DATA MINING TECHNIQUES
Review of Probability Theory

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Review of Probability Theory

Based on "Review of Probability Theory" from CS 229 Machine Learning, Stanford University
(Handout posted on the course website)

Elements of Probability

- Sample space Ω: the set of all the outcomes of an experiment
- Event space F: a collection of possible outcomes of an experiment. $F \subseteq \Omega$.
- Probability measure: a function P: $F \rightarrow R$ that satisfies the following properties:
  - $P(A) \geq 0 \ \forall \ A \in F$
  - $P(\Omega) = 1$
  - If $A_1, A_2, \ldots$ are disjoint events, then $P(\cup_i A_i) = \sum_i P(A_i)$

Properties of Probability

- If $A \subseteq B \implies P(A) \leq P(B)$
- $P(A \cap B) \leq \min\ (P(A), P(B))$
- $P(A \cup B) \leq P(A) + P(B)$ (Union Bound)
- $P(\Omega \setminus A) = 1 - P(A)$
- If $A_1, \ldots, A_k$ is a disjoint partition of Ω, then

$$\sum_{i=1}^{k} P(A_k) = 1$$

Conditional Probability

- A conditional probability $P(A|B)$ measures the probability of an event A after observing the occurrence of event B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Two events A and B are independent iff $P(A|B) = P(A)$ or equivalently, $P(A \cap B) = P(A)P(B)$

Conditional Probability Examples

- A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?
- In New England, 84% of the houses have a garage and 65% of the houses have a garage and a back yard. What is the probability that a house has a backyard given that it has a garage?

Independent Events Examples

- What's the probability of getting a sequence of 1,2,3,4,5,6 if we roll a dice six times?
- A school survey found that 9 out of 10 students like pizza. If three students are chosen at random with replacement, what is the probability that all three students like pizza?

Random Variable

A random variable X is a function that maps a sample space Ω to real values. Formally,

$$X : \Omega \longrightarrow R$$

Examples:

- Rolling one dice
  $X =$ number on the dice at each roll
- Rolling two dice at the same time
  $X =$ sum of the two numbers

Random Variable

A random variable can be continuous. E.g.,

- $X =$ the length of a randomly selected phone call
  (What's the Ω?)
- $X =$ amount of coke left in a can marked 12oz
  (What's the Ω?)

Probability Mass Function

If X is a discrete random variable, we can specify a probability for each of its possible values using the probability mass function (*PMF*). Formally, a *PMF* is a function p: $\Omega \longrightarrow R$ such that

$$p(x) = P(X = x)$$

- Rolling a dice:
  $p(X = i) = \frac{1}{6} \quad i = 1, 2, \ldots, 6$
- Rolling two dice at the same time:
  $X =$ sum of the two numbers
  $p(X = 2) = \frac{1}{36}$

Probability Mass Function

- $X \sim Bernoulli(p),\ p \in [0,1]$

$$p(x) = \begin{cases} p & \text{if} \quad x = 1 \\ 1-p & \text{if} \quad x = 0 \end{cases}$$

- $X \sim Binomial(n,p),\ p \in [0,1]$ and $n \in Z^+$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- $X \sim Geometric(p),\ p > 0$

$$p(x) = p(1-p)^{x-1}$$

- $X \sim Poisson(\lambda),\ \lambda > 0$

$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

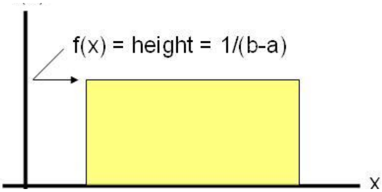
Probability Density Function

- If X is a continuous random variable, we can NOT specify a probability for each of its possible values (why?)
- We use a probability density function *PDF* to describe the relative likelihood for a random variable to take on a given value
- A (*PDF*) specifies the probability of X takes a value within a range. Formally, a *PDF* is a function $f(x)$: $\Omega \longrightarrow R$ such that

$$P(a < X < b) = \int_a^b f(x)dx$$

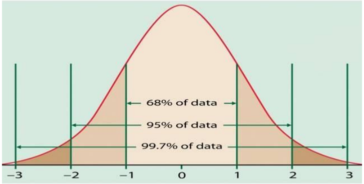
Probability Density Function

- $X \sim$ uniform on $[a, b]$:



$$f(x) = \frac{1}{b-a}$$

- $X \sim N(\mu, \sigma)$:



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

Joint Probability Mass Function

If we have two discrete random variables X, Y, we can define their joint probability mass function (PMF) $p_{XY} : R^2 \longrightarrow [0, 1]$ as:

$$p(x, y) = P(X = x, Y = y)$$

where $p(x, y) \leq 1$ and $\sum_{x \in X} \sum_{y \in Y} p(x, y) = 1$

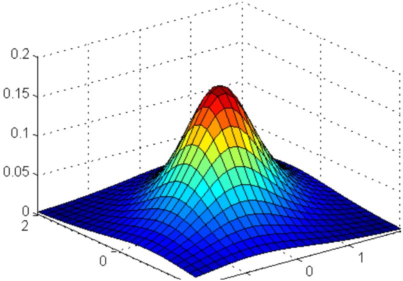
- X, Y: rolling two dice
  $p(x, y) = \frac{1}{36} \quad x, y = 1, 2, \ldots, 6$
- X: rolling one dice $\quad Y$: drawing a colored ball
  $p(6, green) = ? \quad p(5, red) = ?$

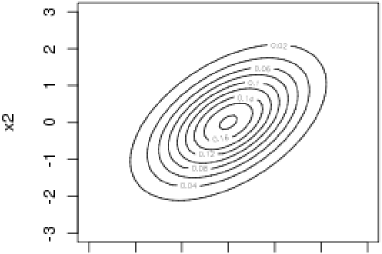
Joint Probability Density Function

If we have two continuous random variables X, Y, we can define their joint probability density function (PDF) $f_{XY} : R^2 \longrightarrow [0, 1]$ as:

$$P(a < X < b, c < Y < d) = \int_c^d \int_a^b f(x, y)dxdy$$

- 2D Gaussian





Marginal Probability Mass Function

How does the joint *PMF* over two discrete variables relate to the *PMF* for each variable separately? It turns out that

$$p(x) = \sum_{y \in Y} p(x, y)$$

- X, Y: rolling two dice

$$p(x, y) = \frac{1}{36} \quad x, y = 1, 2, \ldots, 6$$

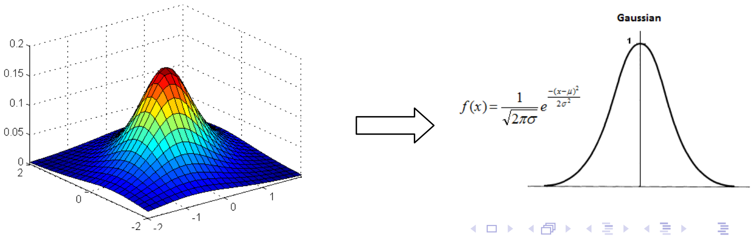
$$p(x) = \sum_{y=1}^{6} p(x, y) = \frac{1}{6}$$

Marginal Probability Density Function

Similarly, we can obtain a marginal *PDF* (also called marginal density) for a continuous random variable from a joint *PDF*:

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

- Integrating out one variable in the 2D Gaussian gives a 1D Gaussian in either dimension



Conditional Probability Distribution

A conditional probability distribution defines the probability distribution over Y when we know that X must take on a certain value x

- Discrete case: conditional *PMF*

$$p(y|x) = \frac{p(x,y)}{p(x)} \Longleftrightarrow p(x,y) = p(y|x)p(x)$$

- Continuous case: conditional *PDF*

$$f(y|x) = \frac{f(x,y)}{f(x)} \Longleftrightarrow f(x,y) = f(y|x)f(x)$$

Marginal vs. Conditional

- **Marginal probability:**

$i \backslash j$	1	2	3	4	5	6	$p_X(i)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p_Y(j)$	1/6	1/6	1/6	1/6	1/6	1/6	

- **Conditional probability: probability of rolling a 2**

$i \backslash j$	1	2	3	4	5	6	$p_X(i)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p_Y(j)$	1/6	1/6	1/6	1/6	1/6	1/6	

Bayes Rule

- We can express the joint probability in two ways:

$$p(x,y) = p(y|x)p(x)$$
$$p(x,y) = p(x|y)p(y)$$

- Bayes rule:

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)} \qquad \text{(discrete)}$$

$$f(y|x) = \frac{f(x|y)f(y)}{f(x)} \qquad \text{(continuous)}$$

Bayes Rule Application

A patient underwent a HIV test and got a positive result. Suppose we know that

- Overall risk of having HIV in the population is 0.1%
- The test can accurately identify 98% of HIV infected patients
- The test can accurately identify 99% of healthy patients

What's the probability the person indeed infected HIV?

Bayes Rule - Application

We have two random variables here:

- $X \in \{+, -\}$: the outcome of the HIV test
- $C \in \{Y, N\}$: the patient has HIV or not

We want to know: $P(C{=}\mathbf{Y}|X{=}+)$?

Apply Bayes rule:

$$P(C{=}Y|X{=}+) = \frac{P(X{=}+|C{=}Y)P(C{=}Y)}{P(X{=}+)}$$

$P(X{=}+|C{=}Y) = 0.98$ $\quad$ $P(C{=}Y) = 0.001$

$P(X{=}+) = 0.98 * 0.001 + (1\text{-}0.99) * 0.999 = 0.01097$

Answer: $0.98 * 0.001/0.01097 = 8.9\%$

Bayes Rule Terminology

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

$P(Y)$: prior probability or, simply, prior

$P(X|Y)$: conditional probability or, likelihood

$P(X)$: marginal probability

$P(Y|X)$: posterior probability or, simply, posterior

Independence

Two random variables X and Y are independent iff

- For discrete random variables
  $$p(x, y) = p(x)p(y) \quad \forall x \in X, y \in Y$$
- For discrete random variables
  $$p(y|x) = p(y) \qquad \forall y \in Y \text{ and } p(x) \neq 0$$
- For continuous random variables
  $$f(x, y) = f(x)f(y) \quad \forall x, y \in R$$
- For continuous random variables
  $$f(y|x) = f(y) \qquad \forall y \in R \text{ and } f(x) \neq 0$$

Multiple Random Variables

Extend to multiple random variables :

- Joint Distribution (discrete):

$$p(x_1, \ldots, x_n) = P(X1 = x_1, \ldots, X_n = x_n)$$

- Conditional Distribution (chain rule - discrete)

$$p(x_1, \ldots, x_n) = p(x_n|x_1, \ldots, x_{n-1})p(x_1, \ldots, x_{n-1})$$

$$= p(x_n|x_1, \ldots, x_{n-1})p(x_{n-1}|x_1, \ldots, x_{n-2})p(x_1, \ldots, x_{n-2})$$

$$= p(x_1) \prod_{i=2}^{n} p(x_i|x_1, \ldots, x_{i-1})$$

(continuous case can be defined similarly using PDF)

Multiple Random Variables

- Independence:

  Discrete case: $X_1, \ldots, X_n$ are independent iff

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p(x_i)$$

  Continuous case: $X_1, \ldots, X_n$ are independent iff

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i)$$

Multiple Random Variables

- Bayes rule:

Discrete case:

$$p(x_n|x_1,\ldots,x_{n-1}) = \frac{p(x_1,\ldots,x_{n-1}|x_n)p(x_n)}{p(x_1,\ldots,x_{n-1})}$$

Continuous case:

$$f(x_n|x_1,\ldots,x_{n-1}) = \frac{f(x_1,\ldots,x_{n-1}|x_n)f(x_n)}{f(x_1,\ldots,x_{n-1})}$$

Probabilistic View of a Dataset

What about a dataset $S = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_N, y_N)\}$?

- We can view S as $d+1$ random variables where d is the number of attributes in $\mathbf{x}$, i.e.

$$X_1,\ X_2,\ \ldots,\ X_d,\ Y$$

- Uncover(model) $p(x_1, x_2, \ldots, x_d, y)$ from the training data
- For ANY $(x_1, x_2, \ldots, x_n)$, we will compute:

$$P(y = 0 | x_1, x_2, \ldots, x_n)\ ?$$

$$P(y = 1 | x_1, x_2, \ldots, x_n)\ ?$$

That is predicting y from $\mathbf{x}$!