Categories of Timed Stochastic Relations

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Motivation

- Probabilistic process calculi (e.g. stochastic CCS)
  - Probabilistic choice
- Stochastic process calculi (e.g. stochastic π calculus)
  - Probabilistic delay on actions
  - Take the first enabled communication
- Probabilistic vs. "stochastic"
  - Categorical models for first-order languages

1. Adding delay to categorical models of iteration
2. Adding delay to the category of stochastic relations

Adding delay to categorical models of iteration

Monadic models of iteration

- First-order imperative language of loops

$$
\begin{aligned}
S ::= &\ \mathsf{skip} \\
\mid &\ S; S \\
\mid &\ \mathsf{let}\ v = E\ \mathsf{in}\ S \\
\mid &\ v := E \\
\mid &\ \mathsf{if}\ E\ \mathsf{then}\ S\ \mathsf{else}\ S \\
\mid &\ \mathsf{while}\ E\ \mathsf{do}\ S
\end{aligned}
$$

- Monadic state-transformer semantics

$$[\![S]\!] : [\![\Gamma]\!] \to T[\![\Gamma]\!] \qquad (\Gamma \vdash S)$$

- T models nontermination/failure (at least)

Monadic models of iteration

$\mathbf{C}$

- Finite products
  - State spaces: $[\![\Gamma]\!] = [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$
- Finite coproducts, distributive category
  - $[\![\text{bool}]\!] = 1 + 1$
  - $X \times (1 + 1) \longrightarrow (X + 1) \times (X + 1)$

$\mathbf{C}_T$

- Partially additive [Manes,Arbib 86]
  - Loops

Iteration

- $\mathbf{Par} \cong \mathbf{Set}_{-_\perp}$ semantics

$$[\![S]\!] : [\![\Gamma]\!] \to [\![\Gamma]\!]_\perp$$

- Unrollings of loop body

$$\begin{aligned}[\![\text{while } E \text{ do } S]\!] = \biguplus\ & [\![\neg E]\!]!, \\ & [\![E]\!]!; [\![S]\!]; [\![\neg E]\!]!, \\ & [\![E]\!]!; [\![S]\!]; [\![E]\!]!; [\![S]\!]; [\![\neg E]\!]!, \\ & \vdots \end{aligned}$$

  - Infinite summation
  - Partially defined

Iteration: partially additive categories

- Summation on arrows
  - Partial functions $\sum_{X,Y}$ on countable subets of $\mathbf{D}(X, Y)$
  - $\{f\}_{i \in I}$ *summable* if $\sum \{f\}_{i \in I}$ defined
  - ...

- Examples
  - $\mathbf{Par}$ – disjoint domains, graph union
  - $\mathbf{Rel}$ – graph union (not partial)
  - $\mathbf{CPO}_{\perp}$ – directed sets, lub

- Zero arrows: $0_{X,Y} = \sum_{X,Y} \varnothing$
  - Failure effect

Iteration: partially additive categories

Every $X \xrightarrow{f} X+Y$ decomposes as

$$f = \sum \left\{ X \xrightarrow{f_1} X \xrightarrow{\iota_1} X+Y,\ X \xrightarrow{f_2} Y \xrightarrow{\iota_2} X+Y \right\}$$

and gives the *iterate*

$$X \xrightarrow{f^\dagger} Y = \sum_{n<\omega} X \xrightarrow{f_1^n} X \xrightarrow{f_2} Y$$

$$[\![\text{while } E \text{ do } S]\!] = [\![\Gamma]\!] \xrightarrow{\left([\![\Gamma]\!] \xrightarrow{[\![E]\!]?} [\![\Gamma]\!]+[\![\Gamma]\!] \xrightarrow{[\![S]\!] + \eta} T[\![\Gamma]\!]+T[\![\Gamma]\!] \right)^\dagger} T[\![\Gamma]\!]$$

Delay

- Time taken by computation
- Delay effect: $- \times M$ monad (M monoid)

$$[\![\text{wait } E]\!] = [\![\Gamma]\!] \xrightarrow{\langle 1, [\![E]\!] \rangle} [\![\Gamma]\!] \times M$$

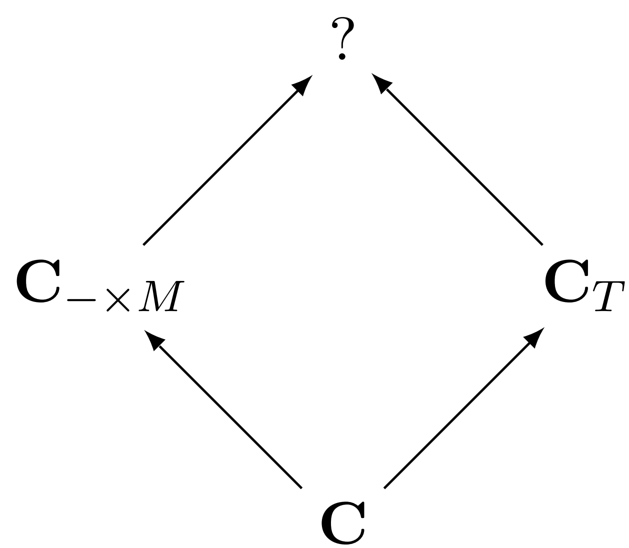
- Not impure monoids in $\mathbf{C}_T$

$$m : M \times M \to TM$$
$$e : 1 \to TM$$

- Pure monoids in $\mathbf{C}$

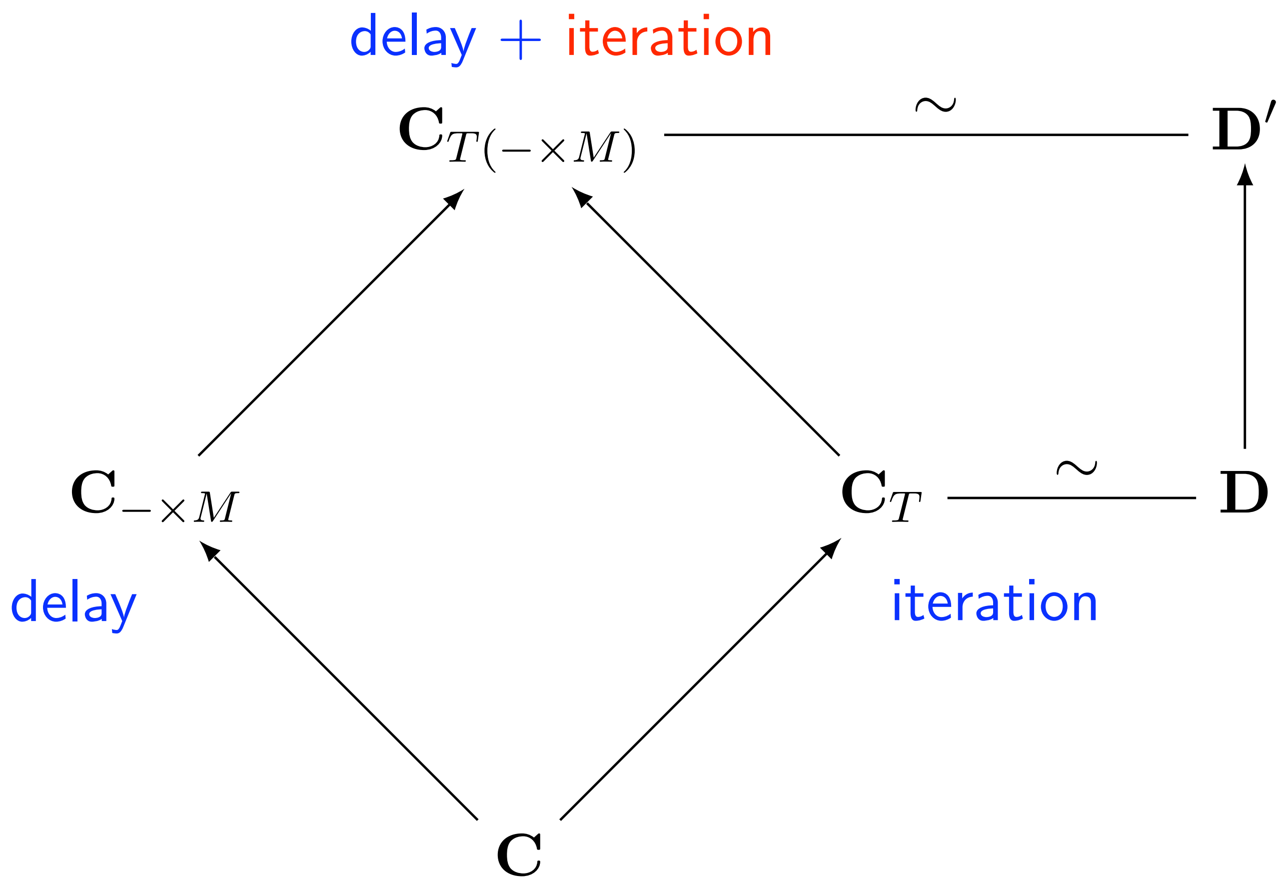
$$m : M \times M \to M$$
$$e : 1 \to M$$

Delay and T



- $(- \times M) \cdot T$ – coarse-grained timing
- $T \cdot (- \times M)$ – fine-grained timing, failure from T
  - Assume distributive law $\lambda_X : TX \times M \to T(X \times M)$
  - Strong monad suffices

Delay and iteration



Lifting partial additivity

Definition

Given $\mathbf{D}$ and $\mathbf{D}'$ partially additive, $F : \mathbf{D} \to \mathbf{D}'$ *preserves partial additivity* iff

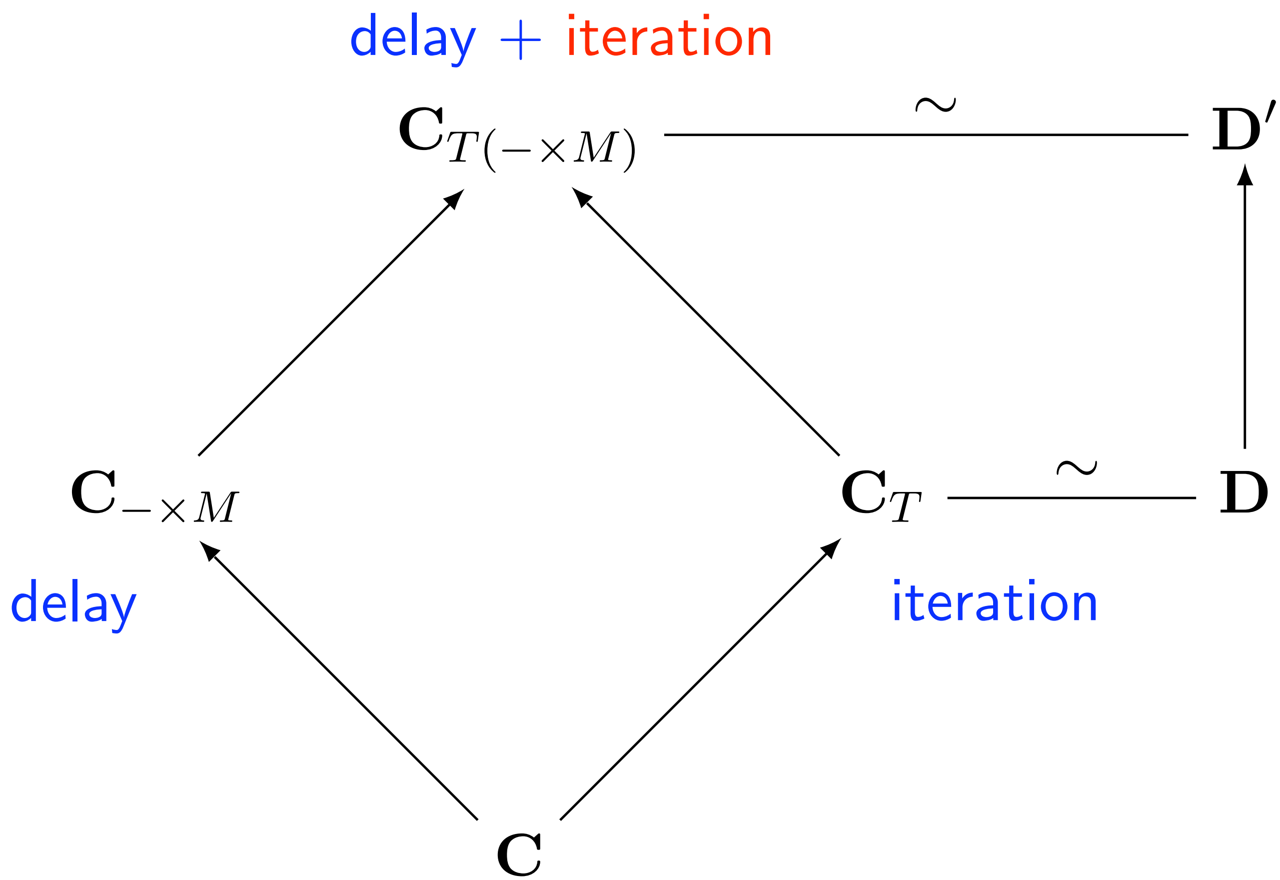
- $\{f_i\}$ summable $\Rightarrow$ $\{Ff_i\}$ summable
- $F(\sum f_i) = \sum F f_i$

Proposition

If $S : \mathbf{D} \to \mathbf{D}$ preserves partial additivity then $\mathbf{D}_S$ is partial additive where

- $\left\{ X_S \xrightarrow{(f_i)_S} Y_S \right\}$ *summable iff* $\left\{ X \xrightarrow{f_i} SY \right\}$ *summable*
- $\sum X_S \xrightarrow{(f_i)_S} Y_S = \left(\sum X \xrightarrow{f_i} SY \right)_S$

Lifting partial additivity



Lifting monads

Proposition

If S distributes over T, then S lifts to a monad $\overline{S} : \mathbf{C}_T \to \mathbf{C}_T$ st.

$$\mathbf{C}_{TS} \cong (\mathbf{C}_T)_{\overline{S}}$$

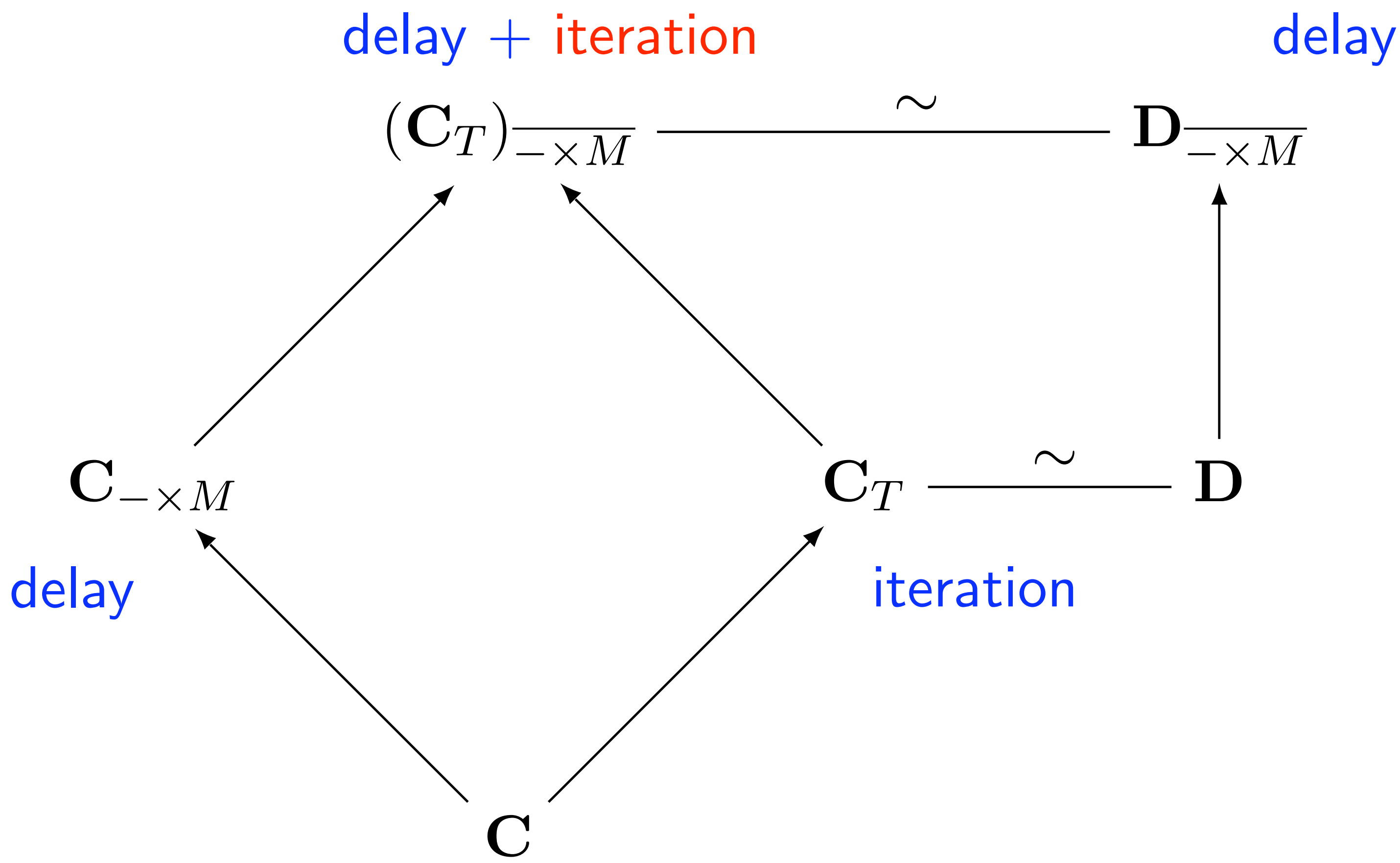
The monad:

$$\overline{S}\left(X_T \xrightarrow{f_T} Y_T \right) = (SX)_T \xrightarrow{\left(SX \xrightarrow{Sf} STY \xrightarrow{\lambda_Y} TSY \right)_T} (SY)_T$$

$$X_T \xrightarrow{\eta^{\overline{S}}_{X_T}} (SX)_T = \left(X \xrightarrow{\eta^{TS}_X} TSX \right)_T$$

$$(SSX)_T \xrightarrow{\mu^{\overline{S}}_{X_T}} (SX)_T = \left(SSX \xrightarrow{(\eta^T \circ \mu^S)_X} TSX \right)_T$$

Lifting monads



Lifting partial additivity

Theorem

Let $S, T : \mathbf{C} \to \mathbf{C}$ be monads with $\mathbf{C}_T$ partially additive. If S distributes over T and $\overline{S} : \mathbf{C}_T \to \mathbf{C}_T$ perserves partial additivity, then $\mathbf{C}_{TS}$ is partially additive.

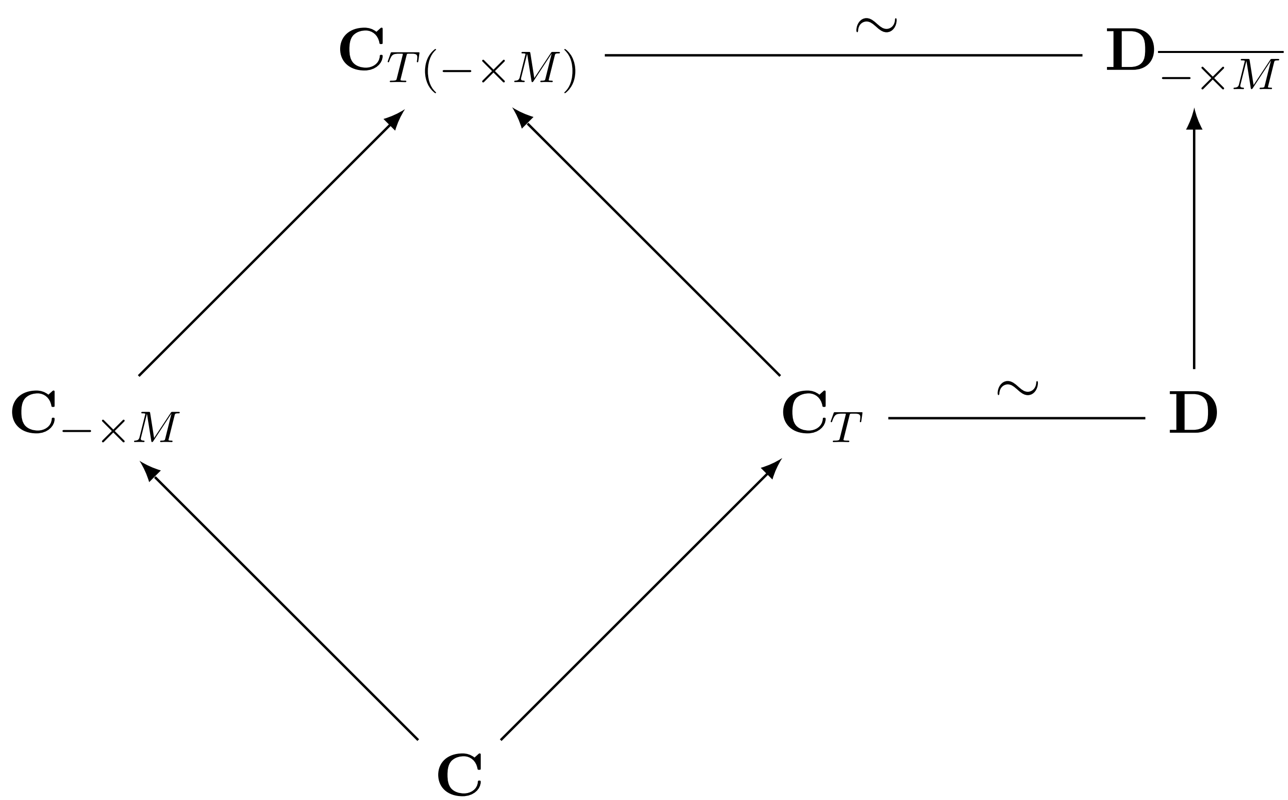
Corollary

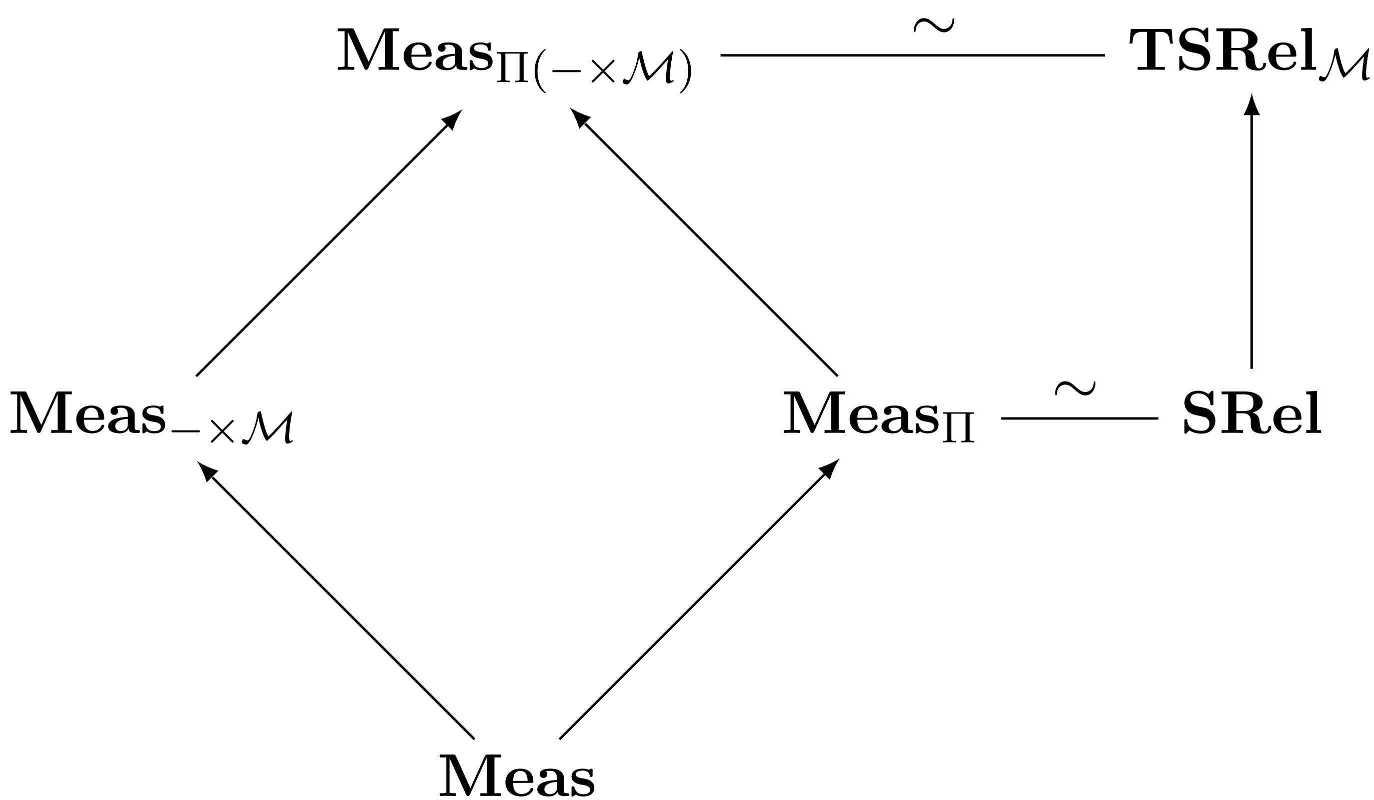
Let $\mathbf{C}$ have finite products with monoid M, let $T : \mathbf{C} \to \mathbf{C}$ be a strong monad, and $\mathbf{C}_T$ be partially additive. Then $T(- \times M) : \mathbf{C} \to \mathbf{C}$ is a monad and, if $\overline{- \times M} : \mathbf{C}_T \to \mathbf{C}_T$ preserves partial additivity, then $\mathbf{C}_{T(- \times M)}$ is partially additive.

Par

- $-_\perp : \mathbf{Set} \to \mathbf{Set}$ strong
- $\overline{- \times M} : \mathbf{Par} \to \mathbf{Par}$ preserves partial additivity
- $\mathbf{Par}_{\overline{- \times M}}$ models iteration and delay
    - $[\![S]\!] : [\![\Gamma]\!] \to ([\![\Gamma]\!] \times M)_\perp$

Adding delay to the category of stochastic relations


$\mathbf{C}_{T(-\times M)}$
$\sim$
$\mathbf{D}_{\overline{-\times M}}$
$\mathbf{C}_{-\times M}$
$\mathbf{C}_T$
$\sim$
$\mathbf{D}$
$\mathbf{C}$


$\mathbf{Meas}_{\Pi(-\times\mathcal{M})}$
$\sim$
$\mathbf{TSRel}_{\mathcal{M}}$
$\mathbf{Meas}_{-\times\mathcal{M}}$
$\mathbf{Meas}_{\Pi}$
$\sim$
$\mathbf{SRel}$
$\mathbf{Meas}$

$\mathbf{Meas}$: a category for probability

- Probability distribution / probability measure

$$
\begin{aligned}
\mathbb{N} &\to [0,1] \\
\mathbb{R} &\to [0,1] \\
\mathcal{P}\mathbb{R} &\to [0,1] \\
\Sigma_{\mathbb{R}} &\to [0,1] \qquad (\Sigma_{\mathbb{R}} \subseteq \mathcal{P}\mathbb{R})
\end{aligned}
$$

- Measurable space—σ-algebra of *observable* events

$$(X, \Sigma_X)$$

- Measurable function

$$
\begin{gathered}
f : (X, \Sigma_X) \to (Y, \Sigma_Y) \\
f^{-1} : \Sigma_Y \to \Sigma_X
\end{gathered}
$$

- Category of measurable spaces: $\mathbf{Meas}$

$\mathbf{SRel}$: Stochastic relations

- Stochastic relation / transition function / sub-Markov kernel

$$f : X \times \Sigma_Y \rightarrow [0,1]$$
$$f(x,-) \text{ sub-probability measure}$$
$$f(-,B) \text{ measurable function}$$

- $\mathbf{SRel}$
  - Objects: measurable spaces (X, Σ_X)
  - Arrows: $f : X \rightarrow Y$ is a stochastic relation $X \times \Sigma_Y \rightarrow [0,1]$
  - Composition: ...

- More concisely

$$f : X \rightarrow \Pi Y \in \mathbf{Meas}$$

where $\Pi Y = \{$sub-probability measures on $Y\}$

- $\Pi : \mathbf{Meas} \rightarrow \mathbf{Meas}$ monad [Giry 81]

- $\mathbf{SRel} \cong \mathbf{Meas}_{\Pi}$

SRel: Stochastic relations

- Composition $\sim$ existential join of relations

$$f : X \to \Pi Y \qquad\qquad g : Y \to \Pi Z$$
$$f : X \times \Sigma_Y \to [0,1] \qquad\qquad g : Y \times \Sigma_Z \to [0,1]$$

$$gf(x, C) = \int_Y f(x, dy)\ g(y, C)$$

- Discrete case:

$$f : X \times Y \to [0,1] \qquad\qquad g : Y \times Z \to [0,1]$$

$$gf(x, z) = \sum_{y \in Y} f(x, y)\ g(y, z)$$

$\mathbf{SRel}$ for probabilistic while languages

- $\mathbf{Meas}$ has finite products, finite coproducts, and distributivity
  - (Think: topological spaces)
- $\mathbf{SRel}$ is partially additive $\rightsquigarrow$ iteration [Panangaden 99]
- $\mathbf{SRel}$ models probabilistic behavior

$$[\![S_1 +_p S_2]\!] = [\![\Gamma]\!] \xrightarrow{\sum\{(1-p)[\![S_1]\!],\ p[\![S_2]\!]\}} \Pi[\![\Gamma]\!]$$

$\mathbf{SRel}$ with delay

- $\Pi : \mathbf{Meas} \to \mathbf{Meas}$ strong:

$$t_{X,Y} : X \times \Pi Y \to \Pi(X \times Y)$$
$$(x, \nu) \mapsto \delta_x \times \nu$$

- $\Pi(- \times \mathcal{M}) : \mathbf{Meas} \to \mathbf{Meas}$ monad
- $\overline{- \times \mathcal{M}} : \mathbf{SRel} \to \mathbf{SRel}$ preserves partial additivity
- $\mathbf{SRel}_{\overline{- \times \mathcal{M}}}$ partially additive

- $\mathbf{SRel}_{\overline{- \times \mathcal{M}}}$ models probabilistic behavior, iteration, and delay
- Let $\mathbf{TSRel}_{\mathcal{M}} \cong \mathbf{SRel}_{\overline{- \times \mathcal{M}}}$

$\mathbf{TSRel}_{\mathcal{M}}$: Timed stochastic relations

- Composition: existential join on states, accumulate delay

$$f : X \to \Pi(Y \times \mathcal{M}) \qquad g : Y \to \Pi(Z \times \mathcal{M})$$
$$f : X \times \Sigma_{Y \times \mathcal{M}} \to [0, 1] \qquad g : Y \times \Sigma_{Z \times \mathcal{M}} \to [0, 1]$$

$$gf(x, C) = \int_{Y \times \mathcal{M}} \int_{Z \times \mathcal{M}} f(x, dy, da))\ g(y, dz, db))\ \chi_C(z, m(b, a))$$

- Discrete case:

$$f : X \times Y \times \mathcal{M} \to [0, 1] \qquad g : Y \times Z \times \mathcal{M} \to [0, 1]$$

$$gf(x, z, c) = \sum_{y \in Y, a \in \mathcal{M}} \sum_{z \in Z, b \in \mathcal{M}} f(x, y, a)\ g(y, z, b)\ \chi_{\{c\}}(m(b, a))$$

$\mathbf{TSRel}_{\mathcal{M}}$ for timed probabilistic while languages

- Models delay

$$[\![\text{wait } E]\!] = [\![\Gamma]\!] \xrightarrow{\langle 1, [\![E]\!]\rangle} [\![\Gamma]\!] \times \mathcal{M} \xrightarrow{\eta^{\Pi}} \Pi([\![\Gamma]\!] \times \mathcal{M})$$

$$[\![\text{pwait } E]\!] = [\![\Gamma]\!] \xrightarrow{\langle 1, [\![E]\!]\rangle} [\![\Gamma]\!] \times \Pi\mathcal{M} \xrightarrow{t} \Pi([\![\Gamma]\!] \times \mathcal{M})$$

- Models probabilistic behavior

$$[\![S_1 +_p S_2]\!] = [\![\Gamma]\!] \xrightarrow{\sum\{(1-p)[\![S_1]\!],\ p[\![S_2]\!]\}} \Pi([\![\Gamma]\!] \times \mathcal{M})$$

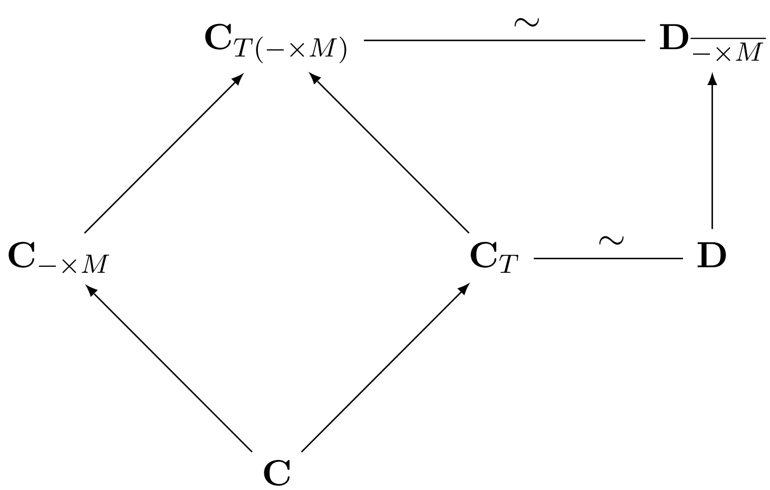
$$[\![v \leftarrow E]\!] = [\![\Gamma]\!] \times [\![\tau]\!] \times [\![\Gamma']\!] \xrightarrow{\langle \pi_1, [\![E]\!], \pi_3\rangle} [\![\Gamma]\!] \times \Pi[\![\tau]\!] \times [\![\Gamma']\!] \xrightarrow{\hat{t};\ \Pi\eta^{-\times\mathcal{M}}} \Pi([\![\Gamma]\!] \times [\![\tau]\!] \times [\![\Gamma']\!] \times \mathcal{M})$$

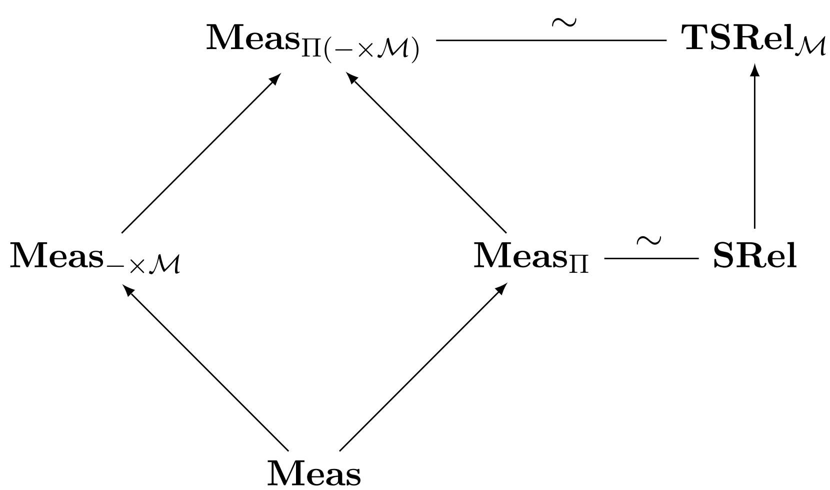
- wait and $\leftarrow$ primitive

$$[\![\text{pwait } E]\!] = [\![\text{let } v = 0 \text{ in } v \leftarrow E; \text{wait } v]\!] \qquad (v \notin \mathrm{fv}(E))$$

$$[\![S_1 +_p S_2]\!] = [\![\text{let } v = \text{true in } v \leftarrow \text{bern(p)}; \text{if } v \text{ then } S_1 \text{ else } S_2]\!] \qquad (v \notin \mathrm{fv}(S_1, S_2))$$

Summary





Thanks!