sigma_1, \Sigma_2, v_1', v_2'$.

It suffices to show that $\Sigma; \cdot; \cdot \vdash f^\times\ (r\ [A]\ v) \approx r\ [B]\ (f^\times\ v) : B \times B$ by Theorem 8.4 (soundness). We prove each conjunct separately.

Left: We are required to show that $\Sigma; \cdot; \cdot \vdash f^\times\ (r\ [A]\ v) \preceq r\ [B]\ (f^\times\ v) : B \times B$.

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}[\![\Sigma]\!]$
- $(W, \rho) \in \mathcal{D}[\![\cdot]\!]$
- $(W, \gamma) \in \mathcal{G}[\![\cdot]\!]\,\rho$

We have that $\rho = \emptyset$ and $\gamma = \emptyset$ by definition.

It suffices to show that $(W, f^\times\ (r\ [A]\ v), r\ [B]\ (f^\times\ v)) \in \mathcal{E}[\![B \times B]\!]\,\rho$.

By the Fundamental Property (Theorem 7.17), we have $\Sigma; \cdot; \cdot \vdash r \preceq r : \forall X.\, X \times X \to X \times X$. We instantiate this with $W, \emptyset, \emptyset$, noting that

- $W \in \mathcal{S}[\![\Sigma]\!]$ by definition of $\mathcal{S}[\![\Sigma]\!]$
- $(W, \emptyset) \in \mathcal{D}[\![\cdot]\!]$ by definition $\mathcal{D}[\![\cdot]\!]$
- $(W, \emptyset) \in \mathcal{G}[\![\cdot]\!]\,\emptyset$ by definition of $\mathcal{G}[\![\cdot]\!]\,\emptyset$

We then have $(W, r, r) \in \mathcal{E}[\![\forall X.\, X \times X \to X \times X]\!]\,\emptyset$.

Assume that $W.\Sigma_i \triangleright r \longmapsto^n \Sigma_i' \triangleright r_v$ where $n < W.j$. Otherwise, we have what we are required to show.

By the definition of $\mathcal{E}[\![\forall X.\, X \times X \to X \times X]\!]\,\emptyset$, we have that there exists W' such that

- $W' \sqsupseteq_n W$
- $W'.\Sigma_1 = \Sigma_1'$
- $W'.\Sigma_2 = \Sigma_2'$
- $(W', r_v, r_v) \in \mathcal{V}[\![\forall X.\, X \times X \to X \times X]\!]\,\emptyset$

Choose α such that $\alpha \notin W'.\Sigma_1$ and $\alpha \notin W'.\Sigma_2$.

Let $R = \{(W, v_3, v_4) \in \mathrm{Atom}^{\mathrm{val}}_{W'.j}[A, B] \mid \Sigma \triangleright f\ v_3 \longmapsto^* \Sigma' \triangleright v_3' \ \wedge\ (W, v_3', v_4) \in \mathcal{V}[\![B]\!]\,\emptyset\}$ and let $W_2 = W' \boxplus (\alpha, A, B, R)$.

By Lemma 5.18 (instantiation steps) there exists e_1 such that

$$\begin{array}{lcl} W'.\Sigma_1 \triangleright r_v\ [A] & \longrightarrow & W'.\Sigma_1, \alpha{:=}A \triangleright (e_1 : \alpha \times \alpha \to \alpha \times \alpha \overset{+\alpha}{\Longrightarrow} A \times A \to A \times A) \\ W'.\Sigma_2 \triangleright r_v\ [B] & \longrightarrow & W'.\Sigma_2, \alpha{:=}B \triangleright (e_1 : \alpha \times \alpha \to \alpha \times \alpha \overset{+\alpha}{\Longrightarrow} B \times B \to B \times B) \end{array}$$

Instantiate $(W', r_v, r_v) \in \mathcal{V}[\![\forall X.\, X \times X \to X \times X]\!]\,\emptyset$ with $W', A, B, R, e_1, e_1, \alpha$, noting that

- $W' \sqsupseteq W'$ by reflexivity
- $W'.\Sigma_1; \cdot \vdash A$ by weakening
- $W'.\Sigma_2; \cdot \vdash B$ by weakening
- $R \in \mathrm{Rel}_{W'.j}[A, B]$
- $W'.\Sigma_1 \triangleright r_v\ [A] \longrightarrow W'.\Sigma_1, \alpha{:=}A \triangleright (e_1 : \alpha \times \alpha \to \alpha \times \alpha \overset{+\alpha}{\Longrightarrow} A \times A \to A \times A)$
- $W'.\Sigma_2 \triangleright r_v\ [B] \longrightarrow W'.\Sigma_2, \alpha{:=}B \triangleright (e_1 : \alpha \times \alpha \to \alpha \times \alpha \overset{+\alpha}{\Longrightarrow} B \times B \to B \times B)$

We have that $(W_2, e_1, e_1) \in \mathcal{E}\,[\![X \times X \rightarrow X \times X]\!]\,\emptyset[X \mapsto \alpha]$.
Then, by Lemma 5.17 (compositionality) we have that

$$(W_2, e_1, e_1) \in \mathcal{E}\,[\![\alpha \times \alpha \rightarrow \alpha \times \alpha]\!]\,\emptyset$$

Assume that $W_2.\Sigma_1 \triangleright e_1 \longmapsto^m \Sigma_3 \triangleright v_3$ where $m < W_2.j$. Otherwise, we have what we are required to show.
Instantiate $(W_2, e_1, e_1) \in \mathcal{E}\,[\![\alpha \times \alpha \rightarrow \alpha \times \alpha]\!]\,\emptyset$ with m, Σ_3, v_3. There exist W_3, Σ_4, v_4 such that

- $W_2.\Sigma_2 \triangleright e_1 \longmapsto^* \Sigma_4 \triangleright v_4$
- $W_3 \sqsupseteq_m W_2$
- $W_3.\Sigma_1 = \Sigma_3$
- $W_3.\Sigma_2 = \Sigma_4$
- $(W_3, v_3, v_4) \in \mathcal{V}\,[\![\alpha \times \alpha \rightarrow \alpha \times \alpha]\!]\,\emptyset$

By the operational semantics, we have that $W.\Sigma_2 \triangleright f^{\times}\ v \longmapsto^* \Sigma, \Sigma_1, \Sigma_2 \triangleright \langle v_1', v_2' \rangle$.
Let $W'' = (W.j, (W.\Sigma_1, \Sigma_1), (W.\Sigma_2, \Sigma_1), W.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\,[\![\Sigma_1(\alpha)]\!]\,\emptyset \rfloor_{W.j} \mid \alpha \in \mathrm{dom}(\Sigma_1)\})$. Note that $W'' \sqsupseteq W$.
By the Fundamental Property (Theorem 7.17), we have that $\Sigma, \Sigma_1; \cdot; \cdot \vdash v_1' \preceq v_1' : B$. We instantiate this with $W'', \emptyset, \emptyset$, noting that

- $W'' \in \mathcal{S}\,[\![\Sigma, \Sigma_1]\!]$
- $(W'', \emptyset) \in \mathcal{D}\,[\![\cdot]\!]$
- $(W'', \emptyset) \in \mathcal{G}\,[\![\cdot]\!]\,\emptyset$

Therefore, since values related in $\mathcal{E}\,[\![B]\!]\,\emptyset$ are related in $\mathcal{V}\,[\![B]\!]\,\emptyset$, we have that $(W'', v_1', v_1') \in \mathcal{V}\,[\![B]\!]\,\emptyset$.
Let $W_3' = (W_3.j, (W_3.\Sigma_1, \Sigma_1), (W_3.\Sigma_2, \Sigma_1), W_3.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\,[\![\Sigma_1(\alpha)]\!]\,\emptyset \rfloor_{W_3.j} \mid \alpha \in \mathrm{dom}(\Sigma_1)\})$.

Note that $(W_3', (v_1 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v_1' : B \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\,[\![\alpha]\!]\,\emptyset$ since $W_3'.\kappa(\alpha) = \lfloor R \rfloor_{W_3.j}$, $\Sigma \triangleright f\ v_1 \longmapsto^* \Sigma, \Sigma_1 \triangleright v_1'$, and $(W_3', v_1', v_1') \in \mathcal{V}\,[\![B]\!]\,\emptyset$ by Lemma 5.6 (monotonicity) since $W_3' \sqsupseteq W''$.
Likewise, let $W''' = (W.j, (W.\Sigma_1, \Sigma_2), (W.\Sigma_2, \Sigma_2), W.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\,[\![\Sigma_2(\alpha)]\!]\,\emptyset \rfloor_{W.j} \mid \alpha \in \mathrm{dom}(\Sigma_2)\})$. Note that $W''' \sqsupseteq W$.
By the Fundamental Property (Theorem 7.17), we have that $\Sigma, \Sigma_2; \cdot; \cdot \vdash v_2' \preceq v_2' : B$. We instantiate this with $W''', \emptyset, \emptyset$, noting that

- $W''' \in \mathcal{S}\,[\![\Sigma, \Sigma_2]\!]$
- $(W''', \emptyset) \in \mathcal{D}\,[\![\cdot]\!]$
- $(W''', \emptyset) \in \mathcal{G}\,[\![\cdot]\!]\,\emptyset$

Therefore, since values related in $\mathcal{E}\,[\![B]\!]\,\emptyset$ are related in $\mathcal{V}\,[\![B]\!]\,\emptyset$, we have that $(W''', v_1', v_1') \in \mathcal{V}\,[\![B]\!]\,\emptyset$.
Let $W_3'' = (W_3.j, (W_3.\Sigma_1, \Sigma_2), (W_3.\Sigma_2, \Sigma_2), W_3.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\,[\![\Sigma_2(\alpha)]\!]\,\emptyset \rfloor_{W_3.j} \mid \alpha \in \mathrm{dom}(\Sigma_2)\})$.

Note that $(W_3'', (v_2 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v_2' : B \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\,[\![\alpha]\!]\,\emptyset$ since $W_3''.\kappa(\alpha) = \lfloor R \rfloor_{W_3.j}$, $\Sigma \triangleright f\ v_2 \longmapsto^* \Sigma, \Sigma_1 \triangleright v_2'$, and $(W_3'', v_2', v_2') \in \mathcal{V}\,[\![B]\!]\,\emptyset$ by Lemma 5.6 (monotonicity) since $W_3'' \sqsupseteq W'''$.
Let $W_3''' = (W_3.j, (W_3.\Sigma_1, \Sigma_1, \Sigma_2), (W_3.\Sigma_2, \Sigma_1, \Sigma_2), W_3.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\,[\![\Sigma_1(\alpha)]\!]\,\emptyset \rfloor_{W_3.j} \mid \alpha \in \mathrm{dom}(\Sigma_1)\} \cup \{\alpha \mapsto \lfloor \mathcal{V}\,[\![\Sigma_2(\alpha)]\!]\,\emptyset \rfloor_{W_3.j} \mid \alpha \in \mathrm{dom}(\Sigma_2)\})$.
Note that $W_3''' \sqsupseteq W_3'$ and $W_3''' \sqsupseteq W_3''$ by definition.

We then have $(W_3''', (v_1 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v_1' : B \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\,[\![\alpha]\!]\,\emptyset$ and $(W_3''', (v_2 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v_2' : B \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\,[\![\alpha]\!]\,\emptyset$ by Lemma 5.6 (monotonicity).

Let $v_\alpha = \langle (v_1 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v_2 : A \overset{-\alpha}{\Longrightarrow} \alpha) \rangle$ and let $v_\alpha' = \langle (v_1' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_2' : B \overset{-\alpha}{\Longrightarrow} \alpha) \rangle$.
Therefore, we have $(W_3''', v_\alpha, v_\alpha') \in \mathcal{V}\,[\![\alpha \times \alpha]\!]\,\emptyset$ by definition.
We instantiate $(W_3, v_3, v_4) \in \mathcal{V}\,[\![\alpha \times \alpha \rightarrow \alpha \times \alpha]\!]\,\emptyset$ with $W_3''', v_\alpha, v_\alpha'$, noting that

- $W_3''' \sqsupseteq W_3$ by definition

- $(W_3''', v_\alpha, v'_\alpha) \in \mathcal{V}\llbracket \alpha \times \alpha \rrbracket \emptyset$

We then have that $(W_3''', v_3\ v_\alpha, v_4\ v'_\alpha) \in \mathcal{E}\llbracket \alpha \times \alpha \rrbracket \emptyset$

Assume that $W_3'''.\Sigma_1 \triangleright v_3\ v_\alpha \longmapsto^k \Sigma_5 \triangleright v_5$ where $k < W_3.j$. Otherwise, we have what we are required to show. If $k \geq W_3.j$ then $k \geq W_3.j = W_2.j - m = W.j - n - m$ so $k + n + m \geq W.j$ and we vacuously have what we are required to show.

Instantiate $(W_3''', v_3\ v_\alpha, v_4\ v'_\alpha) \in \mathcal{E}\llbracket \alpha \times \alpha \rrbracket \emptyset$ with k, Σ_5, v_5. There exist W_4, Σ_6, v_6 such that

- $W_3'''.\Sigma_2 \triangleright v_4\ v'_\alpha \longmapsto^* \Sigma_6 \triangleright v_6$
- $W_4 \sqsupseteq_k W_3'''$
- $W_4.\Sigma_1 = \Sigma_5$
- $W_4.\Sigma_2 = \Sigma_6$
- $(W_4, v_5, v_6) \in \mathcal{V}\llbracket \alpha \times \alpha \rrbracket \emptyset$

By the definition of $\mathcal{V}\llbracket \alpha \times \alpha \rrbracket \emptyset$, we have that $v_5 = \langle (v'_5 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v''_5 : A \overset{-\alpha}{\Longrightarrow} \alpha) \rangle$ and $v_6 = \langle (v'_6 : B \overset{-\alpha}{\Longrightarrow} \alpha), (v''_6 : B \overset{-\alpha}{\Longrightarrow} \alpha) \rangle$ where $(W_4, (v'_5 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v'_6 : B \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\llbracket \alpha \rrbracket \emptyset$ and $(W_4, (v''_5 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v''_6 : B \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\llbracket \alpha \rrbracket \emptyset$. By the definition of $\mathcal{V}\llbracket \alpha \rrbracket \emptyset$, since $W_4.\kappa(\alpha) = R$, we further have that $\Sigma_5 \triangleright f\ v'_5 \Downarrow v'_7$ and $\Sigma_5 \triangleright f\ v''_5 \Downarrow v''_7$ where $(W_4, v'_7, v'_6) \in \mathcal{V}\llbracket B \rrbracket \emptyset$ and $(W_4, v''_7, v''_6) \in \mathcal{V}\llbracket B \rrbracket \emptyset$.

By the operational semantics, we then have that

$$\begin{array}{llllll}
\Sigma \triangleright f^\times\ (r\ [A]\ v) & \longmapsto^{k+n+m} & \Sigma_5 \triangleright f^\times\ (v_5 : \alpha \overset{+\alpha}{\Longrightarrow} A) & \longmapsto^{l} & \Sigma_5, \Sigma_1, \Sigma_2 \triangleright v_6 \\
\Sigma \triangleright r\ [B]\ (f^\times\ v) & & & \longmapsto^* & \Sigma_6 \triangleright v_6
\end{array}$$

Recall that it suffices to show that $(W, f^\times\ (r\ [A]\ v), r\ [B]\ (f^\times\ v)) \in \mathcal{E}\llbracket B \times B \rrbracket \rho$.

We apply Lemma 5.14 (anti-reduction), noting that

- $W_4 \sqsupseteq W$ by transitivity of extension
- $W.j \leq W_4.j + k + n + m$
- $\Sigma \triangleright f^\times\ (r\ [A]\ v) \longmapsto^{k+n+m} \Sigma_5 \triangleright f^\times\ (v_5 : \alpha \overset{+\alpha}{\Longrightarrow} A)$
- $\Sigma \triangleright r\ [B]\ (f^\times\ v) \longmapsto^* \Sigma_6 \triangleright v_6$

Then it suffices to show that $(W_4, f^\times\ (v_5 : \alpha \overset{+\alpha}{\Longrightarrow} A), \langle v'_6, v''_6 \rangle) \in \mathcal{E}\llbracket B \times B \rrbracket \rho$.

Assume that $W_4.\Sigma_1 \triangleright f^\times\ (v_5 : \alpha \overset{+\alpha}{\Longrightarrow} A) \longmapsto^{m'} \Sigma_5 \triangleright v_7$ where $m' < W_4.j$. Otherwise, we have what we are required to show.

Let $W_5 = (W_4.j - m', W_4.\Sigma_1, W_4.\Sigma_2, \lfloor W_4.\kappa \rfloor_{W_4.j - m'})$.

By Lemma 5.14 (anti-reduction), it suffices to show that $(W_5, v_7, \langle v'_6, v''_6 \rangle) \in \mathcal{E}\llbracket B \times B \rrbracket \rho$, which we have from the definition since $v_7 = \langle v'_7, v''_7 \rangle$ and $(W_5, v'_7, v'_6) \in \mathcal{V}\llbracket B \rrbracket \emptyset$ and $(W_5, v''_7, v''_6) \in \mathcal{V}\llbracket B \rrbracket \emptyset$ by Lemma 5.6 (monotonicity).

Right: We are required to show that $\Sigma; \cdot; \cdot \vdash r\ [B]\ (f^\times\ v) \preceq f^\times\ (r\ [A]\ v) : B \times B$.

Consider arbitrary W, ρ, γ such that

- $W \in \mathcal{S}\llbracket \Sigma \rrbracket$
- $(W, \rho) \in \mathcal{D}\llbracket \cdot \rrbracket$
- $(W, \gamma) \in \mathcal{G}\llbracket \cdot \rrbracket \rho$

We have that $\rho = \emptyset$ and $\gamma = \emptyset$ by definition.

It suffices to show that $(W, r\ [B]\ (f^\times\ v), f^\times\ (r\ [A]\ v)) \in \mathcal{E}\llbracket B \times B \rrbracket \rho$.

By the Fundamental Property (Theorem 7.17), we have $\Sigma; \cdot; \cdot \vdash r \preceq r : \forall X. X \times X \rightarrow X \times X$. We instantiate this with $W, \emptyset, \emptyset$, noting that

- $W \in \mathcal{S}\llbracket \Sigma \rrbracket$ by definition of $\mathcal{S}\llbracket \Sigma \rrbracket$
- $(W, \emptyset) \in \mathcal{D}\llbracket \cdot \rrbracket$ by definition $\mathcal{D}\llbracket \cdot \rrbracket$

- $(W, \emptyset) \in \mathcal{G}\llbracket \cdot \rrbracket \emptyset$ by definition of $\mathcal{G}\llbracket \cdot \rrbracket \emptyset$

We then have $(W, r, r) \in \mathcal{E}\llbracket \forall X.\, X \times X \to X \times X \rrbracket \emptyset$.

Assume that $W.\Sigma_i \triangleright r \longmapsto^n \Sigma'_i \triangleright r_v$ where $n < W.j$. Otherwise, we have what we are required to show.

By the definition of $\mathcal{E}\llbracket \forall X.\, X \times X \to X \times X \rrbracket \emptyset$, we have that there exists W' such that

- $W' \sqsupseteq_n W$
- $W'.\Sigma_1 = \Sigma'_1$
- $W'.\Sigma_2 = \Sigma'_2$
- $(W', r_v, r_v) \in \mathcal{V}\llbracket \forall X.\, X \times X \to X \times X \rrbracket \emptyset$

Choose α such that $\alpha \notin W'.\Sigma_1$ and $\alpha \notin W'.\Sigma_2$.

Let $R = \{(W, v_3, v_4) \in \mathrm{Atom}^{\mathrm{val}}_{W'.j}[B, A] \mid \Sigma \triangleright f\ v_4 \longmapsto^* \Sigma' \triangleright v'_4 \ \wedge\ (W, v_3, v'_4) \in \mathcal{V}\llbracket B \rrbracket \emptyset\}$ and let $W_2 = W' \boxplus (\alpha, B, A, R)$.

By Lemma 5.18 (instantiation steps) there exists e_1 such that

$$\begin{array}{lcl} W'.\Sigma_1 \triangleright r_v\ [B] & \longrightarrow & W'.\Sigma_1, \alpha{:=}B \triangleright (e_1 : \alpha \times \alpha \to \alpha \times \alpha \overset{+\alpha}{\Longrightarrow} B \times B \to B \times B) \\ W'.\Sigma_2 \triangleright r_v\ [A] & \longrightarrow & W'.\Sigma_2, \alpha{:=}A \triangleright (e_1 : \alpha \times \alpha \to \alpha \times \alpha \overset{+\alpha}{\Longrightarrow} A \times A \to A \times A) \end{array}$$

Instantiate $(W', r_v, r_v) \in \mathcal{V}\llbracket \forall X.\, X \times X \to X \times X \rrbracket \emptyset$ with $W', B, A, R, e_1, e_1, \alpha$, noting that

- $W' \sqsupseteq W'$ by reflexivity
- $W'.\Sigma_1;\ \cdot \vdash B$ by weakening
- $W'.\Sigma_2;\ \cdot \vdash A$ by weakening
- $R \in \mathrm{Rel}_{W'.j}[B, A]$
- $W'.\Sigma_1 \triangleright r_v\ [B] \longrightarrow W'.\Sigma_1, \alpha{:=}B \triangleright (e_1 : \alpha \times \alpha \to \alpha \times \alpha \overset{+\alpha}{\Longrightarrow} B \times B \to B \times B)$
- $W'.\Sigma_2 \triangleright r_v\ [A] \longrightarrow W'.\Sigma_2, \alpha{:=}A \triangleright (e_1 : \alpha \times \alpha \to \alpha \times \alpha \overset{+\alpha}{\Longrightarrow} A \times A \to A \times A)$

We have that $(W_2, e_1, e_1) \in \mathcal{E}\llbracket X \times X \to X \times X \rrbracket \emptyset[X \mapsto \alpha]$.

Then, by Lemma 5.17 (compositionality) we have that

$$(W_2, e_1, e_1) \in \mathcal{E}\llbracket \alpha \times \alpha \to \alpha \times \alpha \rrbracket \emptyset$$

Assume that $W_2.\Sigma_1 \triangleright e_1 \longmapsto^m \Sigma_3 \triangleright v_3$ where $m < W_2.j$. Otherwise, we have what we are required to show.

Instantiate $(W_2, e_1, e_1) \in \mathcal{E}\llbracket \alpha \times \alpha \to \alpha \times \alpha \rrbracket \emptyset$ with m, Σ_3, v_3. There exist W_3, Σ_4, v_4 such that

- $W_2.\Sigma_2 \triangleright e_1 \longmapsto^* \Sigma_4 \triangleright v_4$
- $W_3 \sqsupseteq_m W_2$
- $W_3.\Sigma_1 = \Sigma_3$
- $W_3.\Sigma_2 = \Sigma_4$
- $(W_3, v_3, v_4) \in \mathcal{V}\llbracket \alpha \times \alpha \to \alpha \times \alpha \rrbracket \emptyset$

By the operational semantics, we have that $W.\Sigma_1 \triangleright f^{\times}\ v \longmapsto^* \Sigma, \Sigma_1, \Sigma_2 \triangleright \langle v'_1, v'_2 \rangle$.

Let $W'' = (W.j, (W.\Sigma_1, \Sigma_1), (W.\Sigma_2, \Sigma_1), W.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\llbracket \Sigma_1(\alpha) \rrbracket \emptyset \rfloor_{W.j} \mid \alpha \in \mathrm{dom}(\Sigma_1))\})$. Note that $W'' \sqsupseteq W$.

By the Fundamental Property (Theorem 7.17), we have that $\Sigma, \Sigma_1; \cdot; \cdot \vdash v'_1 \preceq v'_1 : B$. We instantiate this with $W'', \emptyset, \emptyset$, noting that

- $W'' \in \mathcal{S}\llbracket \Sigma, \Sigma_1 \rrbracket$
- $(W'', \emptyset) \in \mathcal{D}\llbracket \cdot \rrbracket$
- $(W'', \emptyset) \in \mathcal{G}\llbracket \cdot \rrbracket \emptyset$

Therefore, since values related in $\mathcal{E}\llbracket B\rrbracket\,\emptyset$ are related in $\mathcal{V}\llbracket B\rrbracket\,\emptyset$, we have that $(W'', v_1', v_1') \in \mathcal{V}\llbracket B\rrbracket\,\emptyset$.
Let $W_3' = (W_3.j, (W_3.\Sigma_1, \Sigma_1), (W_3.\Sigma_2, \Sigma_1), W_3.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\llbracket \Sigma_1(\alpha)\rrbracket\,\emptyset\rfloor_{W_3.j} \mid \alpha \in \mathrm{dom}(\Sigma_1))\})$.

Note that $(W_3', (v_1' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_1 : A \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\llbracket \alpha\rrbracket\,\emptyset$ since $W_3'.\kappa(\alpha) = \lfloor R\rfloor_{W_3.j}$, $\Sigma \triangleright f\ v_1 \longmapsto^* \Sigma, \Sigma_1 \triangleright v_1'$, and $(W_3', v_1', v_1') \in \mathcal{V}\llbracket B\rrbracket\,\emptyset$ by Lemma 5.6 (monotonicity) since $W_3' \sqsupseteq W''$.
Likewise, let $W''' = (W.j, (W.\Sigma_1, \Sigma_2), (W.\Sigma_2, \Sigma_2), W.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\llbracket \Sigma_2(\alpha)\rrbracket\,\emptyset\rfloor_{W.j} \mid \alpha \in \mathrm{dom}(\Sigma_2))\})$.
Note that $W''' \sqsupseteq W$.
By the Fundamental Property (Theorem 7.17), we have that $\Sigma, \Sigma_2; \cdot; \cdot \vdash v_2' \preceq v_2' : B$. We instantiate this with $W''', \emptyset, \emptyset$, noting that

- $W''' \in \mathcal{S}\llbracket \Sigma, \Sigma_2\rrbracket$
- $(W''', \emptyset) \in \mathcal{D}\llbracket \cdot\rrbracket$
- $(W''', \emptyset) \in \mathcal{G}\llbracket \cdot\rrbracket\,\emptyset$

Therefore, since values related in $\mathcal{E}\llbracket B\rrbracket\,\emptyset$ are related in $\mathcal{V}\llbracket B\rrbracket\,\emptyset$, we have that $(W''', v_1', v_1') \in \mathcal{V}\llbracket B\rrbracket\,\emptyset$.
Let $W_3'' = (W_3.j, (W_3.\Sigma_1, \Sigma_2), (W_3.\Sigma_2, \Sigma_2), W_3.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\llbracket \Sigma_2(\alpha)\rrbracket\,\emptyset\rfloor_{W_3.j} \mid \alpha \in \mathrm{dom}(\Sigma_2))\})$.

Note that $(W_3'', (v_2' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_2 : A \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\llbracket \alpha\rrbracket\,\emptyset$ since $W_3''.\kappa(\alpha) = \lfloor R\rfloor_{W_3.j}$, $\Sigma \triangleright f\ v_2 \longmapsto^* \Sigma, \Sigma_1 \triangleright v_2'$, and $(W_3'', v_2', v_2') \in \mathcal{V}\llbracket B\rrbracket\,\emptyset$ by Lemma 5.6 (monotonicity) since $W_3'' \sqsupseteq W'''$.
Let $W_3''' = (W_3.j, (W_3.\Sigma_1, \Sigma_1, \Sigma_2), (W_3.\Sigma_2, \Sigma_1, \Sigma_2), W_3.\kappa \cup \{\alpha \mapsto \lfloor \mathcal{V}\llbracket \Sigma_1(\alpha)\rrbracket\,\emptyset\rfloor_{W_3.j} \mid \alpha \in \mathrm{dom}(\Sigma_1))\} \cup \{\alpha \mapsto \lfloor \mathcal{V}\llbracket \Sigma_2(\alpha)\rrbracket\,\emptyset\rfloor_{W_3.j} \mid \alpha \in \mathrm{dom}(\Sigma_2))\})$.
Note that $W_3''' \sqsupseteq W_3'$ and $W_3''' \sqsupseteq W_3''$ by definition.

We then have $(W_3''', (v_1' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_1 : A \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\llbracket \alpha\rrbracket\,\emptyset$ and $(W_3''', (v_2' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_2 : A \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\llbracket \alpha\rrbracket\,\emptyset$ by Lemma 5.6 (monotonicity).

Let $v_\alpha = \langle (v_1 : A \overset{-\alpha}{\Longrightarrow} \alpha), (v_2 : A \overset{-\alpha}{\Longrightarrow} \alpha)\rangle$ and let $v_\alpha' = \langle (v_1' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_2' : B \overset{-\alpha}{\Longrightarrow} \alpha)\rangle$.
Therefore, we have $(W_3''', v_\alpha', v_\alpha) \in \mathcal{V}\llbracket \alpha \times \alpha\rrbracket\,\emptyset$ by definition.
We instantiate $(W_3, v_3, v_4) \in \mathcal{V}\llbracket \alpha \times \alpha \to \alpha \times \alpha\rrbracket\,\emptyset$ with $W_3''', v_\alpha', v_\alpha$, noting that

- $W_3''' \sqsupseteq W_3$ by definition
- $(W_3''', v_\alpha', v_\alpha) \in \mathcal{V}\llbracket \alpha \times \alpha\rrbracket\,\emptyset$

We then have that $(W_3''', v_3\ v_\alpha', v_4\ v_\alpha) \in \mathcal{E}\llbracket \alpha \times \alpha\rrbracket\,\emptyset$
Assume that $W_3'''.\Sigma_1 \triangleright v_3\ v_\alpha' \longmapsto^k \Sigma_5 \triangleright v_5$ where $k < W_3.j$. Otherwise, we have what we are required to show. If $k \geq W_3.j$ then $k \geq W_3.j = W_2.j - m = W.j - n - m$ so $k + n + m \geq W.j$ and we vacuously have what we are required to show.
Instantiate $(W_3''', v_3\ v_\alpha', v_4\ v_\alpha) \in \mathcal{E}\llbracket \alpha \times \alpha\rrbracket\,\emptyset$ with k, Σ_5, v_5. There exist W_4, Σ_6, v_6 such that

- $W_3'''.\Sigma_2 \triangleright v_4\ v_\alpha \longmapsto^* \Sigma_6 \triangleright v_6$
- $W_4 \sqsupseteq_k W_3'''$
- $W_4.\Sigma_1 = \Sigma_5$
- $W_4.\Sigma_2 = \Sigma_6$
- $(W_4, v_5, v_6) \in \mathcal{V}\llbracket \alpha \times \alpha\rrbracket\,\emptyset$

By the definition of $\mathcal{V}\llbracket \alpha \times \alpha\rrbracket\,\emptyset$, we have that $v_5 = \langle (v_5' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_5'' : B \overset{-\alpha}{\Longrightarrow} \alpha)\rangle$ and $v_6 = \langle (v_6' : A \overset{-\alpha}{\Longrightarrow} \alpha), (v_6'' : A \overset{-\alpha}{\Longrightarrow} \alpha)\rangle$ where $(W_4, (v_5' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_6' : A \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\llbracket \alpha\rrbracket\,\emptyset$ and $(W_4, (v_5'' : B \overset{-\alpha}{\Longrightarrow} \alpha), (v_6'' : A \overset{-\alpha}{\Longrightarrow} \alpha)) \in \mathcal{V}\llbracket \alpha\rrbracket\,\emptyset$. By the definition of $\mathcal{V}\llbracket \alpha\rrbracket\,\emptyset$, since $W_4.\kappa(\alpha) = R$, we further have that $\Sigma_5 \triangleright f\ v_6' \Downarrow v_7'$ and $\Sigma_5 \triangleright f\ v_6'' \Downarrow v_7''$ where $(W_4, v_5', v_7') \in \mathcal{V}\llbracket B\rrbracket\,\emptyset$ and $(W_4, v_5'', v_7'') \in \mathcal{V}\llbracket B\rrbracket\,\emptyset$.
By the operational semantics, we then have that

$$
\begin{array}{llllll}
\Sigma \triangleright r\ [B]\ (f^\times\ v) & \longmapsto^l & \Sigma_5 \triangleright r\ [B]\ \langle v_1', v_2'\rangle & \longmapsto^{k+n+m} & \Sigma_5 \triangleright v_5 \\
\Sigma \triangleright f^\times\ (r\ [A]\ v) & \longmapsto^* & \Sigma_6 \triangleright f^\times\ (v_6 : \alpha \overset{+\alpha}{\Longrightarrow} A) & \longmapsto^* & \Sigma_6, \Sigma_1, \Sigma_2 \triangleright v_5
\end{array}
$$

Recall that it suffices to show that $(W, f^\times\ (r\ [A]\ v), r\ [B]\ (f^\times\ v)) \in \mathcal{E}\llbracket B \times B\rrbracket\,\rho$.
We apply Lemma 5.14 (anti-reduction), noting that

- $W_4 \sqsupseteq W$ by transitivity of extension
- $W.j \leq W_4.j + k + n + m$
- $\Sigma \triangleright r\ [B]\ (f^{\times}\ v) \longmapsto^{k+n+m+l} \Sigma_5 \triangleright v_5$
- $\Sigma \triangleright f^{\times}\ (r\ [A]\ v) \longmapsto^{*} \Sigma_6 \triangleright f^{\times}\ (v_6 : \alpha \overset{+\alpha}{\Longrightarrow} A)$

Then it suffices to show that $(W_4, \langle v_5', v_5'' \rangle, f^{\times}\ (v_6 : \alpha \overset{+\alpha}{\Longrightarrow} A)) \in \mathcal{E}\,[\![B \times B]\!]\,\rho$.
By Lemma 5.14 (anti-reduction), it suffices to show that $(W_4, \langle v_5', v_5'' \rangle, \langle v_7', v_7'' \rangle) \in \mathcal{E}\,[\![B \times B]\!]\,\rho$.
which we have from the definition since $(W_5, v_5', v_7') \in \mathcal{V}\,[\![B]\!]\,\emptyset$ and $(W_5, v_5'', v_7'') \in \mathcal{V}\,[\![B]\!]\,\emptyset$ by Lemma 5.6 (monotonicity).

□