CS 7880 Graduate Cryptography	October 1, 2015

Lecture 6: PRG with 1-bit stretch implies arbitrary stretch

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1 Topic Covered

Creating a PRG in 3 steps:

- Creating a PRG with constant stretch from a PRG with stretch 1
- Creating a PRG with polynomial stretch from a PRG with stretch 1
- Creating a PRG with stretch 1 from a OWF

2 Increasing the stretch of a PRG

2.1 Previous Definitions

Definition 1 We define computational indistinguishability $X \approx Y$ between ensembles $X = \{X_n\}_{n\in\mathbb{N}}$ and $Y = \{Y_n\}_{n\in\mathbb{N}}$ as $\forall$ PPT D, $\exists\, \varepsilon(n) = \mathsf{negl}(n)$ such that

$$|\Pr[D(X_n) = 1] - \Pr[D(Y_n) = 1]| \leq \varepsilon(n)$$

$\diamondsuit$

Definition 2 A function $G : \{0,1\}^* \to \{0,1\}^*$ is a PRG with stretch $\ell(n)$ if

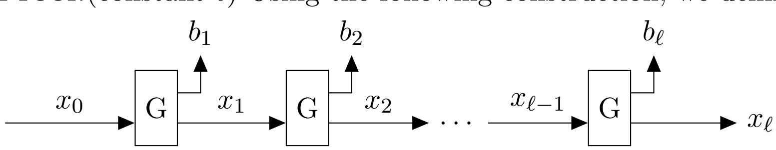
$$G(U_n) \approx U_{n+\ell(n)}$$

where U_m denotes the uniform distribution over $\{0,1\}^m$. $\diamondsuit$

2.2 Increasing stretch from 1 to a constant

Theorem 1 *If $\exists$ PRG G with 1-bit stretch, then $\forall\, \ell(n) = \mathsf{poly}(n), \exists$ PRG G^ℓ with $\ell(n)$-bit stretch.*

Proof:(constant ℓ) Using the following construction, we define $G^\ell(x_0) = (b_1, b_2, \ldots\ b_\ell, x_\ell)$



Or in psuedocode:

$$G^\ell(x_0) = \begin{cases} \textbf{for } i \in \{1, \ldots\ \ell\} \\ \quad (x_i, b_i) := G(x_{i-1}) \\ \textbf{output } (b_1, \ldots\ b_\ell, x_\ell) \end{cases}$$

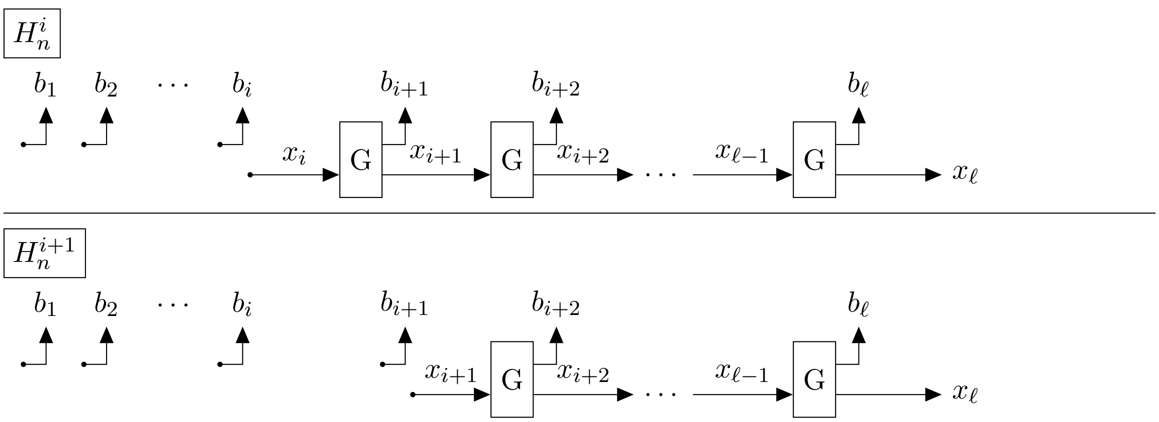
To prove this is a PRG, we need to show that if we could break G^ℓ then we could break G.

Recall:
Hybrid argument: If $X \approx Y$ and $Y \approx Z$, then $X \approx Z$.

We will define some hybrid, in-between distributions then show that every step of the chain is computationally indistinguishable from the next. We define:

$$
\begin{array}{ll}
H_n^0 := G^\ell(U_n) & \\
\hline
 & b_1, \dots b_i \leftarrow \{0,1\} \\
H_n^i := & x_i \leftarrow \{0,1\}^n \\
 & (b_{i+1}, \dots b_\ell, x_\ell) := G^{\ell-i}(x_i) \\
\hline
H_n^\ell := U_{n+\ell} &
\end{array}
$$

We want to show that any two adjacent hybrids are indistinguishable. Here's a representation of the difference between two:



Claim 1 $\forall\, i \in \{0, 1, \dots \ell - 1\}$, $H^i \approx H^{i+1}$

Idea: If we can distinguish between hybrids, we can distinguish between $(x_{i+1}, b_{i+1}) = G(x_i)$ and (x_{i+1}, b_{i+1}) being uniformly random. This is the only difference between Hybrids H^i and H^{i+1}.

Proof: We define a PPT function f_i as

$$
f_i(x_{i+1}, b_{i+1}) = \begin{cases}
b_1, \dots b_i \leftarrow \{0,1\}^n \\
\textbf{for } j \in \{i+2, \dots \ell\} \\
\quad (x_j, b_j) := G(x_{j-1}) \\
\textbf{output } (b_1, \dots b_\ell, x_\ell)\}
\end{cases}
$$

We note that the distribution of $f_i(U_{n+1})$ is related to H, in particular

$$
f_i(U_{n+1}) \equiv H_n^{i+1} \quad \text{and}
$$
$$
f_i(G(U_n)) \equiv H_n^i
$$

Where "≡" means equal distributions.
Last time we claimed that if $X \approx Y$ and f is a PPT function then $f(X) \approx f(Y)$. By this claim and assumption of security of G, we know $H^i \approx H^{i+1}$. Now we know

$$H^0 \approx H^1 \approx \cdots \approx H^\ell$$

and by the hybrid argument

$$G^\ell(U_n) \equiv H^0 \approx H^\ell \equiv U_{n+\ell}$$

Which proves G^ℓ is a PRG.

□

2.3 Increasing stretch from 1 to a polynomial

However, that proof only works for constant ℓ. We now want to extend the proof to any polynomial $\ell(n)$. (Side note: we are only dealing with the cases where $\ell(n)$ is computible in polynomial time.) We use almost the exact same construction as last time, just changing ℓ to $\ell(n)$:

$$G^\ell(x_0) = \begin{cases} \textbf{for } i \in \{1, \ \ldots \ \ell(n)\} \\ \quad (x_i, b_i) := G(x_{i-1}) \\ \textbf{output } (b_1, \ \ldots \ b_{\ell(n)}, x_{\ell(n)}) \end{cases}$$

The analysis is almost the same, but now our hybrids look like:

$$\{H_n^i\}_{n\in\mathbb{N}, i\in\{0, \ \ldots \ \ell(n)-1\}}$$

Claim 2 *If for all polynomials $i(n)$ such that $i(n) \in \{0, \ \ldots \ \ell(n) - 1\}$ we have*

$$\{H_n^{i(n)}\}_{n\in\mathbb{N}} \approx \{H_n^{i(n)+1}\}_{n\in\mathbb{N}}$$

then

$$\{H_n^0\}_{n\in\mathbb{N}} \approx \{H_n^{\ell(n)}\}_{n\in\mathbb{N}}$$

We need this claim because while we could use the hybrid argument for a known number of ensembles, now the number of hybrid ensembles depends on n.

Proof: Let D be a PPT distinguisher between $\{H_n^0\}_{n\in\mathbb{N}}$ and $\{H_n^{\ell(n)}\}_{n\in\mathbb{N}}$.

$$\begin{aligned} & \left|\Pr[D(H_n^0) = 1] - \Pr[D(H_n^{\ell(n)}) = 1]\right| \\ = & \left|\sum_{i=0}^{\ell(n)-1} \Pr[D(H_n^i) = 1] - \Pr[D(H_n^{i+1}) = 1]\right| \\ \le & \sum_{i=0}^{\ell(n)-1} \underbrace{\left|\Pr[D(H_n^i) = 1] - Pr[D(H_n^{i+1}) = 1]\right|}_{\delta_n^i} \\ \le & \ \ell(n) \cdot \left|\Pr[D(H_n^{i^*(n)}) = 1] - Pr[D(H_n^{i^*(n)+1}) = 1]\right| \end{aligned}$$

Where $i^*(n) = \underset{i \in \{0, \ldots \ell(n)-1\}}{\arg\max} \delta_n^i$.
Essentially, we are bounding every term in the sum by the worst case term. Since by assumption, $\left|\Pr[D(H_n^{i^*(n)}) = 1] - Pr[D(H_n^{i^*(n)+1}) = 1]\right|$ is negligable, we can conclude that $\ell(n) \cdot \mathsf{negl}(n)$ is also negligable. $\square$

To prove $\{H_n^{i(n)}\}_{n\in\mathbb{N}} \approx \{H_n^{i(n)+1}\}_{n\in\mathbb{N}}$ would be the same as proving $H^i \approx H^{i+1}$ in the fixed ℓ case (Claim 1), but there is an additional difficulty: $i(n)$ may not be efficiently computable.
There are at least two ways different ways we could deal with this:

1. Use the non-uniform model of computation, which equips a TM with some fixed lookup value of n. This can also be viewed as a family of algorithms indexed by n.
2. Instead of changing our model of computation, we can make a stronger claim by using a weaker assumption:

Claim 3 *Let I_n be uniform over $\{0, \ldots \ell(n-1)\}$. If $H^{I_n} \approx H^{I_n+1}$ then $H_n^0 \approx H_n^{\ell(n)}$*

Proof: (Similar to Claim 2).

$$
\begin{aligned}
&\left|\Pr[D(H_n^0) = 1] - \Pr[D(H_n^{\ell(n)}) = 1]\right| \\
&= \left|\sum_{i=0}^{\ell(n)-1} \Pr[D(H_n^i) = 1] - \Pr[D(H_n^{i+1}) = 1]\right| \\
&= \left|\sum_{i=0}^{\ell(n)-1} \Pr[D(H_n^{I_n}) = 1 \mid I_n = i] - \Pr[D(H^{I_n+1}) = 1 \mid I_n = i]\right| \\
&= \ell(n) \cdot \left|\sum_{i=0}^{\ell(n)-1} Pr[D(H_n^{I_n}) = 1, I_n = i] - \Pr[D(H^{I_n+1}) = 1, I_n = i]\right| \\
&= \ell(n) \cdot \left|Pr[D(H_n^{I_n}) = 1] - \Pr[D(H^{I_n+1}) = 1]\right| \\
&= \mathsf{negl}(n)
\end{aligned}
$$

$\square$

Now to finish the proof that G^ℓ is a PRG we need to show $H^{I_n} \approx H^{I_n+1}$
We change our definition of f_i to f_{I_n}

$$
f_{I_n}(x, b) = \begin{cases}
\textbf{pick } i \leftarrow I_n \\
(x_{i+1}, b_{i+1}) := (x, b) \\
b_1, \ldots b_{I_n} \leftarrow \{0,1\}^n \\
\textbf{for } j \in \{I_n + 2, \ldots \ell\} \\
\quad (x_j, b_j) := G(x_{j-1}) \\
\textbf{output } (b_1, \ldots b_\ell, x_\ell)\}
\end{cases}
$$

The rest of the proof is identical to before. Using Claim 3, we know

$$G^{\ell(n)}(U_n) \equiv H_n^0 \approx H_n^{\ell(n)} \equiv U_{n+\ell(n)}$$

Which shows G^ℓ is a PRG for any computible $l(n) = \mathsf{poly}(n)$.

3 Creating a PRG from a OWF

We've shown that PRG's of larger stretch can be constructed from a PRG with 1-bit stretch. Now we need to construct such a PRG from a OWF. It's slightly suprising that this can be done, since the requirement of uniformity doesn't seem to be provided by a OWF.

DEFINITION 3 A OWF $f : \{0,1\}^* \to \{0,1\}^*$ is a one way permutation (OWP) when both

- $|f(x)| = |x| \quad \forall\, x$
- $\forall\, x \neq x',\ f(x) \neq f(x')$

$\diamondsuit$

Note that this definition implies that f is one-to-one and onto.

Idea: We construct $G = (f(x), \mathsf{hc}(x))$ for some $\mathsf{hc} : \{0,1\}^* \to \{0,1\}$.
We want to exploit the fact that there is some information in x that is unknown and hard to recover.
As a first attempt, would defining $\mathsf{hc}(x) = x[1]$ produce a good PRG? Unfortunately, this won't work for arbitrary OWP f. As a counterexample, let f' be a OWP, and $f(x) = (x[1], f'(x[2\ \ldots\ n]))$. We can show that $f'(x)$ is a valid OWP, since a preimage of f' would result in a preimage of f, but $G(x) = (f(x), hc(x))$ would always output equal first and last bits, so G could be easily distinguished from U_n, and wouldn't be a PRG.

To be continued: finding a good $\mathsf{hc}(x)$...