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Temporal Bayesian Knowledge Bases – Reasoning about uncertainty with temporal constraints

Eugene Santos Jr. [a], Deqing Li [a,*], Eunice E. Santos [b], John Korah [b]

[a] Thayer School of Engineering, Dartmouth College, Hanover, NH, USA
[b] Institute of Defense & Security, University of Texas in El Paso, El Paso, TX, USA

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ABSTRACT

Time is ubiquitous. Accounting for time and its interaction with change is crucial to modeling the dynamic world, especially in domains whose study of data is sensitive to time such as in medical diagnosis, financial investment, and natural language processing, to name a few. We present a framework that incorporates both uncertainty and time in its reasoning scheme. It is based on an existing knowledge representation called Bayesian Knowledge Bases. It provides a graphical representation of knowledge, time and uncertainty, and enables probabilistic and temporal inferencing. The reasoning scheme is probabilistically sound and the fusion of temporal fragments is well defined. We will discuss some properties of this framework and introduce algorithms to ensure groundedness during the construction of the model. The framework has been applied to both artificial and real world scenarios.

1. Introduction

Time is ubiquitous. Everything changes over time. Time is a point of reference for all human activities including their perceptions. Thus, managing time is an essential part of complex systems in various domains such as medical diagnosis, financial investment and natural language processing. For example, in medical diagnosis, temporal information is critical in cases like "After taking medication, a patient recovers from a cold within one week." In financial investment, similar cases like "The investment only pays off after one year" also require the representation of time. Finally, to encode the information "Peter came to the party 10 min before John" for natural language processing, a temporal representation is necessary. Things can change both according to some stimulus, e.g. taking medicines can help in the recovery from a disease, and according to time, e.g. symptoms start to improve in a week. Therefore, considering either of these in isolation is not sufficient. Without considering the temporal element, critical constraints due to temporal order, duration of events and continuous change, which play a large part in human activities, are neglected. As such, a number of temporally infeasible solutions will be incorrectly presented. Although traditional knowledge representations model change according to causality, most of them neglect the reactions according to time. On the other hand, temporal models usually ignore the importance of causality in the representation of relations.

A temporal reasoning scheme should consist of three components: temporal reasoning, abductive reasoning, and reasoning under uncertainty (Santos, 1996). Abductive reasoning infers the best set of hypotheses to a given consequence. This is important not just in diagnostic situations but also for simulations and predictions. The foundations of abductive reasoning provide a concrete and formal definition for inferencing in these systems. Using the same examples listed above – the patient recovers from a cold because he took the medication, the investor makes money because he invested in stock A, and Peter and John came to the party because they were invited by the party holder. Uncertainty can be introduced into traditional reasoning schemes by using probability or other representations such as fuzzy logic, rule refinement, analysis of what-if scenarios, and so forth, to connect related events. Uncertainty occurs when information is incomplete or nondeterministic. In fact, almost everything in the real-world is uncertain because exceptions are always possible. To see why these components are necessary in a temporal formalism, consider the following scenario:

If a person who has the flu virus is an adult, they are able to infect others from the time symptoms develop up until the 5th day of showing symptoms. On the other hand, if they are a child, they are contagious as long as they are sick. Additionally, it is sometimes possible for people to spread the flu virus before showing symptoms or after symptoms improve. Symptoms tend to develop before the 4th day of catching the flu and to improve 7 days after the virus gets into the body.

This scenario shows that causal and temporal relationships and related uncertainties are both common and critical information that need to be captured.

* Corresponding author. Tel.: +1 603 646 2577; fax: +1 603 646 2277.
E-mail addresses: Eugene.Santos.Jr@Dartmouth.edu (E. Santos Jr.), Deqing.Li@Dartmouth.edu (D. Li), eesantos@utep.edu (E. E. Santos), jkorah@utep.edu (J. Korah).

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The framework we present in this paper is called Temporal Bayesian Knowledge Bases (TBKBs), which incorporates temporal reasoning, abductive reasoning and reasoning under uncertainty. It is based on an existing knowledge representation called *Bayesian Knowledge Bases* (BKBs) (Santos & Santos, 1999). To the best of our knowledge, few approaches incorporate all three components into their temporal reasoning schemes. Most deal with the relationships between temporal events but neglect the causal relationship and underlying uncertainty between them. Approaches that do incorporate elements from all three components typically suffer from problems with their knowledge representations such as the inability to infer the duration of events (e.g. Temporal Bayesian Networks (Tawfik & Neufeld, 1994)), and the high complexity in representing highly dynamic events (e.g. Dynamic Bayesian Networks (Ghahramani, 1997)). TBKBs inherit the components of BKBs, which include abductive reasoning and reasoning under uncertainty, and have the advantages of BKBs' representational power such as in managing incompleteness and cycles. The TBKB framework incorporates time by introducing another level of constraints to BKBs. As will be demonstrated, its representation of temporal and atemporal knowledge is probabilistically sound and admits an effective reasoning scheme accounting for both time and uncertainty.

In the next section, we briefly review the existing approaches for temporal reasoning and compare and contrast them with our framework. Then before detailing the formulation of TBKBs in Section 4, we introduce BKBs and related functions in Section 3. The reasoning scheme for TBKBs is next provided in Section 5. We also provide some provable properties of the TBKB framework and introduce an algorithm to construct TBKBs with feasible temporal solutions in Section 6. Lastly, our conclusion and future work are presented in Section 7.

2. Literature review

Temporal reasoning has been studied since the 1960s and the first research results can be traced back to situation calculus (McCarthy & Hayes, 1969). Situation calculus represents the concept of change by use of state-based representations. But it was not until the 1970s that time was represented explicitly and temporal relations were interpreted and reasoned as constraints on temporal entities. Two of the most influential theories about time are Allen's *Interval Algebra* (IA) (Allen, 1983) and McDermott's *temporal logic* (McDermott, 1982). Allen was the first to propose a systematic theory of time intervals that included the qualitative relationship between intervals and algorithms to propagate temporal constraints. In IA, any pair of time intervals can be related in one of thirteen ways which are *equals, precedes, meets, during, starts, finishes, overlaps*, and their inverses. Since then, there have been various debates over choosing the primitives of time as either intervals or points. *Point algebra* (PA) (Vilain & Kautz, 1986) models the relationship among points in time using a combination of the three basic operations: <,> and =. PA is still chosen often due to its simplicity, but is not as expressive as IA. In research such as Vilain (1982), Meiri (1996) and Campos, Juarez, Palma, Marin, and Palacios (2011), the authors combine points and intervals and propose composition rules that relate two time intervals (points). In Lin, Shan, Liu, Qu, and Ren (2005), the authors extend PA by representing the end points as intervals in order to model nondeterministic time points. For this paper, temporal information about an event is represented as intervals and a TBKB is said to represent a temporal interval network. However, to deal with quantitative information, we explicitly consider the end points of intervals as the variables we wish to constrain. Some aspects of IA that cannot be expressed in PA can be represented using disjunctions, such as the relationship of $\{b, bi\}$ represented as

$$x^+ > y^- \vee x^- < y^+.$$

We extend the qualitative relationships in IA by adding metric information to the end points. For events that take place instantaneously, the end points can be represented as $(t_0, t_0 + \varepsilon)$, where t_0 is the start point and ε is a value arbitrarily close, but not equal, to zero and is held constant throughout the model. In this way, our framework subsumes and extends the expressive power of both PA and IA.

There are two components in any framework for temporal reasoning: temporal representation and reasoning engine. The major forms of temporal representation include *temporal arguments* (Haugh, 1987), *modal temporal logic* (Pnueli, 1977) and *reified temporal logic* (Allen, 1984; Haddawy, 1996; McDermott, 1982). As the most classic approach for temporal representation, temporal arguments represent time as a parameter of the atemporal proposition without changing the original syntax of the proposition, while in modal temporal logic, each possible world is associated with a different time, and the worlds are connected with temporal operators. To maintain the efficiency of reasoning over temporal arguments while increasing its expressiveness, reified temporal logic was developed. It uses a "truth predicate" to take an atemporal expression. It also takes a temporal reference as an argument so as to indicate that the atemporal expression is true during the time period specified in the temporal reference.

To reason over temporal constraints, graph-based formalisms and probabilistic temporal formalisms are gaining popularity (Tawfik & Neufeld, 1994). Graphs provide a visual representation of relationships such as causality and temporal sequence by encoding knowledge in vertices and relationships in arcs, while probabilistic temporal formalisms address the problem of uncertainty. Some approaches to representing time in graphs include *Temporal Constraint Networks* (TCN) (Dechter, Meiri, & Pearl, 1991), *Dynamic Bayesian Networks* (DBN) (Ghahramani, 1997), *Temporal Bayesian Networks* (TBN) (Tawfik & Neufeld, 1994), and *Probabilistic Temporal Networks* (PTNs) (Santos & Young, 1999). TCN is a graph-based formalism of the *Constraint Satisfaction Problem* (CSP) for temporal reasoning. Time points are denoted as nodes and temporal constraints are encoded in arcs. Temporal reasoning is performed by finding solutions to CSPs. However, TCN does not consider the causal relationship between events. In a DBN, conditional dependencies are represented between nodes both within a time slice and across time slices. The original sequence is unrolled to future sequences and they are connected by arc(s) denoting transition(s). Since temporal relationships are not explicitly represented in DBN, their representation is limited. For example, the *Conditional Probability Table* (CPT) will explode if the time duration is long, making it hard to represent relationships between variables with long intervening times. It has been argued that only two time-slices are needed to be saved in the CPT if the model is time-invariant. However, time-invariant situations are rarely observed in the real world. Next, based on Bayesian Networks, TBNs represent probabilities as functions of time. Both continuous and discrete change of interaction with time can be represented. Probabilities of events at a particular time point can be inferred by convolution of the probability density functions. PTNs also use BNs to construct temporal references. In PTNs, time durations are represented as a new parameter to each node. The conditional probabilities of each node consider all time intervals of interest. However, since both TBN and PTN are constructed from BNs, they require complete specification of probability distributions as well as an underlying acyclic network. In contrast, TBKBs, though similarly based on directed graphs (also called correlation-graphs), handles incompleteness and cycles while admitting potentially more efficient reasoning algorithms. Besides, both TBN and PTN require complete temporal information of an random variable (r.v.) over the time of interest. Also, TBN does not allow relative temporal distances between r.v.s, and PTN only allows qualitative temporal relationships.

In reality, time primitives may be indeterminate, which requires the modeling of uncertainty. Popular techniques are probability theory, possibility theory and preference. Recent studies (including Ryabov & Trudel, 2004; Mouhoub & Liu, 2008) assign probabilities to the qualitative relations between time intervals indicating the likelihood of the existence of the relations, and in Ryabov (2001), algorithms are proposed to infer the likelihood of temporal relations based on the uncertainty in the end points. On the other hand, possibility theory is used in Campos et al. (2011) and in Badaloni and Giacomin (2000) to model and reason over the uncertainty in relations. In Badaloni and Giacomin (2000), a priority degree is attached to each constraint, indicating the necessity of its satisfaction, and in Campos et al. (2011), a 5-tuple fuzzy number indicates the necessary and possible time intervals. The above literature deals with IA, while in Tsamardinos (2002), the author uses a normal distribution to model the expectation of the relation between a controllable point and an uncontrollable one. Instead of using probabilities, some researchers regard uncertainty as preference, such as in Peintner and Pollack (2004), which adds preference to the *Disjunctive Temporal Problem* (DTP) and CSP, and in Khatib, Morris, Morris, and Rossi (2001), which attaches preference to CSP and the *Simple Temporal Problem* (STP). Due to the uncertainty in time, some events are controllable while others are contingent. With regards to the controllability of a TCN, Vidal and Fargier (1999) classified TCN into strong controllable (controllable under any condition), weak controllable (controllable under at least one condition) and dynamically controllable (the future is controllable according to the past decisions and observations). However, as we mentioned in the introduction, none of these representations capture the uncertainty in causality. Within the few models that deal with both temporal elements and causality, *the Architecture for Intensive Care Unit Decision Support* (Juarez, Campos, Palma, & Marin, 2008) solves the problem of temporal diagnostics for highly complex conceptual environments by proposing a temporal-causal model, in which the temporal uncertainty is handled by fuzzy numbers, and the *Causal Temporal Constraint Network* (CTCN) (Fernandez-Leal, Moret-Bonillo, & Mosqueira-Rey, 2009) extends the TCN with causal constraints. However, causal constraints in CTCN are limited to qualitative measures and its complexity of reasoning is high since it propagates the causal constraints in addition to the temporal constraints during reasoning. According to Kim (1995), uncertainty in time can be categorized as *observation uncertainty*, *nondeterministic action laws*, *persistence duration uncertainty* and *action time uncertainty*. In our framework, the first two kinds of uncertainty by Kim (1995) are addressed in a probabilistic manner through BKBs, and we deal with the latter two using unary and binary temporal constraints.

Reasoning over time involves two types of tasks: *temporal consistency maintenance* and *temporal question answering*. Temporal consistency maintenance is conducted when the knowledge is updated. When a new temporal constraint is added or an existing constraint is deleted, the constraint must be propagated through all the related time elements. If the newly inferred constraint is in conflict with the original one, some action should be taken to deal with the conflict. Typically, approaches will ignore the newly inferred constraint and stay with the original one. Temporal question answering is about retrieving temporal information from the knowledge representation system based on the existing knowledge, such as asking for the duration of a state or an event, or asking yes-or-no questions like whether event1 happens before event2. The reasoning scheme of TBKBs deals with both tasks. Algorithms are presented in Section 7 to eliminate conflict between original and updated temporal knowledge. Belief revision infers the temporal assignment of each event, which assigns the best time duration to each event, and belief updating infers the posterior probability of a temporal event within temporal constraints. For a more detailed review of historical temporal reasoning systems please refer to Vila (1994) and Chittaro and Montanari (2000).

3. Bayesian Knowledge Bases (BKBs)

Many of the temporal reasoning formalisms are constructed by attaching a temporal reference to the atemporal assertions. The atemporal assertions that we use are based on BKBs. BKBs are a general class of probabilistic networks that are an alternative to Bayesian Networks. The formation and reasoning of BKBs are defined in Santos and Santos (1999).

The structure of a BKB is a directed graph with nodes and arcs, which is also called a *correlation-graph*. Two types of node can be identified in a correlation-graph: I-node and S-node. We show an example of a *correlation-graph* in Fig. 1 where I-nodes are denoted as white ovals and S-nodes as black dots. An I-node represents a pairing of a random variable and its state such as $C = c$ where C is the random variable and c is one of its possible instantiations. An S-node represents the prior probability (of root I-nodes) or the conditional probability of an I-node such as $P(A = a|C = c') = 0.8$. Arcs denote conditional dependencies between nodes. The formal definition of a *correlation-graph* is as follows:

Definition. A *correlation-graph* is a directed graph $G = (I \cup S, E)$ in which $I \cap S = \Phi$, $E \subset \{I \times S\} \cup \{S \times I\}$, and $\forall\ a \in S$, there exists a unique $b \in I$such that $a \to b \in\ E$.

Φ denotes the empty set, I denotes the set of all I-nodes in graph G, S denotes the set of all S-nodes, E denotes the set of all edges, and $a \to b$ denotes a directed edge connecting node a to node b. For each S-node a in a correlation graph G, we denote the set of all I-nodes that point to a as $Tail_G(a)$, i.e. $Tail_G(a) = \{b \in I | b \to a \in E\}$. Similarly the $Head\ _G(a)$ is the I-node that a points to in G, i.e. $Head_G(a) = \{b \in I | a \to b \in E\}$. Two sets of I-nodes, I_1 and I_2 are said to be *mutually exclusive* if there is an I-node $v = i_1$ in I_1 and an I-node $v = i_2$ in I_2 such that $i_1 \neq i_2$.

A BKB is a correlation-graph with additional constraints. These constraints enable BKBs to maintain the semantics of probability theory.

Definition. A *Bayesian Knowledge Base (BKB)* is a tuple $B = (G, w)$ where $G = (I \cup S, E)$ is a correlation-graph, and $w{:}S \to [0, 1]$ such that

1. $\forall a \in S, Tail_G(a)$ contains at most one instantiation of each r.v.
2. For distinct S-nodes a, $b \in S$ that support the same I-node, $Tail_G(a)$ and $Tail_G(b)$ are mutually exclusive.
3. $\forall A \subseteq S$ s.t. (i) $Head_G(a_1)$ and $Head_G(a_2)$ are mutually exclusive, (ii) $Tail_G(a_1)$ and $Tail_G(a_2)$ are not mutually exclusive for all a_1 and a_2 in A, $\sum_{a \in A} w(a) \leqslant 1$.

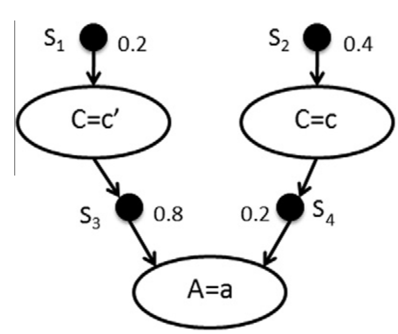

Fig. 1. Example of a BKB fragment.

From the definition of BKBs, it is easy to see that Fig. 1 is a valid BKB fragment, because (1) every S-node has only one I-node which contains only one instantiation of a r.v. in its tail, (2) s_3 and s_4 support the same I-node $A = a$, and their tails ($C = c'$ and $C = c$) are mutually exclusive, (3) s_1 and s_2 are mutually exclusive without tail (no prior information means prior information is not mutually exclusive), and $0.2 + 0.4 < 1$. To model the flu spread scenario introduced earlier using a BKB, we translate the scenario into the following if-then rules but ignore the temporal constraints for now. If a person is an adult, he has a 12% chance catching the flu. If he caught the flu he is very likely to show symptoms. And if an adult has developed symptoms of the flu, there is an 83% chance that he is contagious. The same causalities apply to children but with different probabilities. Fig. 2 shows the BKB with all the if-then rules encoded.

Probabilistic inferencing on BKBs can be done using Bayesian updating. Bayesian updating calculates the probability of each I-node given some evidence. However, a task more relevant to temporal reasoning is to find out the most probable state of the world under constraints. This is called Bayesian revision. To perform both types of inferencing, we make use of a special type of subgraph of a BKB called an *inference*. An inference is formally defined as follows:

Definition. Let $B = (G,w)$ be a BKB with correlation-graph $G = (I \cup S,E)$. A subgraph $\tau = (I' \cup S',E')$ of G, where $I' \subseteq I$, $S' \subseteq S$ and $E' \subseteq E$, is called an *inference* over B if

1. τ is acyclic.
2. (Well-supported) $\forall a \in I'$, $\exists b \in S'$ s.t. $b \to a \in E'$.
3. (Well-founded) $\forall b \in S'$, $Tail_\tau(b) = Tail_G(b)$.
4. (Well-defined) $\forall b \in S'$, $Head_\tau(b) = Head_G(b)$.
5. There is at most one I-node corresponding to any given random variable in I'.

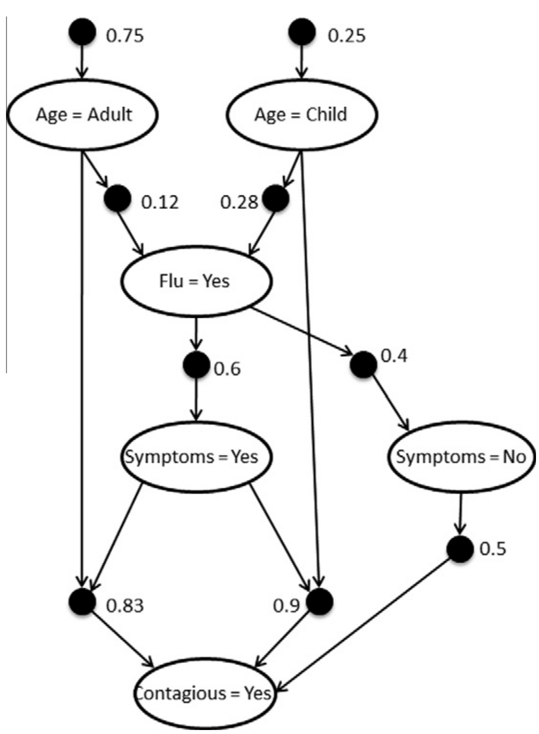

Fig. 2. BKB of the flu spread scenario.

Bayesian revision then simply finds the inference with the highest joint probability $P(A_{k+1} = a_{k+1},\ldots,A_n = a_n|A_1 = a_1,\ldots,A_k = a_k)$ where $A_1 = a_1,\ldots,A_k = a_k$ are evidence. Taking Fig. 1 as an example, the subgraph containing $C = c$, $A = a$ and their tail S-nodes is an inference (called I_1), and so is the subgraph with $C = c'$, $A = a$, and their tail S-nodes (called I_2). However, $C = c$ and $C = c'$ cannot be in the same inference because there can be at most one I-node corresponding to any given r.v in an inference. Then after comparing the joint probabilities of the two inferences, we find that $I_1(P(I_1) = P(C = c') * P(A = a|C = c') = 0.2 * 0.8 = 0.16)$ is more probable than $I_2(P(I_2) = P(C = c) * P(A = a|C = c) = 0.4 * 0.2 = 0.08)$. Bayesian updating on an I-node, on the other hand, can be regarded as the accumulation of all the possible worlds that include the I-node, and the summing of their joint probabilities. Still referring back to Fig. 1, the posterior probability of $A = a$ is obtained by summing up the joint probabilities of I_1 and I_2, which is $0.16 + 0.08 = 0.24$. In practice, inferencing is always conducted under constraints like evidence, which are I-nodes that already have been observed. In this case, inferences that do not include the evidence should be eliminated.

As the alternative to BNs, BKBs are more flexible knowledge representations for they manage incompleteness and cycles which are not allowed in BNs. The BKB also admits fusion algorithms to merge or add fragments as necessary, which significantly improves efficiency and achieves dynamism during construction. Details of the fusion algorithms can be found in Santos, Wilkinson, and Santos (2009b).

4. Bayesian Knowledge Bases with temporal constraints

A preliminary formulation of TBKBs can be found in Santos, Li, and Wilkinson (2009a). Here we provide more context and details to the readers. TBKBs are formed by imposing temporal constraints on BKBs. The temporal constraints include temporal assignments to I-nodes and temporal relationships between connected I-nodes. Temporal assignments denote the assignment of time during which an event (I-node) occurs. They are encoded by an interval of time which was first used by Allen (1983). In this paper, we call it a *temporal interval*. We now give a formal definition of *temporal interval*.

Definition. A *temporal interval* is a pair (t^-,t^+) where t^- represents the start time, t^+ represents the end time, $t^- < t^+$, and $t^-, t^+ \in \mathfrak{R}$.

Allen (1983) first defined thirteen relationships between any two possible temporal intervals. Borrowing the ideas from Allen, we also encode relationships between temporal intervals of any connected I-nodes, but are not restricted to the thirteen relationships. Instead we quantify temporal relationships by *disjunctive linear systems* of temporal intervals. Disjunctive linear systems were defined in Santos, Santos, and Kim (2006). Here we tailor the definition for the semantics of temporal reasoning.

Definition. A *disjunctive linear system (DLS)* L is a set $\{A_1,A_2,\ldots,A_n\}$ where each A_i is an individual system of linear equations and inequalities. $V(L)$ represents the variable set for L such that a variable v is in $V(L)$ IFF it has a nonzero coefficient in some linear system A_i in L. A *solution* to L is a complete assignment of values to $V(L)$ that satisfies some A_i in L.

In a TBKB, $V(L)$ is the set of start and end points of all related temporal intervals. To ease the work in construction and also to capture temporal relationship in a more natural manner, temporal relationships are defined locally, that is, only on I-nodes that are connected by a single S-node. For instance, it is more straightforward to infer direct relationships such as the relationship between getting the flu and showing symptoms rather than trying to imagine the relationship between getting the flu and infecting others or any other indirect relationships. Since the temporal relationships are defined on I-nodes with direct causal influence, we call them *temporal causal relationships*.

Definition. A *temporal causal relationship* (TCR) r_a describes a disjunctive set of temporal relationships between a set of I-nodes X that are adjacent to a single S-node a. r_a is a temporal causal relationship iff r_a can be expressed as a DLS of time intervals of I-nodes in X.

Suppose there are n I-nodes in X, whose temporal intervals are denoted as (t_x^-, t_x^+), then r_a is a TCR on the nodes in X iff r_a can be expressed as $\{A_1, A_2, \ldots, A_n\}$ where

$$A_i = \begin{cases} \alpha_{i,1,1}t_1^- + \alpha_{i,1,2}t_1^+ + \cdots + \alpha_{i,1,2n-1}t_n^- + \alpha_{i,1,2n}t_n^+ \Delta c_{i,1} \\ \ldots\ldots \\ \alpha_{i,j,1}t_1^- + \alpha_{i,j,2}t_1^+ + \cdots + \alpha_{i,j,2n-1}t_n^- + \alpha_{i,j,2n}t_n^+ \Delta c_{i,j} \\ \ldots\ldots \end{cases}$$

where $\alpha_{i,j,2k-1}$ denotes the coefficient of t_k^-, $\alpha_{i,j,2k}$ denotes the coefficient of t_k^+, $c_{i,j}$ denotes the constant in the jth linear inequality, and Δ is one of the following relations: <, $\leqslant$, >, $\geqslant$ or =. For example, to represent the TCR on the causal rule "Symptoms tend to develop before the 4th day of catching the flu and to improve 7 days after the virus gets into the body", we identify the relevant r.v.s which are *Flu = Yes*, *Symptoms = Yes* and *Symptoms = No* whose temporal intervals are denoted by $(t^-_{Flu=Y}, t^+_{Flu=Y})$, $\left(t^-_{Symp=Y}, t^+_{Symp=Y}\right)$ and $(t^-_{Symp=N}, t^+_{Symp=N})$. The dependency between getting the flu and showing symptoms is simply represented by $\left\{t^-_{Symp=Y} <= t^-_{Flu=Y} + 4\right\}$. However, the relationship between "not showing symptoms" and getting the flu is represented by a DLS because "not showing symptoms" happens both at the beginning and at the end of having flu. Symptoms developing before the 4th day of catching the flu means that the situation of no symptom ends at latest 4 days after the beginning of catching the flu, while symptoms improving 7 days after the virus gets into the body implies that the beginning of no symptom is 7 days after the beginning of catching the flu at the earliest. By combining the two relationships through disjunction, we obtained the TCR displayed in Fig. 3.

As we can see from the definition, TCRs denote not only the relationship between different time intervals, but also the constraint on a single interval by expressing the relationship between the start and the end points.

With the definition of temporal interval and temporal causal relationship, we can now define a TBKB.

Definition. A *Temporal Bayesian Knowledge Base (TBKB)* is a pair $T = (B,R)$ where $B = (G,w)$ is a Bayesian Knowledge Base with $G = (I \cup S, E)$, and R is a set of temporal causal relationships defined on S, i.e. $\forall\ r_a \in R$, $a \in S$.

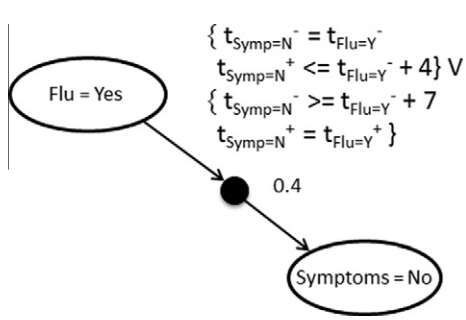

Fig. 3. Temporal causal relationship representing "Symptoms tend to develop before the 4th day of catching the flu and to improve 7 days after the virus gets into the body".

Intuitively, a TBKB is a BKB with the constraints of temporal intervals and temporal causal relationships. To model the flu spread scenario into a TBKB, we encode all the temporal relationships into TCRs on related S-nodes. Besides the constraints illustrated in the scenario, we further notice some implicit dependencies, namely a cause(s) has to occur before the effect in order to establish the causality (Pearl, 2000). In this scenario, symptoms appear only after the virus enters the body and disappear before the person has fully recovered. As a result of combining all the dependencies, we constructed the TBKB shown in Fig. 4. Temporal relationships are absent from some S-nodes such as *Age = Adult*. This may be due to the inapplicability of temporal constraints on the nodes or the unavailability of information with regard to the temporal constraints.

5. Reasoning with Temporal Bayesian Knowledge Bases

Reasoning with temporal models should consider both the causality and the temporal relationship between events in order to make sure that the solution not only satisfies causal relationships but also complies with time constraints. Reasoning over a TBKB can be conducted through either Bayesian revision or Bayesian updating. Bayesian revision finds the most probable inference that is consistent with the pre-defined temporal causal relationships, while Bayesian updating accumulates all possible inferences with regard to a variable. Such an inference is called a *temporal inference*. Below we define temporal assignment and temporal inference.

Notation. Let π_v denote the set of all states of random variable v.

Definition. a is a *temporal assignment* (TA) IFF a is a triple (v, γ, ε) where v is a random variable, γ is a temporal interval, and $\varepsilon \in \pi_v$.

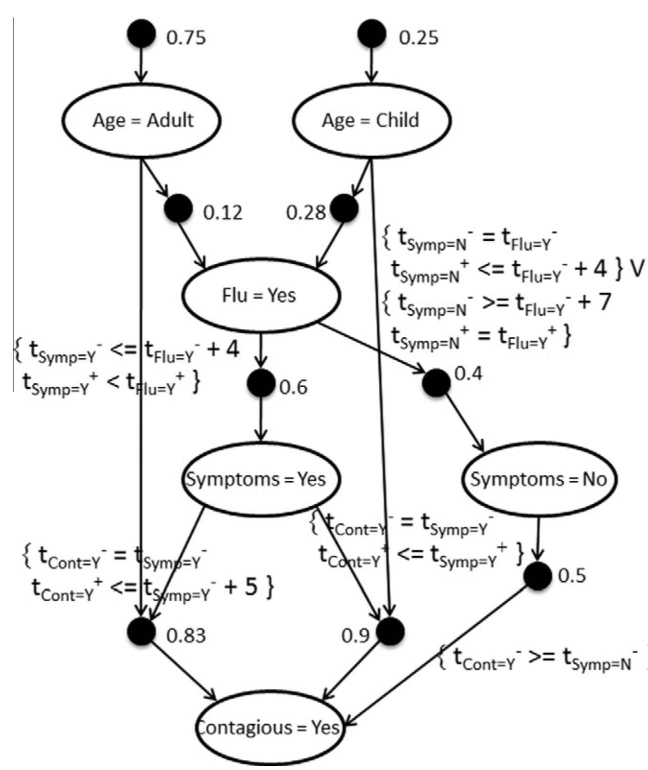

Fig. 4. A TBKB example showing causes of spreading flu.

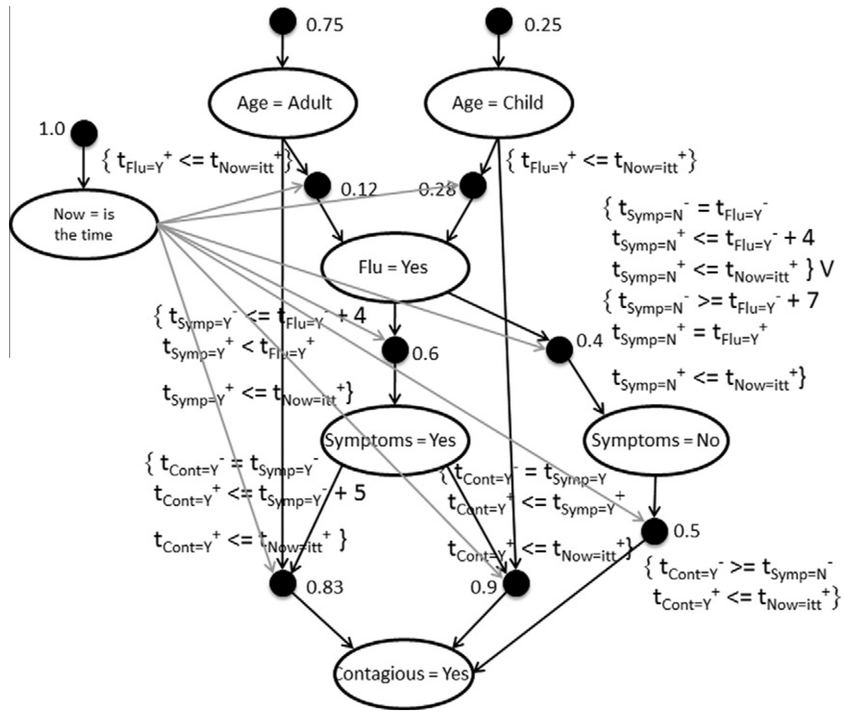

Fig. 5. TBKB with reference to current time.

Definition. Let $T = (B,R)$ be a temporal Bayesian knowledge-Base. A *temporal inference* over T is a pair $\tau = (\tau', \alpha)$ where $\tau' = (I' \cup S', E')$ is an inference over B,

$$\alpha = \{\alpha_i = (v_i, \gamma_i, x_i) | \cup_i x_i = I' \quad \text{and} \quad \cap_i x_i = \Phi\}$$

is a set of temporal assignments that is a solution to all DLSs representing the TCRs on S'.

Next is the problem of determining whether an inference is consistent with the entire collection of the TCRs. To do that, we first define a *Cluster Constraint System* (CCS) (Santos et al., 2006), which denotes the set of TCRs that need to be satisfied.

Definition. A *Cluster Constraint System C* is a set $\{L_1, L_2, \ldots, L_n\}$ where each L_i is a DLS. A *solution* to C is an assignment of the variables in C to values such that the assignment is a solution to each L_i. Thus C is a conjunction of DLSs.

As such, the problem is reduced to finding a solution to the CCS formed by the TCRs on the inference. Although the complexity of solving a CCS is NP-hard at worst, it can be approximated by convex envelopes, which solves the problem in polynomial time. Furthermore, in practice not all causal relationships are constrained by time. Most of them are either atemporal or follow the simple temporal order where causes precede consequences. If a solution to the CCS exists, the inference is said to be *feasible*, and the answer is said to be the *temporal assignments* of the I-nodes. Below we define *feasible inference*.

Definition. A *feasible inference* is an inference whose I-nodes either do not have temporal intervals due to lack of TCRs on connected S-node (s) or whose temporal intervals are feasible solutions to the CCS formed by the TCRs on the inference.

Sometimes we observe facts before conducting reasoning. The observed facts may be the occurrence of some event or prior information about the time interval of some event, which are in general called evidence. In this case, we further restrict the inferences by evidence and require that all feasible inferences should include the evidence. We now present examples on conducting inferencing using the flu spread model. Say we want to ask what the most probable state of a person is if he caught the flu on day 0 and spread the flu to others on day 10. We first identify the evidence implied in the statement. Evidence can be both temporal and atemporal. Atemporal evidence include *Flu* = *Yes* and *Contagious* = *Yes* while temporal evidence include $t^-_{Flu=Y} = 0$ and $t^-_{Con=Y} <= 10 <= t^+_{Con=Y}$. Assume that it is well known that the duration of flu symptoms is 7 days, and thus another piece of evidence is available: $t^+_{Symp=Y} - t^-_{Symp=Y} = 7$. The most probable complete inference is that *Age* = *Adult*, *Flu* = *Yes*, *Symptoms* = *Yes*, *and Contagious* = *Yes* with joint probability

$$\begin{aligned}
&P(Age = Adult, Flu = Yes, Symptoms = Yes, Contagious = Yes) \\
&\qquad = P(Contagious = Yes | Age = Adult \& Symptoms = Yes)^* \\
&P(Symptoms = Yes | Flu = Yes)^* \\
&P(Flu = Yes | Age = Adult)^* \\
&P(Age = Adult) = 0.045.
\end{aligned}$$

However, this inference is not consistent with the TCRs. More specifically, an adult getting the flu on day 0 is contagious until day 9, and thus is unable to spread the flu on day 10. The next most probable state of world is that *Age* = *Child*, *Flu* = *Yes*, *Symptoms* = *Yes*, *and Contagious* = *Yes* with joint probability 0.038. It turns out to be a feasible inference with the temporal assignments $(-\infty,\infty)$, $(0,\infty)$, $(4,11)$, and $(4,11)$ as a feasible solution. By using the symbol ∞, we mean that any number that forms a valid temporal interval will satisfy the inferencing. Moreover, temporal assignments need not be unique, e.g. *Symptoms* = *Yes* is possible at both $(3,10)$ and $(4,11)$. In the case of the I-node *Flu* = *yes*, there are infinitely many valid assignments. In our current approach, we arbitrarily choose one of them, but may improve the scheme in the future.

The algorithm of Bayesian updating is built on top of that of Bayesian revision. Specifically, to update the probability of a node, Bayesian revision is first conducted on the node in order to collect all the possible worlds within temporal evidence. Then the joint

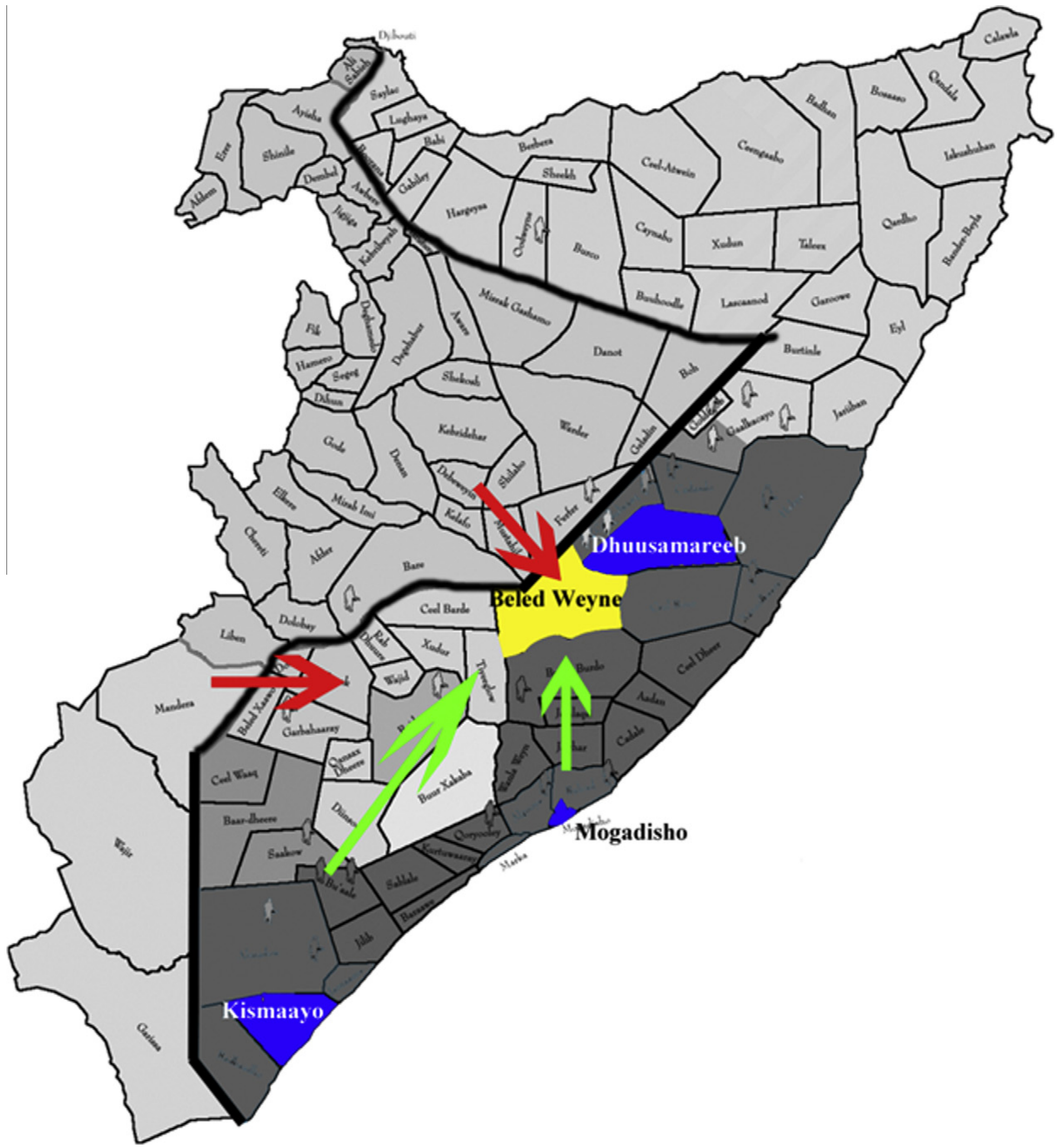

Fig. 6. Scenario on how ICU respond to the invasion of Ethiopia, red arrows indicates invasion of Ethiopia, green arrows indicate reinforcement from ICU. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

probabilities of the worlds are accumulated to measure the posterior probability of the node. Say now we want to update the posterior probabilities on the age of the patient given the same temporal and atemporal evidence as before. For the case of adults, although it is infeasible for an adult to have flu on day 0, be contagious on day 10, and at the same time show symptoms at any point, it is possible for him to get the flu without showing any symptoms. Thus the following inference is still feasible.

$$\begin{aligned} &P(Age = Adult, Flu = Yes, Symptoms = No, Contagious = Yes) \\ &\qquad = P(Contagious = Yes|Symptoms = No)^* \\ &P(Symptoms = No|Flu = Yes)^* \\ &P(Flu = Yes|Age = Adult)^* \\ &P(Age = Adult) = 0.018. \end{aligned}$$

Since this is the only feasible inference containing $Age = Adult$ based on the evidence, the posterior probability of $Age = Adult$ is 0.018. On the other hand, for the case of a child, both the inference about showing symptoms and that about not showing symptoms are feasible, so the updated probability of $Age = Child$ is $P(Age = Adult, Flu = Yes, Symptoms = Yes, Contagious = Yes) + P(Age = Adult, Flu = Yes, Symptoms = No, Contagious = Yes) = 0.038 + 0.014 = 0.052$. Note that the probability of $Age = Adult$ and that of $Age = Child$ do not have to sum up to 1 due to incomplete information.

We have been assuming that the current time is the day of our latest knowledge, which is day 10. If we go back a day, then according to the TBKBs in Fig. 4, we would not have known that the patient would be contagious on day 10, and would have expected the patient to be an adult rather than a child. Therefore, reasoning results might be different at different time points given the same

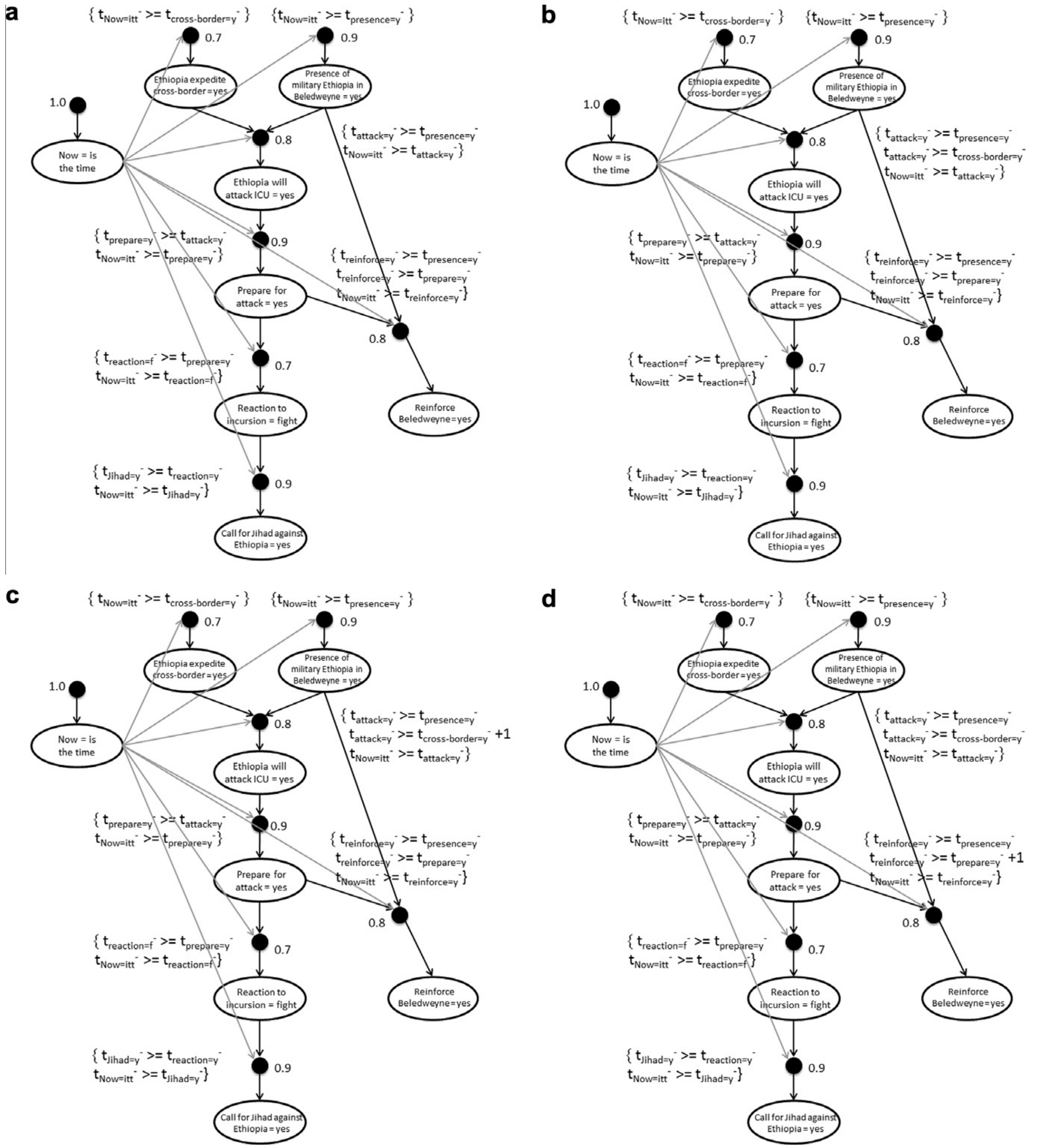

Fig. 7. (a) TBKB model of Dhuusamareeb, (b) TBKB model of Kismaayo, (c) TBKB model of Mogadisho under TFG tactic I, (d) TBKB model of Mogadisho under TFG tactic II.

information. Having access to it is important because decision making is based on people's perception specific to a given moment rather than on the objective truth. To enable the capability of switching time reference, we introduce a reference to the current time, called the NOW variable. To indicate that the current time is day 9, we add an I-node called "Now" with a single state "is the time" and give its prior probability as 1.0. To relate all temporal events with the time reference, for any temporal I-node, we attach the NOW variable to each of its tail S-node, specify on the S-node the temporal relationship that the original I-node finishes before or at the same time of NOW, and keep the weight on the S-nodes unchanged. During inferencing, a new piece of temporal evidence is imposed to indicate that NOW starts and finishes at day 9. The TBKB of our flu example with the reference to current time would layout as Fig. 5.

Although we used a toy example to demonstrate the functions of TBKBs, we will now demonstrate how TBKBs can be applied to real world problems such as complex military and political fields (Pioch, Melhuish, Seidel, & Li, 2009). We leverage previous work in analyzing group stability during the 2006 Somali civil war

(Santos et al., 2011) and describe how TBKBs can be used to model adversarial intent. In 2006, an Islamic organization called Islamic Courts Union (ICU) ousted the government forces of the internationally recognized Transitional Federal Government (TFG) by gaining control over most of southern Somalia, and implemented extreme Sharia Law. After a struggle that lasted most of 2006, TFG with the help of Ethiopia, the African Union, and the United States, drove ICU out of the country and resumed its control over Somalia. Due to the cultural complexities and the clan-based social structures, the situation in Somalia is highly volatile with political affiliations changing rapidly. The clan affiliations have splintered the once united nation, with each region having its own agenda. Also, due to geographical, cultural and religious factors, the time taken by different regions to respond to events varies greatly. Thus, the temporal element plays an important role in how the sequence of events progresses, which in turn has a great impact on the formation and dissolution of groups and political affiliations. To capture this dynamism of different regions within the control of ICU over time, we use TBKB to model the intent of the regional leaders for one specific scenario as an example. The scenario, occurring over 3 days, starts with Ethiopian forces being massed across the border, threatening the region Beledweyne since mid July. Under the control of ICU, the region Dhuusamareeb has been preparing to fight back against Ethiopia. As Ethiopian forces mobilize, ICU is apprehensive about an invasion and calls for emergency reinforcements from regions Mogadisho and Kismaayo. The scenario is depicted in Fig. 6, where the red arrows depict the Ethiopian mobilization and the green arrows depict the ICU reinforcements from Kismaayo and Mogadisho.

In the model, ICU and TFG are represented by a BKB illustrating some general beliefs of the organizations respectively. Since the regions under ICU and TFG control also retain the respective BKBs, regional behaviors evolve as they move from one organization to another. As events occur, TBKBs describing the events (examples are shown in Fig. 7) are fused in the models of the regions of interest. The models are simplified in the paper for the purpose of demonstration. To model adversarial intent of TFG in this scenario, we look at two hypothetical tactics that could have been pursued by TFG to break the consensus between the three regions (Dhuusamareeb, Mogadisho and Kismaayo) and weaken the ICU. We term these tactics as Tactic I and Tactic II. In Tactic I, TFG delays Mogadisho's belief about when Ethiopia is going to invade by spreading the wrong information. In Tactic II, TFG delays Mogdaisho's action of reinforcing Beledweyne by hindering the advance of their troops. For both cases, we built the TBKBs for the beliefs of the three regions shown in Fig. 7. Basically, Dhuusamareeb believes that Ethiopia has already displayed its intention of invasion when they started to harass the border. Kismaayo and Mogadisho do not believe that Ethiopia will invade until it expedites its troop mobilization. However, through tactic I, TFG tricks Mogadisho by delaying its belief about the invasion of Ethiopia by 1 day, and through tactic II, TFG forces it to delay its reinforcement to Beledweyne by 1 day. The NOW variable is used to enable the fine measurement of the intent of the regions over a period of time. A belief revision is then performed on the TBKBs of the three regions on these three consecutive days to understand change in beliefs and intent. On the first day, TFG identifies the presence of Ethiopian forces on the border of Beledweyne. On the second day, Ethiopia's cross-border expedition is reported. There is no new information on the third day. Information about these events is encoded as atemporal and temporal evidence before performing belief revision. The result of belief revision in Tables 1 and 2 show that Dhuusamareeb declared Jihad against Ethiopia and reinforced Beledweyne from the first day. Due to the disinformation in tactic I, Mogadisho delays its belief of Ethiopian invasion by one day, which results in the delay of all its actions including the declaration of Jihad and the reinforcement. On the other hand, since tactic II only delays its action to reinforce Beledweyne, it holds the same belief to fight against the incursion as Kismaayo on day 2. Therefore, it is clear that with tactic I, the three regions demonstrate a larger deviation in their beliefs as compared to tactic II, and thus, tactic I is a better strategy to create friction within ICU. If we had not considered temporal information and enforced the evidence that Ethiopian forces are present on the border of Beledweyne and poised to invade on the BKBs of the three regions, then their behavior would turn out to be the same, resulting in the same response to Ethiopia's invasion. More specifically, all of them would declare Jihad, fight against Ethiopia, and reinforce Beledweyne at the same time. It would appear that the tactics we proposed would make no difference to their responses. By representing and using temporal information, we are able to eliminate temporally infeasible situations and provide a world with finer granularity over time. This level of granularity is critical in the evaluation of the dynamism of events, which cannot be achieved with atemporal models. Clearly, temporal BKB can be a powerful tool in the hands of an analyst. Other analyses on the rise and collapse of ICU can be found in (Santos et al., 2011).

Table 1
reasoning result of TBKBs of three regions with TFG tactic I.

	Reaction = fight	Declare Jihad = yes	Reinforce beledweyne = yes
Day 1			
Kismaayo	*n*	*n*	*n*
Dhuusamareeb	*y*	*y*	*y*
Mogadisho	*n*	*n*	*n*
Day 2			
Kismaayo	*y*	*y*	*y*
Dhuusamareeb	*y*	*y*	*y*
Mogadisho	*n*	*n*	*n*
Day 3			
Kismaayo	*y*	*y*	*y*
Dhuusamareeb	*y*	*y*	*y*
Mogadisho	*y*	*y*	*y*

Table 2
Reasoning result of TBKBs of three regions with TFG tactic II.

	Reaction = fight	Declare Jihad = yes	Reinforce beledweyne = yes
Day 1			
Kismaayo	*n*	*n*	*n*
Dhuusamareeb	*y*	*y*	*y*
Mogadisho	*n*	*n*	*n*
Day 2			
Kismaayo	*y*	*y*	*y*
Dhuusamareeb	*y*	*y*	*y*
Mogadisho	*y*	*y*	*n*
Day 3			
Kismaayo	*y*	*y*	*y*
Dhuusamareeb	*y*	*y*	*y*
Mogadisho	*y*	*y*	*y*

6. Temporal BKB fusion

There are cases in which we want to fuse several small pieces of TBKBs into a larger one. For example, when several engineers are involved in the construction of a TBKB, they may disagree on the causal or temporal relationships. They may have different ideas with respect to the magnitude of the conditional probabilities. A more common case is that we want to build or refine the TBKB on the fly so that the TBKB is more adaptive to the dynamic world. However, fusing TBKBs is not as straightforward as it seems to be.

It may introduce contradictions, loops or mutual exclusion problems without a carefully designed algorithm. A fusing algorithm for BKBs was proposed in Santos et al. (2009b), in which an input BKB is parameterized with a new type of I-node called a source node before getting fused with other BKBs. Based on the BKB fusion algorithm, we proposed an algorithm to fuse TBKBs in Santos et al. (2009a). The algorithm can be described as follows: Each input TBKB is referred to as a TBKB fragment. In a TBKB fragment σ, each S-node whose head is a state of r.v. A is parameterized with a source node $S_A = \sigma$, indicating that this conditional rule comes from fragment σ. Then we add an S-node to the tail of the source node, whose weight represents the reliability of the fragment. To fuse the fragments, we take the union of the I-nodes but preserve the original S-nodes together with the temporal relationships encoded on them even if they refer to the same rules. In this way, all the conditional probabilities and temporal relationships from individual sources are preserved, possible loops in inferences are broken because only one source node is allowed during belief revision, and mutual exclusion problems are prevented.

An example of TBKB fusion is shown in Fig. 8 (Santos et al., 2009a). The two fragments in Fig. 8(a) are merged into a bigger fragment in Fig. 8(b). In Fig. 8(b), each S-node is parameterized with a source node indicating which fragment it comes from. For the experimental validation in the paper, equal reliabilities were assigned to the fragments. While the common I-node $A = a$ is merged, all S-nodes remain separated as well as the source nodes.

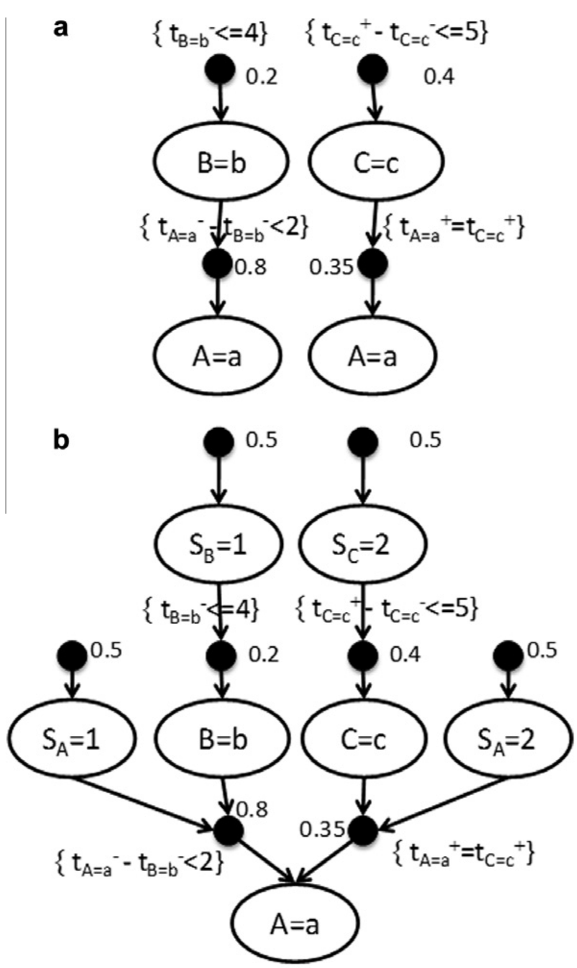

Fig. 8. Example of TBKF fusion. (a) Original BKB fragments. (b) Fused BKB fragment.

In practice, belief revision on a fused BKB or TBKB selects rules coming from single sources while belief updating calculates a weighted sum of the rules coming from different sources.

7. Properties of Temporal Bayesian Knowledge Bases – algorithms and proofs

In this section, we discuss some important provable properties of TBKBs. The first property we are interested in is the probabilistic soundness. We show that TBKBs are probabilistically sound.

Theorem 1. *For any set of mutually incompatible temporal inferences R′ in a TBKB T,* $\sum_{\tau' \in R'} P(\tau') \leqslant 1$.

Proof. Let B be a BKB. Let $T = (B, R)$.

Let A be the set of all sets of mutually incompatible inferences in B.

Let $\overline{A}$ be the set of all sets of mutually incompatible temporal inferences in T.

Let A' be the set of the underlying atemporal inferences of all sets of mutually incompatible temporal inferences in T.

Let I be the set of all inferences in B.

Let I' be the set of the underlying atemporal inferences of all temporal inferences in T.

Since $\forall\ \tau$ *s.t.* $\tau' = (\tau, \alpha)$ is a temporal inference over T, $\tau$is an inference over B, which means $\forall \tau \in I'$, $\tau \in I$ so $I' \subseteq I$.

We also know that $\forall\ a \in A$, $a \subseteq I$, and $\forall\ a' \in A'$, $a' \subseteq I'$. Thus, $A' \subseteq A$.

According to Santos and Santos (1999), $\forall\ a \in A, \sum_{\tau \in a} P(\tau) \leqslant 1$, so $\forall a' \in A', \sum_{\tau' \in a'} P(\tau') \leqslant 1$.

Since $\forall \bar{a} \in \overline{A}, \exists a' \in A'$ s.t.$\sum_{\bar{\tau} \in \bar{a}} P(\bar{\tau}) = \sum_{\tau' \in a'} P(\tau')$, $\forall\ \bar{a} \in \overline{A}$, $\sum_{\bar{\tau} \in \bar{a}} P(\bar{\tau}) \leqslant 1$. □

Another property of TBKBs is groundedness. If an S-node is grounded, it means it is always included in an inference. If the S-node remains grounded and unaltered, the initial probability assigned to it is semantically preserved no matter how the rest of the network is changed (Santos, Santos, & Shimony, 2003). We first generalize the definition of groundedness to TBKB, and then prove that groundedness is preserved in terms of TBKB fusion.

Definition. A node $a \in I \cup S$ in a TBKB is said to be *grounded* if there exists a temporal inference τ over the TBKB such that a is in τ. A TBKB T is said to be *grounded* if $\forall\ a \in I \cup S$, a is grounded.

Lemma 1. *In TBKB fusion, if a subgraph of a fused TBKB contains an inference of an original TBKB and for each S-node in the inference an I-node indicating the source of the inference is added on top of it and an S-node indicating the reliability of the source is added on top of the I-node, the subgraph is a temporal inference in the fused TBKB.*

Proof. Let T be the fused TBKB with correlation-graph G. Let T_i be the ith original TBKB with correlation-graph G_i.

Let $\tau = (\bar{\tau}, \alpha)$ be a temporal inference in T_i where $\bar{\tau}$ is the underlying atemporal inference and α is the temporal assignments to the temporal constraints.

Let $\tau' = (\bar{\tau}', \alpha')$ be the new subgraph in T where $\bar{\tau}'$ is the underlying atemporal subgraph and α' is the temporal assignments to the temporal constraints.

We know that source nodes do not introduce new temporal constraints. Thus $\alpha = \alpha'$. Now the problem is reduced to proving that the underlying atemporal subgraph is an inference in the fused TBKB.

Let I be the set of I-nodes in T. Let I_τ be the set of I-nodes in inference τ.

Let S be the set of S-nodes in T. Let S_τ be the set of S-nodes in inference τ.

Let δ_b be the source node on top of S-node b.

Let $r(a)$ denote the random variable of I-node a.

1. Obviously τ' is acyclic since each source node only connects with one S-node, and thus no loops are created.
2. To prove well-supportedness, $\forall\ a \in I_{\tau'}$, there are 2 cases.
   Case 1: a belongs to T_i.
   Since $S_\tau \subseteq S_{\tau'}$ and $\exists b \in S_\tau$ s.t. $b \to a \in E_\tau$, so $\exists b \in S_{\tau'}$ s.t. $b \to a \in E_{\tau'}$.
   Case 2: a is a source node, which means $\exists x \in S_\tau$ s.t. $a = \delta_x$.
   Let b be the S-node of a.
   Since b is in $\tau\prime, b \to a \in E_{\tau'}$.
   Therefore, τ' is well supported.
3. To prove well-foundedness, $\forall b \in S_{\tau'}$ there are 3 cases.
   Case 1: b is a conditional probability in T_i.
   $Tail_{\tau'}(b) = Tail_\tau(b) = Tail_{G_i}(b) = Tail_G(b)$.
   Case 2: b is a prior probability in T_i.
   $Tail_{\tau'}(b) = \delta_b$. Since δ_b is in $\tau', Tail_{\tau'}(b) = Tail_G(b)$.
   Case 3: b is the S-node of a source node in T.

   $Tail_{\tau'}(b) = \Phi$.

   Therefore, τ' is well founded.
4. To prove well-definedness, $\forall b \in S_{\tau'}$ there are 2 cases.
   Case 1: b belongs to T_i.

   $Head_{\tau'}(b) = Head_\tau(b) = Head_{G_i}(b) = Head_G(b)$.

   Case 2: b is the S-node of a source node in T, which means$\exists x \in S_\tau$ s.t. $Head_{\tau'}(b) = \delta_x$.
   Since δ_x is in $\tau\prime, Head_{\tau'}(b) = Head_G(b)$.
   Therefore, τ' is well defined.
5. To prove there is at most one I-node corresponding to any given random variable in $I_{\tau'}$, there are 2 cases.
   Case 1: the I-node belongs to T_i.
   Then there is only one I-node to any r.v. in $I_{\tau'}$.
   Case 2: the I-node is a source node.
   Prove by contradiction.
   Let δ_a^p denote the I-node $\delta_a = p$.
   Suppose $\exists a \in I_\tau$ s.t. $\delta_a^p, \delta_a^q \in I_{\tau'}$ and $p \neq q. \exists b_p \in S_{\tau'}$ s.t. $b_p = Head(\delta_a^p)$ and $\exists b_q \in S_{\tau'}$ s.t. $b_q = Head(\delta_a^q)$. Then $Head(b_p) = Head(b_q) = a$.
   Let A_p be the set of I-nodes in $Tail(b_p)$ except δ_a^p. Let A_q be the set of I-nodes in $Tail(b_q)$ except δ_a^q.
   According to mutual exclusiveness (Santos & Santos, 1999), $A_p \neq \Phi$, $A_q \neq \Phi$, and $\exists a_p \in A_p$, $a_q \in A_q$ s.t. $r(a_p) = r(a_q)$.
   According to well-foundedness, since $b_p \in S_{\tau'}$, so $a_p \in I_{\tau'}$, and since $b_q \in S_{\tau'}$, so $a_q \in I_{\tau'}$.
   Since a_p and a_q are not source nodes, so a_p, $a_q \in I_\tau$, which is in contradiction with the definition of BKB. □

Theorem 2. *In TBKB fusion, if the input TBKBs are grounded, then the fused TBKB T' is grounded.*

Proof. If the input TBKBs are grounded, then $\forall\ a \in I \cup S$ such that I and S belong to an input TBKB T, there exists a temporal inference τ over T such that a is in τ. According to Lemma 1, there exists a subgraph τ' in the fused TBKB T', such that τ is contained in τ' and τ' is a temporal inference. Therefore, if the input TBKBs are grounded, the fused TBKB is always grounded. □

A more important feature of groundedness is whether a BKB continues to be grounded after being turned into a TBKB by incorporating temporal constraints. Unfortunately, it is not true although the preservation of groundedness during the transformation in the opposite direction is true. Thus, we want to investigate how to construct a grounded TBKB from a grounded BKB. Below we propose an algorithm, but before that we need to define a *conjunctive cycle*.

Definition. A cycle C is a *conjunctive cycle* if C is an undirected cycle, the node with the highest level in C is an S-node b, and $|Tail_C(b)| > 1$.

A local algorithm for checking whether a TBKB is grounded given that the underlying BKB is grounded:

1. For each atemporal inference
   1.1 For each S-node, make sure its TCR is feasible.
   1.2 While there is a conjunctive cycle C
      1.2.1 For each intermediate node A (node that is neither root or leaf) in C
         1.2.1.1 Eliminate A by propagating the temporal relationship on $Tail(A)$ to the temporal relationship on $Head(A)$ to form the new temporal relationship $\hat{r}_{\mathrm{Head(A)}}$, and connecting its parent I-nodes with its child I-nodes.
         1.2.1.2 If $\hat{r}_{\mathrm{Head(A)}}$ is not a feasible temporal relationship, the TBKB is not grounded and the algorithm stops.

      1.2.2 Eliminate all nodes that are not involved in any cycle.

2. The TBKB is grounded and the algorithm stops.

We claim that any TCR can be propagated as in step 1.2.1.1. Recall that a TCR is represented as a disjunction of the following system.

$$A_i = \begin{cases} \alpha_{i,1,1}t_1^- + \alpha_{i,1,2}t_1^+ + \cdots + \alpha_{i,1,2n-1}t_n^- + \alpha_{i,1,2n}t_n^+ \Delta c_{i,1} \\ \ldots\ldots \\ \alpha_{i,j,1}t_1^- + \alpha_{i,j,2}t_1^+ + \cdots + \alpha_{i,j,2n-1}t_n^- + \alpha_{i,j,2n}t_n^+ \Delta c_{i,j} \\ \ldots\ldots \end{cases}$$

where t_n represents the nth r.v. connected by the S-node, $c_{i,j}$ is a constant, and Δ is one of the following relations: <, $\leqslant$, >, $\geqslant$ or =. We can further restrict Δ to $\leqslant$ and $\geqslant$ for the ease of propagation without any distortion of the original relationship in the following way. $X > Y$ is transformed into $X \geqslant Y + \varepsilon$, $X < Y$ is transformed into $X \leqslant Y + \varepsilon$, $X = Y$ is transformed into $X \geqslant Y$ & $X \leqslant Y$ where ε represents an arbitrarily small positive number.

Say the node to be eliminated is n with temporal interval (t_n^-, t_n^+). We can represent each disjunctive component of the TCR on $Tail(n)$ "$r_{\mathrm{Tail(n)}}$" by

$$\begin{aligned} &\alpha_{i,j,1}t_1^- + \alpha_{i,j,2}t_1^+ + \cdots + \alpha_{i,j,2n-1}t_n^- + \alpha_{i,j,2n}t_n^+ \Delta c_{i,j} \\ &\Rightarrow \alpha_{i,j,2n-1}t_n^- + \alpha_{i,j,2n}t_n^+ \Delta c_{i,j} - \alpha_{i,j,1}t_1^- - \alpha_{i,j,2}t_1^+ - \cdots \\ &\quad - \alpha_{i,j,2n-3}t_{n-1}^- - \alpha_{i,j,2n-2}t_{n-1}^+ \\ &\Rightarrow \frac{\alpha_{i,j,2n-1}}{\alpha_{i,j,2n}} t_n^- + t_n^+ \Delta \big(c_{i,j} - \alpha_{i,j,1}t_1^- - \alpha_{i,j,2}t_1^+ - \cdots \\ &\quad - \alpha_{i,j,2n-3}t_{n-1}^- - \alpha_{i,j,2n-2}t_{n-1}^+\big) * \frac{1}{\alpha_{i,j,2n}} \ \text{if } \alpha_{i,j,2n} > 0. \\ &\Rightarrow \frac{\alpha_{i,j,2n-1}}{\alpha_{i,j,2n}} t_n^- + t_n^+ \overline{\Delta} \big(c_{i,j} - \alpha_{i,j,1}t_1^- - \alpha_{i,j,2}t_1^+ - \cdots \\ &\quad - \alpha_{i,j,2n-3}t_{n-1}^- - \alpha_{i,j,2n-2}t_{n-1}^+\big) * \frac{1}{\alpha_{i,j,2n}} \end{aligned}$$

where $\overline{\Delta}$ represents the inverse of Δ, if $\alpha_{i,j,2n} < 0$.

It is represented with respect to t_n- if $\alpha_{i,j,2n} = 0$.

A disjunctive component of the TCR on $Head(n)$ "$r_{\mathrm{Head(n)}}$" is

$$\begin{aligned}
&\beta_{i,j,1}t_n^- + \beta_{i,j,2}t_n^+ + \beta_{i,j,3}t_{n+1}^- + \beta_{i,j,4}t_{n+2}^+ + \cdots + \beta_{i,j,2(m-n)+1}t_m^- \\
&\quad + \beta_{i,j,2(m-n)+2}t_m^+ \Delta d_{i,j} \\
&\Rightarrow \beta_{i,j,3}t_{n+1}^- + \beta_{i,j,4}t_{n+2}^+ + \cdots + \beta_{i,j,2(m-n)+1}t_m^- + \beta_{i,j,2(m-n)+2}t_m^+ - d_{i,j}\ \Delta \\
&\quad - \beta_{i,j,1}t_n^- - \beta_{i,j,2}t_n^+ \\
&\Rightarrow -\left(\beta_{i,j,3}t_{n+1}^- + \beta_{i,j,4}t_{n+2}^+ + \cdots + \beta_{i,j,2(m-n)+1}t_m^- + \beta_{i,j,2(m-n)+2}t_m^+ - d_{i,j}\right) \\
&\quad * \frac{1}{\beta_{i,j,2}}\ \overline{\Delta} \frac{\beta_{i,j,1}}{\beta_{i,j,2}} t_n^- + t_n^+ \text{ if } \beta_{i,j,2} > 0. \\
&\Rightarrow -\left(\beta_{i,j,3}t_{n+1}^- + \beta_{i,j,4}t_{n+2}^+ + \cdots + \beta_{i,j,2(m-n)+1}t_m^- + \beta_{i,j,2(m-n)+2}t_m^+ - d_{i,j}\right) \\
&\quad * \frac{1}{\beta_{i,j,2}} \Delta \frac{\beta_{i,j,1}}{\beta_{i,j,2}} t_n^- + t_n^+ \text{ if } \beta_{i,j,2} < 0.
\end{aligned}$$

It is represented with respect to t_n- if $\beta_{i,j,2} = 0$. To propagate $r_{\mathrm{Tail(n)}}$ to $r_{\mathrm{Head(n)}}$, connect (2) with (1),

$$\begin{aligned}
&-\left(\beta_{i,j,3}t_{n+1}^- + \beta_{i,j,4}t_{n+2}^+ + \cdots + \beta_{i,j,2(m-n)+1}t_m^- + \beta_{i,j,2(m-n)+2}t_m^+ - d_{i,j}\right) \\
&\quad * \frac{1}{\beta_{i,j,2}} \Delta_1 \frac{\beta_{i,j,1}}{\beta_{i,j,2}} t_n^- + t_n^+ \Delta_2 \frac{\alpha_{i,j,2n-1}}{\alpha_{i,j,2n}} t_n^- \\
&\quad + t_n^+ \Delta_3 \left(c_{i,j} - \alpha_{i,j,1}t_1^- - \alpha_{i,j,2}t_1^+ - \cdots - \alpha_{i,j,2n-3}t_{n-1}^- - \alpha_{i,j,2n-2}t_{n-1}^+\right) * \frac{1}{\alpha_{i,j,2n}}
\end{aligned}$$

It is easy to get Δ_2 by comparing $\frac{\beta_{i,j,1}}{\beta_{i,j,2}}$ and $\frac{\alpha_{i,j,2n-1}}{\alpha_{i,j,2n}}$. The propagated relationship can be reduced to

$$\begin{aligned}
&-\left(\beta_{i,j,3}t_{n+1}^- + \beta_{i,j,4}t_{n+2}^+ + \cdots + \beta_{i,j,2(m-n)+1}t_m^- + \beta_{i,j,2(m-n)+2}t_m^+ - d_{i,j}\right) * \tfrac{1}{\beta_{i,j,2}} \Delta_4 \\
&\left(c_{i,j} - \alpha_{i,j,1}t_1^- - \alpha_{i,j,2}t_1^+ - \cdots - \alpha_{i,j,2n-3}t_{n-1}^- - \alpha_{i,j,2n-2}t_{n-1}^+\right) * \tfrac{1}{\alpha_{i,j,2n}}
\end{aligned}$$

by the following rules:

if $A \geqslant B \geqslant C \geqslant D$, then $A \geqslant D$;

if $A \leqslant B \leqslant C \leqslant D$, then $A \leqslant D$;

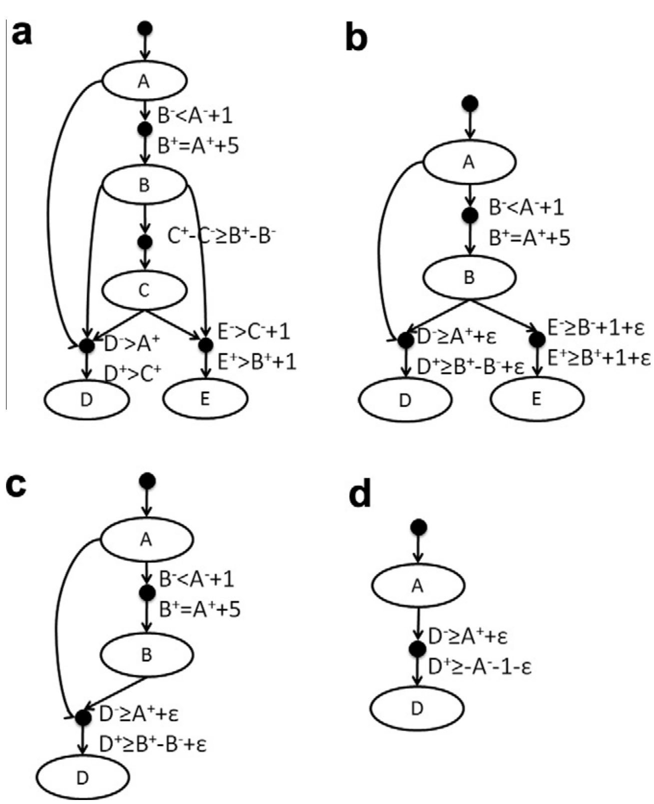

Fig. 9. An example of applying the local algorithm to check the feasibility of a temporal inference. (a) The original temporal inference. (b) Eliminating C and propagating its relationship with its parent. (c) Eliminating of E. (d) Eliminating B and propagating its relationship with its parent.

where A represents $-(\beta_{i,j,3}t_{n+1}^- + \beta_{i,j,4}t_{n+2}^+ + \cdots + \beta_{i,j,2(m-n)+1} t_m^- + \beta_{i,j,2(m-n)+2}t_m^+ - d_{i,j})$, B represents $\frac{\beta_{i,j,1}}{\beta_{i,j,2}} t_n^- + t_n^+$, C represents $\frac{\alpha_{i,j,2n-1}}{\alpha_{i,j,2n}} t_n^- + t_n^+$, and D represents $(c_{i,j} - \alpha_{i,j,1}t_1^- - \alpha_{i,j,2}t_1^+ - \cdots - \alpha_{i,j,2n-3}t_{n-1}^- - \alpha_{i,j,2n-2}t_{n-1}^+) * \frac{1}{\alpha_{i,j,2n}}$.

For any other combination of Δ's, no information can be propagated. In other words, nothing can be said about the relationship between A and D. This rule can be verified by applying twice the multiplication operation in PA (Vilain & Kautz, 1986) to four consecutive time points A, B, C and D, which determines the relation between A and D by the given relations between the intermediate points.

The above algorithm can be easily modified to maintain the groundedness of a TBKB after adding or removing TCRs by checking temporal inferences containing the relevant S-nodes instead of checking all possible temporal inferences. Fig. 9 provides an example in which we apply the local algorithm to check the feasibility of a temporal inference.

Fig. 9(a) shows the original temporal inference. In practice, the way we select which node to eliminate is based on how many cycles it is involved in. We choose to eliminate the node that is involved in the least number of cycles at each time to minimize the effort by propagating to the fewest possible TCRs. Therefore, C is eliminated first in this case. After transformation, *Tail*(D)'s TCR "$r_{\mathrm{Tail(D)}}$" is represented as $D^- \geqslant A^+ + \varepsilon \& D^+ \geqslant C^+ + \varepsilon$, and *Tail*($E$)'s TCR "$r_{\mathrm{Tail(E)}}$" as $E^- \geqslant C^- + 1 + \varepsilon \& E^+ \geqslant B^+ + 1 + \varepsilon$. Combining $r_{\mathrm{Tail(C)}}$ with $r_{\mathrm{Tail(D)}}$, we get $D^- \geqslant A^+ + \varepsilon \& D^+ - \varepsilon \geqslant C^+ \geqslant C^+ - C^- \geqslant B^+ - B^-$, so $D^+ \geqslant B^+ - B^- + \varepsilon$. Combining $r_{\mathrm{Tail(C)}}$ with $r_{\mathrm{Tail(E)}}$, we get $E^+ \geqslant B^+ + 1 + \varepsilon \& E^- - 1 - \varepsilon \geqslant C^-$. Although we do not know what C^- is, we know $C^- \geqslant B^-$ according to the causal relationship. Thus, to eliminate C^-, we obtain $E^- - 1 - \varepsilon \geqslant B^-$, which means $E^- \geqslant B^- + 1 + \varepsilon$. The resulted inference is shown in Fig. 9(b). In Fig. 9(c), E and related arcs are removed since E is not involved in cycle any more. Then, to eliminate B, we first transform *Tail*(B)'s TCR $r_{\mathrm{Tail(B)}}$ into $B^- \leqslant A^- + 1 + \varepsilon \& B^+ \leqslant A^+ + 5 \& B^+ \geqslant A^+ + 5$. Combining $r_{\mathrm{Tail(D)}}$ with $r_{\mathrm{Tail(B)}}$, we get $D^- \geqslant A^+ + \varepsilon \& D^+ - \varepsilon \geqslant B^+ - B^- \leqslant B^+ \leqslant A^+ + 5 \& D^+ - \varepsilon \geqslant B^+ - B^- \leqslant B^+ \geqslant A^+ + 5 \& \varepsilon - D^+ \leqslant B^- - B^+ \leqslant B^- \leqslant A^- + 1 + \varepsilon$. Since the intermediate two relationships do not provide any information, the temporal relationship between D and A becomes $D^- \geqslant A^+ + \varepsilon$ & $D^+ \geqslant -A^- - 1 - \varepsilon$. Since the TCR propagated on the final nodes is feasible, the entire temporal inference is feasible.

8. Conclusion

In this paper, we propose a framework to reason over uncertainty under temporal constraints. The framework is constructed by incorporating temporal constraints on atemporal Bayesian Knowledge Bases. TBKB inferencing and fusion algorithms are extended from BKBs, while preserving the semantics in representing uncertainty in the interaction between variables and uncertainty in the time when these interactions may occur. We have proved that the framework is probabilistically sound and continues to be grounded after fusion. A local algorithm is proposed to ensure that the Temporal Bayesian Knowledge Bases extended from a grounded BKB preserves its groundedness. In summary, TBKBs allow the modeling of higher dimensional knowledge by providing a sound and inexpensive representation and reasoning of time-constraints, which is more powerful than atemporal knowledge representations in modeling real-world scenarios with dynamic interactions.

Sometimes although the time interval is incomplete, some features of it are available such as probability, and more frequently, time intervals as either evidence or constraints are not absolutely certain. Thus, our future work will be focused on the representation of and the reasoning over the uncertainty in time. One possible solution is to represent a time interval using normal distribution instead of discrete numbers. While this may require a new scheme of reasoning and introduce higher complexity, it provides higher dimension in representing time and finer resolution in inferencing. Other improvements on our framework include allowing assignment of multiple intervals to a r.v. to address cases where an event may be true in discontinuous chunks of time and allowing contradicting states of a r.v. to be true as long as the temporal assignments of the states are mutually exclusive.

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