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FIXED-POINT CONSTRUCTIONS IN ORDER-ENRICHED CATEGORIES*

Mitchell WAND
Computer Science Department, Indiana University, Bloomington, IN 47401, U.S.A.

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Abstract. The fixed-point construction of Scott, giving a continuous lattice solution of equations $X \cong T(X)$ where T is an endofunctor on the category of continuous lattices, is extended to categories enriched by partial orderings on the morphism sets. The result allows data structures to be realized not only in the category of continuous lattices, but also in the category of complete lattices, in the category of complete partial orders, or in any of several related categories of partial orders.

1. Introduction

A key feature of lattice-oriented theories of computation is the specification of objects as solutions of fixed-point equations $X \cong T(X)$. When X ranges over the elements of a complete lattice, a canonical solution is supplied by the Tarski fixpoint theorem. Typical applications include languages [3, 26] and programs in assorted variations [6, 11, 25]. Scott defined lattice-theoretic models of the lambda-calculus [19, 21] and of several other structures [18, 20] by solving similar equations where X ranged over the class of continuous lattices. Reynolds [16] showed the existence of canonical solutions for a large class of functors T, and Lawvere [19, p. 129] pointed out that the result in the case $T(X) = [X \to X]$ is a consequence of the fact that certain direct and inverse limits coincided.

In this paper we extend these results from the category of complete lattices to any category on which each morphism set has a well-behaved complete partial ordering. These include the original case of continuous lattices, complete lattices, complete partial orders, powers of these categories, and the category of directed complete relations. Thus many of the repetitious verifications of details are "factored out" into the proof of the general theorem, leaving a smaller portion which must be worked out for each category under consideration. By claifying and separating the properties of the general construction from the properties of the

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individual categories, we hope to give a more elegant analysis of this class of problems.

It is worthwhile to explore the analogy of the standard fixpoint construction. If L is a complete lattice, $f: L \to L$ a continuous function, then one constructs

$$x_0 = \perp,$$
$$x_{k+1} = f(x_k).$$

Then $y = \bigsqcup x_k$ is a fixed point of f, and it is "least" in the sense that if $f(z) \leq z$, then $y \leq z$. To get the fixed-point property, we calculate

$$f(y) = f(\bigsqcup x_k) = \bigsqcup f(x_k) = \bigsqcup x_{k+1} = \bigsqcup x_k = y.$$

The "least" property is obtained by showing that if $f(z) \leq z$ and then $x_k \leq z$ for every k (by induction on k). If L is regarded as a category with $L(x, y) = \{1\}$ if $x \leq y$ and $\emptyset$ otherwise, then least upper bounds are colimits and f is an endofunctor which preserves directed colimits.

Hence, to solve a fixpoint equation in some appropriate category, starting with an initial object a, one sets

$$x_0 = a,$$
$$x_{k+1} = Tx_k,$$
$$y = \operatorname{colim} x_k.$$

Then $Ty = T(\operatorname{colim} x_k) \cong \operatorname{colim} Tx_k \cong \operatorname{colim} x_{k+1} = y$.

The correctness of this construction, in the case where the category has colimits and T preserves ω-colimits, was shown by Smyth and Plotkin [14]. The main new result of this paper, Theorem 3.1, gives a sufficient condition for the existence of these colimits in terms of the existence of limits, which are generally easier to supply. Again we have a "least" property, which says that if z is any object of the category and there is a morphism $Tz \to z$ (analogous to $T(z) \leq z$), then there is a unique morphism $y \to z$ satisfying an appropriate diagram condition. This forces y to be unique up to isomorphism. Last, in Section 4, we give some examples of categories and functors included by the theory.

Our use of enriched categories is also worthy of note. One of the dogmas of category theory is that all of the interesting structure in a category lies in its morphisms [8]. If we are interested in ordered structures, then it becomes plausible to study categories with ordered morphism sets [2, Section 4E].[1] In this case, we are then able to prove theorems about classes of categories rather than single categories.

[1] See also [9], in which category-enriched categories are studied.

2. Preliminaries

We presume familiarity with the standard notions of category, morphism, functor, limit, colimit, and cone [10]. We denote categories with boldface type, e.g., **K**, **KP**, $\boldsymbol{\omega}$. The set of morphisms from object x to object y in category **C** is denoted $\mathbf{C}(x, y)$. We compose morphisms from left to right: if $f \in \mathbf{C}(x, y)$ and $g \in \mathbf{C}(y, z)$, then $fg \in \mathbf{C}(x, z)$. (This will eventually make the subscript conventions more tractable.) We write application from right to left: if $T:\mathbf{C} \to \mathbf{D}$ and $U:\mathbf{D} \to \mathbf{E}$ are functors, and $k \in \mathbf{C}(x, y)$, then $UTk \in \mathbf{E}(UTx, UTy)$; similarly, if ϕ is an I-indexed family and $i \in I$, then ϕi is the element corresponding to i. We will also use center dot ($\cdot$) for composition and add parentheses as needed for clarity. We will say **C** has **D**-(co-)limits iff every $T:\mathbf{D} \to \mathbf{C}$ has a (co-)limit.

Let ω denote the category whose objects are the nonnegative integers, with $\omega(k, n) = \{(k, n)\}$ if $k \leqslant n$ and $= \emptyset$ otherwise.

Proposition 2.1. *ω is the category freely generated by the graph whose set of objects is ω and whose edges are $(k, k+1)$ for each k.*

Let **O** be the category whose objects are partially-ordered sets X such that every ω-chain $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$ of elements of X has a least upper bound and whose morphisms are maps which preserve lub's of ω-chains. Let U be the forgetful functor $\mathbf{O} \to \mathbf{SETS}$. Clearly **O** has finite products under the componentwise ordering.

Proposition 2.2. *Let X and Y be two objects in* **O**, *let $\{x_i\}$ be an ω-chain in X and let $\{y_i\}$ be an ω-chain in Y. Then in $X \times Y$, $(\bigsqcup_i x_i, \bigsqcup_i y_i) = \bigsqcup_i (x_i, y_i)$.*

Definition 2.3. A category **K** is *order-enriched* by giving, for each hom-set $\mathbf{K}(x, y)$, a relation $\sqsubseteq_{(x,y)}$ such that $(\mathbf{K}(x, y), \bigsqcup_{(x,y)})$ is an object of **O** and such that for each x, y, z, the composition map $\mathbf{K}(x, y) \times \mathbf{K}(y, z) \to \mathbf{K}(x, z)$ is a morphism in **O**. We write $\mathbf{K}(x, y)$ for both the hom-set and the object in **O**.

An order-enriched category is just an **O**-category in the sense of [7] or [10, pp. 180–181]. This ordering requirement is weaker than one might expect, as we do not even require that morphism sets have least elements. In fact, every category is order-enriched under the ordering which makes every pair of distinct morphisms incomparable. Our primary interest, of course, is in orders which are nontrivial. Still, **O** is sufficiently close to **SETS** that elementwise arguments are feasible:

Proposition 2.4. *If $f_k \in \mathbf{K}(x, y)$ and $g_k \in \mathbf{K}(y, z)$ are ω-chains of morphisms in an order-enriched category, then $(\bigsqcup_k f_k) \cdot (\bigsqcup_k g_k) = \bigsqcup_k f_k g_k$.*

Proof. Immediate from Proposition 2.2 and the continuity of composition.

Definition 2.5. Given an order-enriched category **K**, let **KP** denote the category whose objects are the objects of **K** and whose morphisms are given by $\mathbf{KP}(x, y) = \mathbf{K}(x, y) \times \mathbf{K}(y, x)$, with $\langle f, g\rangle \cdot \langle f', g'\rangle = \langle ff', g'g\rangle$. The identity morphisms $\langle 1, 1\rangle$ of **KP** will be denoted 1. Let **KR** (the category of **K**-*projections*) be the subcategory of **KP** whose objects are those of **K** and whose morphisms $\mathbf{KR}(x, y)$ consist of pairs $\langle f, g\rangle \in \mathbf{K}(x, y) \times \mathbf{K}(y, x)$ such that $fg = 1$ and $gf \sqsubseteq 1$.

If $\langle f, g\rangle \in \mathbf{KR}(x, y)$, we occasionally refer to f as the *embedding* and g as the *retraction* of $\langle f, g\rangle$.

If **K** is a category of data types, a morphism in $\mathbf{KR}(x, y)$ may be thought of as an injection of the data type x into the "larger" type y [20]. The name "projection", of course, conflicts with the standard notion of projection maps from a product to its components, but the latter notion does not arise in this paper until Section 4. We will occasionally write "projection pair" instead of "projection" for a morphism of **KR**.

Proposition 2.6. *φ is an isomorphism iff $\varphi = \langle f, f^{-1}\rangle$ for some morphism f of* **K**.

Proposition 2.7. (i) *If $\langle f, g\rangle$ and $\langle f', g\rangle$ are projections, then $f = f'$.*
(ii) *If $\langle f, g\rangle$ and $\langle f, g'\rangle$ are projections, then $g = g'$.*

Proof. (i) $f' \sqsubseteq f'gf = f$, and similarly $f \sqsubseteq f'$.
(ii) $g' = g'fg \sqsubseteq g$, and similarly $g \sqsubseteq g'$.

Proposition 2.7 (i) implies that **KR** is isomorphic to the sub-category of **K** whose morphisms are "embeddings", i.e. first elements of projections.[2] Most of our concern is with **K** and **KR**; we use **KP** only occasionally. Dually, by Proposition 2.7 (ii), **KR** is isomorphic to the subcategory of $\mathbf{K}^{\mathrm{op}}$ whose morphisms are "retractions," i.e. second elements of projection pairs.[2]

Definition 2.8. If $\mathbf{K}, \mathbf{K}'$ are order-enriched categories, a functor $T : \mathbf{K} \to \mathbf{K}$ is *continuous on morphism sets* iff for each $x, y \in \mathbf{K}$, the map $\mathbf{K}(x, y) \to \mathbf{K}'(Tx, Ty)$ given by $f \mapsto Tf$ is a morphism of **O**.

This is another way of saying that T is an **O**-functor [7].

Proposition 2.9. *If $T : \mathbf{K} \to \mathbf{K}'$ is continuous on morphism sets, and f_i is a monotonic ω-chain of morphisms, then $\bigsqcup Tf_i = T(\bigsqcup f_i)$.*

Since we will spend a great deal of time manipulating limits, it is worthwhile to review the relevant concepts.

[2] Isomorphic as categories, but not as order-enriched categories.

If T is a functor $\mathbf{D} \to \mathbf{K}$ and x is an object of $\mathbf{K}$, a *cone from x to T* is a family ϕ of morphisms of $\mathbf{K}$, indexed by the objects of $\mathbf{D}$, such that for each object d of $\mathbf{D}$, $\phi d \in \mathbf{K}(x, Td)$, and for each morphism $h \in \mathbf{D}(d, d')$, the following diagram in $\mathbf{K}$ commutes:

$$\begin{tikzcd} & x \arrow[dl, "\phi d"'] \arrow[dr, "\phi d'"] & \\ Td \arrow[rr, "Th"'] & & Td' \end{tikzcd}$$

Typically $\mathbf{D}$ will be ω or ω^{op}. x is the *apex* of the cone.

If ϕ is a cone from x to T and ϕ' is a cone from y to T, then $f \in \mathbf{K}(x, y)$ is a *mediating arrow* from ϕ to ϕ' iff for each object d of $\mathbf{D}$, the following diagram commutes:

$$\begin{tikzcd} x \arrow[d, "f"'] \arrow[dr, "\phi d"] & \\ y \arrow[r, "\phi' d"'] & Td \end{tikzcd}$$

γ is a *limiting cone* of T iff for any cone ϕ to T, there is a unique mediating arrow from ϕ to γ. We often write ϕ^* for this mediating arrow when T is clear from context. We refer to the apex of a limiting cone as $\lim T$. Limits are, of course, unique up to isomorphism. The dual notion is a *cone from T to x*, and a *colimit*.

3. Results

The first theorem establishes a sufficient condition for the category $\mathbf{KR}$ to have ω-colimits. These colimits turn out to coincide with ω^{op}-limits in $\mathbf{K}$.

Theorem 3.1. *Let $\mathbf{K}$ be an order-enriched category with ω^{op}-limits. Then $\mathbf{KR}$ has ω-colimits.*

The proof proceeds by definitions and lemmas.[3]

Definition 3.2. Let $\xi = \{\xi k : k \in \omega\}$ be a family of morphisms in $\mathbf{KR}$ with common codomain x. ξ is said to have *property ρ* iff $\xi k = \langle f_k, g_k \rangle$ and (i) $g_k f_k \sqsubseteq g_{k+1} f_{k+1}$ for $k \in \omega$ and (ii) $\bigsqcup_k g_k f_k = 1$.

[3] The theorem is a refinement of one proved by the author under some additional assumptions about the behaviour of limits in K. Gordon Plotkin showed that the additional conditions could be removed; the present arrangement of the proof is due to D. Lehmann.

Lemma 3.3. *Let* $\mathbf{K}$ *be an order-enriched category with* ω^{op}*-limits, and let* $L:\omega\to\mathbf{KR}$ *be any functor. Then there is an object* L^* *of* $\mathbf{KR}$ *and a* ξ *from* L *to* L^* *which has property* ρ*. Furthermore, the cone formed by the retractions of* ξ *is a limiting cone for the functor* $L':\omega^{\mathrm{op}}\to\mathbf{K}$ *obtained by keeping the retractions and forgetting the embeddings.*

Proof. Let $L:\omega\to\mathbf{KR}$ be given by $n\mapsto Ln$; $(n,m)\mapsto\langle f_{nm}, g_{mn}\rangle(n\leqslant m)$. We will construct colim L. Let $L':\omega^{\mathrm{op}}\to\mathbf{K}$ be $n\mapsto Ln$, $(m,n)\mapsto g_{mn}$ $(m\geqslant n)$. Let $L^*=\lim L'$, with $\gamma:n\mapsto g_{\infty n}$ the limiting cone. The cone γ is shown in Fig. 1.

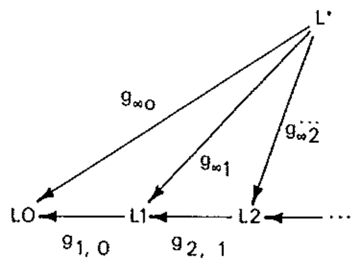

Fig. 1

We must next supply arrows $f_{n\infty}:Ln\to L^*$ which will turn Fig. 1 into a cone from L to L^*. To supply an arrow $f_{n\infty}:Ln\to L^*$, we construct a cone ϕ_n from Ln to L'; then the mediating arrow will serve for $f_{n\infty}$.

For each n, define

$$\phi_n:k\mapsto\begin{cases} g_{nk} & k\leqslant n,\\ f_{nk} & n\leqslant k.\end{cases}$$

To show that ϕ_n is a cone in $\mathbf{K}$ from Ln to T, we must show that if $m\geqslant k$, $(\phi_n m)\cdot g_{mk}=\phi_n k$. (Note that if $m<k$, there is no morphism in ω^{op} and hence nothing to prove.) If $n\geqslant m$, then $n\geqslant k$ and $\phi_n m\cdot g_{mk}=g_{nm}g_{mk}=g_{nk}=\phi_n k$. If $n\leqslant k$, then $m\geqslant n$, so $\phi_n m\cdot g_{mk}=f_{nm}g_{mk}=f_{nk}f_{km}g_{mk}=f_{nk}=\phi_n k$. Since $k\leqslant m$, this takes care of all values of n. So ϕ_n is a cone from Ln to T.

Let $f_{n_\infty}\in\mathbf{K}(Ln,L^*)$ be the mediating arrow $\phi_n\to\gamma$. Thus, $f_{n\infty}g_{\infty k}=\phi_n k$. In particular $f_{n\infty}g_{\infty n}=1_{Ln}$.

Let $\xi n=\langle f_{n\infty}, g_{\infty n}\rangle$. To show that $\{\xi n:n\in\omega\}$ is a cone from L to L^*, we must show that $f_{n\infty}=f_{n,n+1}f_{n+1,\infty}$. But for any k,

$$f_{n,n+1}f_{n+1,\infty}g_{\infty k}=f_{n,n+1}\phi_{n+1}k=\phi_n k=f_{n\infty}g_{\infty k},$$

so the equality holds by uniqueness of the mediating arrow.

For condition (i) of property ρ, we calculate:

$$g_{\infty n}f_{n\infty}=g_{\infty,n+1}g_{n+1,n}f_{n,n+1}f_{n+1,\infty}\sqsubseteq g_{\infty,n+1}f_{n+1,\infty}.$$

For condition (ii) we show that $\bigsqcup_k g_{\infty k} f_{k\infty}$ is a mediating arrow $\gamma \to \gamma$. For any n,

$$\Big(\bigsqcup_k g_{\infty k} f_{k\infty}\Big) g_{\infty n} = \bigsqcup_k g_{\infty k} f_{k\infty} g_{\infty n} = \bigsqcup_{k \geq n} g_{\infty k} g_{kn} = g_{\infty n}.$$

Since 1 is also a mediating arrow $\gamma \to \gamma$, the uniqueness property allows us to conclude $\bigsqcup_k g_{\infty k} f_{k\infty} = 1$. Consequently, $g_{\infty k} f_{k\infty} \sqsubseteq 1$, and $\langle f_{k\infty}, g_{\infty k}\rangle$ is a morphism of **KR**.

Lemma 3.4. *Let* **K** *be an order-enriched category,* $L : \omega \to \mathbf{KR}$ *a functor, and* $\xi : L \to L^*$ *a cone with property* ρ. *Then*

(i) ξ *is a colimiting cone in* **KR**;

(ii) *the retractions of* ξ *are a limiting cone in* **K** *to* $L' : \omega^{\mathrm{op}} \to \mathbf{K}$ *obtained from* L *by keeping the retractions*;

(iii) *the embeddings of* ξ *are a colimiting cone in* **K** *from* $L'' : \omega \to \mathbf{K}$ *obtained from* L *by keeping the embeddings.*

Proof. (ii) and (iii) are dual; we prove (ii). Let $\xi n = \langle f_{n\infty}, g_{\infty n}\rangle$, and let $\{g_{Mn} : n \in \omega\}$ be a cone in **K** from an object M to L'. We claim the mediating arrow is $\bigsqcup_k g_{Mk} f_{k\infty}$. We must first show that the $g_{Mk} f_{k\infty}$ form an ω-chain:

$$g_{Mk} f_{k\infty} = g_{M,k+1} g_{k+1,k} f_{k,k+1} f_{k+1,\infty} \sqsubseteq g_{M,k+1} f_{k+1,\infty}.$$

Hence the indicated lub exists. To verify that this is a mediating arrow we calculate, for any n:

$$\begin{aligned}\Big(\bigsqcup_k g_{Mk} f_{k\infty}\Big) g_{\infty n} &= \bigsqcup_k g_{Mk} f_{k\infty} g_{\infty n} = \bigsqcup_{k \geq n} g_{Mk} f_{k\infty} g_{\infty n} \\ &= \bigsqcup_{k \geq n} g_{Mk} g_{kn} \\ &= g_{Mn}.\end{aligned}$$

So this is a mediating arrow. For uniqueness, let α be any mediating arrow from $\{g_{Mn}\}$ to $\{g_{\infty n}\}$. Then

$$\alpha = \alpha 1 = \alpha \cdot \Big(\bigsqcup_k g_{\infty k} f_{k\infty}\Big) = \bigsqcup_k \alpha g_{\infty k} f_{k\infty} = \bigsqcup_k g_{Mk} f_{k\infty},$$

thus establishing uniqueness.

For (i), let $\{\langle f_{nM}, g_{Mn}\rangle : n \in \omega\}$ be a cone in **KR** from L to some object M. By (ii) and (iii) then exist $g_{M\infty} \in \mathbf{K}(M, L^*)$ and $f_{\infty M} \in \mathbf{K}(L^*, M)$ which uniquely mediate the retractions and embeddings. Hence $\langle f_{\infty M}, g_{M\infty}\rangle$ is the unique mediating arrow $\xi \to \{\langle f_{nM}, g_{Mn}\rangle\}$. It remains only to show that $\langle f_{\infty M}, g_{M\infty}\rangle$ is a morphism of **KR**.

$$\begin{aligned}
f_{\infty M} g_{M\infty} &= \Big(\bigsqcup_k g_{\infty k} f_{k\infty}\Big) f_{\infty M} g_{M\infty} \Big(\bigsqcup_k g_{\infty k} f_{k\infty}\Big)\\
&= \bigsqcup_k g_{\infty k} f_{k\infty} f_{\infty M} g_{M\infty} g_{\infty k} f_{k\infty} \quad \text{(Proposition 2.4)}\\
&= \bigsqcup_k g_{\infty k} f_{kM} g_{Mk} f_{k\infty}\\
&= \bigsqcup_k g_{\infty k} f_{k\infty}\\
&= 1.
\end{aligned}$$

$$\begin{aligned}
g_{M\infty} f_{\infty M} &= g_{Mk} g_{k\infty} f_{\infty k} f_{kM} \quad \text{for any } k\\
&\sqsubseteq g_{Mk} f_{kM}\\
&\sqsubseteq 1.
\end{aligned}$$

Lemmas 3.3 and 3.4 complete the proof of Theorem 3.1.

Theorem 3.5. *Let* **K** *be an order-enriched category with* ω^{op}*-limits, and let* $T:\mathbf{KR}\to\mathbf{KR}$ *preserve property* ρ. *Then* T *preserves* ω*-colimits in* **KR**.

Proof. Immediate from Lemma 3.4 (i).

Theorems 3.1 and 3.5 give us conditions on **K** and T which enable us to apply the general fixed-point construction sketched in Section 1. Our account of this construction follows that of Plotkin and Smyth [14]. If **C** is any category with initial object and $T:\mathbf{C}\to\mathbf{C}$ is any functor, let **PFP**(T) denote the category whose objects are diagrams in **C**:

$$\mathrm{M} \xleftarrow{\eta} \mathrm{TM}$$

and whose morphisms $\eta \to \eta'$ are those morphisms $\sigma \in \mathbf{C}(\mathrm{cod}(\eta), \mathrm{cod}(\eta))$ such that

$$\begin{tikzcd}
\mathrm{M} \arrow[d, "\sigma"'] & \mathrm{TM} \arrow[l, "\eta"'] \arrow[d, "\mathrm{T}\sigma"] \\
\mathrm{M}' & \mathrm{TM}' \arrow[l, "\eta'"]
\end{tikzcd}$$

commutes.

Theorem 3.6 [14]. *Let* C *be a category with* ω*-colimits and an initial object, and let* $T : \mathbf{C} \to \mathbf{C}$ be any functor that preserves ω-colimits. Then $\mathbf{PFP}(T)$ *has an initial object* $\psi : TL^* \to L^*$ *which is an isomorphism in* C.

Proof. Let x_0 be an initial object of C and let θ_0 be the unique morphism in $\mathbf{C}(x_0, Tx_0)$. Define $L : \omega \to \mathbf{C}$ by

$$L0 = x_0, \qquad L(0,1) = \theta_0,$$

$$L(k+1) = TLk, \qquad L(k, k+1) = \theta_k = T\theta_{k-1}.$$

Let $L^* = \operatorname{colim} L$ with ξ the colimiting cone. Next construct a cone μ from TL to L^* by setting $\mu k = \xi(k+1)$. Since T preserves ω-colimits, TL^* is a colimit of TL, with colimiting cone $T\xi$. So we have a unique arrow $\psi \in \mathbf{C}(TL^*, L^*)$ mediating between $T\xi$ and μ, that is, for any k, $T\xi k \cdot \psi = \xi(k+1)$. We claim ψ is the desired initial object.

Let $\eta \in \mathbf{C}(TM, M)$ be any object of $\mathbf{PFP}(T)$. Define a cone ν from L to M by

$$\nu(0) = \alpha, \quad \text{the unique morphism in } \mathbf{C}(x_0, M),$$

$$\nu(k+1) = T\nu k \cdot \eta.$$

To show that ν is a cone, we verify by induction that $\theta_n \cdot \nu(n+1) = \nu n$: For $n = 0$, $\theta_0 \cdot \nu 1 = \theta_0 \cdot T\nu 0 \cdot \eta = \theta_0 \cdot T\alpha \cdot \eta = \alpha = \nu 0$. Assume the identity holds for $n = k$. Then

$$\begin{aligned}
\theta_{k+1} \cdot \nu(k+2) &= T(\theta_k) \cdot T\nu(k+1) \cdot \eta && \text{(definition of } \theta, \nu) \\
&= T(\theta_k \cdot \nu(k+1)) \cdot \eta && (T \text{ is a functor}) \\
&= T\nu k \cdot \eta && \text{(by induction hypothesis)} \\
&= \nu(k+1) && \text{(definition of } \nu).
\end{aligned}$$

We must show that there is a unique morphism σ such that

$$\begin{tikzcd}
TL^* \arrow[r, "\psi"] \arrow[d, "T\sigma"'] & L^* \arrow[d, "\sigma"] \\
TM \arrow[r, "\eta"] & M
\end{tikzcd}$$

commutes. We will show that σ makes the diagram commute iff σ mediates between the cones ξ and ν. Since the mediating arrow exists and is unique, this will complete the proof of initiality.

First, assume σ is the mediating arrow from ξ to ν, that is, $\xi k \cdot \sigma = \nu k$. Since $T\xi$ is a colimiting cone, it will suffice to show that for any k, $T\xi k \cdot \psi \cdot \sigma = T\xi k \cdot T\sigma \cdot \eta$:

$$\begin{aligned}
(T\xi k)\cdot \psi \cdot \sigma &= \xi(k+1)\cdot \sigma && \text{(mediating property of } \psi) \\
&= \nu(k+1) && \text{(mediating property of } \sigma) \\
&= T\nu k \cdot \eta && \text{(definition of } \nu) \\
&= T(\xi k \cdot \sigma)\cdot \eta && \text{(mediating property of } \sigma) \\
&= T\xi k \cdot T\sigma \cdot \eta && (T \text{ is a functor}).
\end{aligned}$$

Last, assume σ makes the square commute. We must show that $\xi k \cdot \sigma = \nu k$. We proceed by induction on k. For $k = 0$, the equation holds by initiality of x_0. Assume $(\xi k)\cdot \sigma = \nu k$. Then

$$\begin{aligned}
\xi(k+1)\cdot \sigma &= T\xi k \cdot \psi \cdot \sigma && \text{(mediating property of } \varphi) \\
&= T\xi k \cdot T\sigma \cdot \eta && \text{(since the square commutes)} \\
&= T(\xi k \cdot \sigma)\cdot \eta && (T \text{ is a functor}) \\
&= T\nu k \cdot \eta && \text{(by induction hypothesis)} \\
&= \nu(k+1).
\end{aligned}$$

Last, we construct an inverse for ψ as follows. Define a cone ν from L to TL^* via

$$\nu 0 = \text{the unique morphism } x_0 \to TL^*,$$

$$\nu(k+1) = T\xi k.$$

Let θ be the mediating arrow from ξ to ν, so $\xi k \cdot \theta = \nu k$. Then

$$\xi(k+1)\cdot \theta \cdot \psi = \nu k \cdot \psi = T\xi k \cdot \psi = \xi(k+1)$$

and

$$T\xi k \cdot \psi \cdot \theta = \xi(k+1)\cdot \theta = \nu(k+1) = T\xi k.$$

Since ξ and $T\xi$ are both colimiting cones, we deduce $\theta\psi = 1$ and $\psi\theta = 1$.

4. Applications

The framework of the previous section says that one should construct domains as follows: Choose a category **K** of domains with ω^{op}-limits, and a ρ-continuous functor $T: \mathbf{KR} \to \mathbf{KR}$ which describes the self-referential properties of the desired data types. One then solves the domain equation $X \cong T(X)$ using Theorem 3.6 (by Theorem 3.1 the colimit object is constructed as an ω^{op}-limit in **K**); the solution obtained is canonical.

This section is devoted to listing some categories **K** with ω^{op}-limits and some ρ-continuous functors T. The choice of **K** and T for a particular application is often a delicate decision which is beyond the scope of this paper; our aim is merely to indicate some of the possibilities.

Example 4.1. A complete lattice is a partial order $(L, \leqslant)$ with the property that if $S \subseteq L$, then S has a least upper bound in L. We say $D \subseteq L$ is *directed* if $D \neq \emptyset$ and any pair of members of D has some upper bound in D. Let **CLD** denote the category of complete lattices with morphisms chosen to be the maps that preserve lubs of directed sets. **CLD** is order-enriched under the ordering $f \sqsubseteq g$ iff $(\forall x)[f(x) \sqsubseteq g(x)]$. Then **CLD** has ω^{op}-limits.

Proof. Let $G : \omega^{\mathrm{op}} \to \mathbf{CLD}$. Denote $G(n, k)$ by $g_{nk} (n \geqslant k)$. Let $L_\infty = \{(x_0, x_1, \ldots)$: $x_i \in Gi$ & $(\forall n \in \omega)(\forall k \in \omega)[n \geqslant k \Rightarrow g_{nk}(x_n) = x_k]\}$ under the ordering $(x_0, x_1, \ldots) \sqsubseteq (y_0, y_1, \ldots)$ iff $(\forall i \in \omega)[x_i \sqsubseteq y_i]$.

To show L_∞ is a complete lattice, let $S \subseteq L_\infty$. Let $S_k = \{x_k : (x_0, x_1, \ldots, x_k, \ldots) \in S\} \subseteq Gk$. Then for each k, S_k has a least upper bound $S_k^* \in Gk$. Let $y_k = \bigsqcup_{n \geqslant k} g_{nk}(S_n^*)$. We claim that $y = (y_0, y_1, \ldots)$ is the least upper bound of S. We must first show that $y \in L_\infty$. If $k \leqslant n$, note $S_k^* = \bigsqcup \{x_k : x \in S\} = \bigsqcup \{g_{nk}(x_n) : x \in S\} \sqsupseteq g_{nk}(\bigsqcup \{x_n : x \in S\}) = g_{nk}(S_n^*)$. Therefore, if $m \geqslant n \geqslant k$, then $g_{nk}(S_n^*) \sqsupseteq g_{nk}(g_{mn}(S_m^*)) = g_{mk}(S_m^*)$, so the terms in the construction of y_k are an ω-chain. Hence, if $n \geqslant k$, then $y_k = \bigsqcup_{m \geqslant k} g_{mk}(S_m^*) = \bigsqcup_{m \geqslant n} g_{mk}(S_m^*) = \bigsqcup_{m \geqslant n} g_{nk}(g_{mn}(S_m^*)) = g_{nk}(y_n)$. So $y = (y_0, y_1, \ldots) \in L_\infty$.

To show that y is the least upper bound of S, we first observe from the definition of y_k that $y_k \sqsupseteq S_k^*$. If $x = (x_0, x_1, \ldots) \in S$, then for each k, $x_k \in S_k$, so $x_k \sqsubseteq S_k^* \sqsubseteq y_k$. Hence y is an upper bound for S in L_∞. Next, let $z = (z_0, z_1, \ldots)$ be another upper bound for S in L_∞. Then for every n, $S_n^* \sqsubseteq z_n$. Now $z \in L_\infty$, so for every $n \geqslant k$, $z_k = g_{nk}(z_n)$. So $z_k = \bigsqcup_{n \geqslant k} g_{nk}(z_n) \sqsupseteq \bigsqcup_{n \geqslant k} g_{nk}(S_n^*) = y_k$. So $y \sqsubseteq z$, and y is the least upper bound. (This construction is of course due to Scott.)

The maps $g_{\infty k} : L_\infty \to Gk : (x_0, x_1, \ldots) \mapsto x_k$ form a cone and preserve lubs of directed sets. To verify the limit property, let $n \mapsto g_{Mn}$ be a cone from M to G. Then for $m \in M$, $(g_{M0}(m), \ldots, g_{Mn}(m), \ldots) \in L_\infty$ since the g_{Mn} form a cone, and $g_{M\infty} : m \mapsto (g_{M0}(m), \ldots, g_{Mn}(m), \ldots)$ is also a morphism in **CLD**.

So $g_{M\infty}$ is a mediating arrow. The uniqueness of $g_{M\infty}$ is assured by the fact that the underlying set of L_∞ is a limit in **SETS**.

As was pointed out by Scott, L_∞ is a subset and sub-poset of $\prod_k Gk$, but not a sublattice; lubs of ω-chains, however, are formed componentwise.

Example 4.2. O, CPC (the full subcategory of objects of **O** with bottom element), and **CPC*** (**CPC** restricted to bottom-preserving maps) [12] all have ω^{op}-limits.

Proof. Mutatis mutandis from the previous proof.

Example 4.3. Any finite product of categories with ω^{op}-limits has ω^{op}-limits.

Thus we can solve systems of several mutually recursive simultaneous domain equations. Another example is Reynolds' category of directed complete relations [17]:

Example 4.4. Let **RCL** denote the category whose objects are triples (L, R, L') where L and L' are complete lattices and $R \subseteq L \times L'$ has the property that if $\Delta \subseteq L \times L'$ is directed and $\Delta \subseteq R$, then lub $\Delta \in R$; the morphisms $(L, R, L') \to (M, S, M')$ of **RCL** are pairs (f, g) where $f \in \mathbf{CLD}(L, M)$, $g \in \mathbf{CLD}(L', M')$ and for all $(x, y) \in L \times M$, if $(x, y) \in R$, then $(f(x), g(y)) \in S$ (Reynolds' category $\mathcal{R}$ is **RCL-R**). Then **RCL** has ω^{op}-limits.

Proof. Let $G : \omega^{\mathrm{op}} \to \mathbf{RCL}$. Denote Gk by (L_k, R_k, L'_k) and $G(n, k)$ by $G_{nk} = (g_{nk}, g'_{nk})$. Let L_∞, L'_∞ be limits of the L_k and L'_k respectively (i.e. of the appropriate functors $\omega^{\mathrm{op}} \to \mathbf{RCL} \to \mathbf{CLD}$) constructed as in Example 4.1, with limiting cones $g_{\infty n}, g'_{\infty n}$, and let $G_{\infty n} = (g_{\infty n}, g'_{\infty n})$. Let

$$R_\infty = \{(x, y) \in L_\infty \times L'_\infty : (\forall n)(G_{\infty n}(x, y) \in R_n)\}.$$

We claim that $(L_\infty, R_\infty, L'_\infty)$ is a limit, with the cone given by the $G_{\infty n}$.

We must first show that this construction makes $(L_\infty, R_\infty, L'_\infty)$ an object of **RCL**. Let $\Delta \subseteq L_\infty \times L'_\infty$ be directed and $\Delta \subseteq R_\infty$. We must show that lub $\Delta \in R$. Let $\Delta_k = \{(x_k, x'_k) : (\exists \delta \in \Delta)[G_{\infty k}(\delta) = (x_k, x'_k)]\}$. Each Δ_k is directed and $\Delta_k \subseteq R_k$, so $\Delta^*_k = \text{lub } \Delta_k \in R_k$. Recalling the construction of lubs in Example 4.1, and using the fact that lubs in product lattices are constructed componentwise, we see that $G_{\infty k}(\text{lub } \Delta) = \bigsqcup_{n \geq k} G_{nk}(\Delta^*_n)$. Now $\Delta^*_n \in R_n$, so $G_{nk}(\Delta^*_n) \in R_k$. Hence $G_{\infty k}(\text{lub } \Delta)$ is a lub of an ω-chain in $L_k \times L'_k$, each of whose elements belongs to R_k. So $G_{\infty k}(\text{lub } \Delta) \in R_k$ for each k. So lub $\Delta \in R_\infty$. Thus R_∞ has the required property.

To verify the limit property, let (M, S, M') be an object of **RCL** and let (g_{Mn}, g'_{Mn}) form a cone from (M, S, M') to G. Since L_∞ and L'_∞ were constructed as limits, there exists a unique pair $(g_{M\infty}, g'_{M\infty})$ of morphisms which will mediate between the morphisms of the cones. It remains only to show that $(g_{M\infty}, g'_{M\infty}) \in \mathbf{RCL}((M, S, M'), (L_\infty, R_\infty, L'_\infty))$. Let $(m, m') \in S$. Then for each k, $(g_{Mk}(m), g'_{Mk}(m')) \in R_k$. But $(g_{Mk}(m), g'_{Mk}(m)) = (g_{\infty k}(g_{M\infty}(m)), g'_{\infty k}(g'_{M\infty}(m))) = G_{\infty k}((g_{M\infty}(m), g'_{M\infty}(m))) \in R_k$. So $(g_{M\infty}(m), g'_{M\infty}(m)) \in R_\infty$, as desired.

This category is typically used for comparing different semantic schemes [17] rather than for constructing domains. Plotkin's **SFP** [13] also appears to have the required properties.

To catch the category of continuous lattices, we need an embedding theorem:

Proposition 4.5. *Let* **C** *be any category with an initial object and* ω*-colimits, and let* $T:\mathbf{C}\to\mathbf{C}$ *be a functor which preserves* ω*-colimits. Let* $\mathbf{C}'$ *be a full subcategory of* **C** *such that*

(i) $\mathbf{C}'$ *is closed under isomorphic copies of objects*;
(ii) $\mathbf{C}'$ *is closed under* T;
(iii) Colim T *is an object of* $\mathbf{C}'$.

Let T' *denote the restriction of* T *to* $\mathbf{C}'$. *Then* $\mathbf{PFP}(T')$ *has an initial object which is an isomorphism in* **C**.

Proof. $\mathbf{PFP}(T')$ is a full subcategory of $\mathbf{PFP}(T)$ which, by (iii), includes the initial object of $\mathbf{PFP}(T)$.

Example 4.6. Let **CONTL** be the full subcategory of **CLD** whose objects are the continuous lattices [19]. Let $T:\mathbf{CLDR}\to\mathbf{CLDR}$ be a ρ-preserving functor such that **CONTL** is closed under T, and let T' denote the restriction of T to **CONTL**. Then $\mathbf{PFP}(T')$ has an initial object which is an isomorphism in **CONTL**.

Proof. By Theorem 3.1, colim T is the limit of the retractions of T; by [19, Proposition 4.1], colim T is a continuous lattice.

For a starting point in the construction, we usually choose an initial object of **KR**:

Proposition 4.7. *For any of the categories* **K** *of Examples 4.1–4.3 the one-point order is initial in* **KR**.

For some constructions, however, the initial object is not the appropriate starting place. The following proposition ensures that we can start with any x_0 so long as we can provide a starting morphism $x_0\to Tx_0$:

Proposition 4.8 (Plotkin). *Let* **C** *be any category with* ω*-colimits, and let* x *be any object of* **C**. *Let* **D** *denote the category whose objects are morphisms* α *of* **C** *whose domain is* x, *and whose morphisms* $\alpha\to\alpha'$ *are those morphisms* $\sigma\in\mathbf{C}(\mathrm{cod}(\alpha),\mathrm{cod}(\alpha'))$ *such that*

$$\begin{tikzcd} & x \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ y \arrow[rr, "\sigma"'] & & y' \end{tikzcd}$$

commutes. Then **D** *has* ω*-colimits, and the forgetful functor* $\mathbf{D}\to\mathbf{C}$ *preserves them. Furthermore, the identity morphism on* x *is an initial object of* **D**.

Given a functor $T:\mathbf{C}\to\mathbf{C}$ and a morphism $\theta_0: x\to Tx$, we can extend T to functor $T':\mathbf{D}\to\mathbf{D}$ via $T'\alpha=\theta_0\cdot T\alpha$. This, in effect, starts the iterative construction at x.

We may now start to consider, for some fixed suitable $\mathbf{K}$, some functors $\mathbf{KR}\to\mathbf{KR}$ which are ρ-preserving.

Proposition 4.9. *The class of ρ-preserving functors $T:\mathbf{KR}\to\mathbf{K'R}$ is closed under composition and includes the projection functors $\mathbf{K}^n\mathbf{R}\to\mathbf{KR}$.*

Proposition 4.10. *Let* $\mathbf{OC}$ *be the graph whose objects are small order-enriched categories* $\mathbf{K}$, *with* $\mathbf{OC}(\mathbf{K},\mathbf{L})$ the set of ρ-preserving functors $\mathbf{KR}\to\mathbf{LR}$. *Then* $\mathbf{OC}$ *is a category.*

The usefulness of this proposition is limited by the fact that most of the interesting categories $\mathbf{K}$ are not small.

Proposition 4.11. *If $T:\mathbf{KP}\to\mathbf{LP}$ is continuous on morphism sets, and has the property that if $T(\langle f,g\rangle)=\langle f',g'\rangle$, then $T(\langle g,f\rangle)=\langle g',f'\rangle$, then the restriction of T to* $\mathbf{KR}$ *is a ρ-preserving functor* $\mathbf{KR}\to\mathbf{LR}$.

Proof. Let $\langle f,g\rangle$ be a projection, and let $T(\langle f,g\rangle)=\langle f',g'\rangle$. Then

$$\begin{aligned}\langle g'f',g'f'\rangle&=\langle g',f'\rangle\cdot\langle f',g'\rangle=T(\langle g,f\rangle)\cdot T(\langle f,g\rangle)\\&=T(\langle g,f\rangle\cdot\langle f,g\rangle)=T(\langle gf,gf\rangle)\sqsubseteq T(\langle 1,1\rangle)=1.\\\langle f'g',f'g'\rangle&=\langle f',g'\rangle\cdot\langle g',f'\rangle=T(\langle f,g\rangle)\cdot T(\langle g,f\rangle)\\&=T(\langle f,g\rangle\cdot\langle g,f\rangle)=T(\langle fg,fg\rangle)=T(\langle 1,1\rangle)=1.\end{aligned}$$

Let $\xi=\{\langle f_k,g_k\rangle:k\in\omega\}$ be family of morphisms with property ρ. Let $T\xi k=\langle f'_k,g'_k\rangle$. We must show that the $g'_kf'_k$ form an ω-chain with a lub of 1.
For the ω-chain, we calculate:

$$\begin{aligned}\langle g'_kf'_k,g'_kf'_k\rangle&=T(\langle g_k,f_k\rangle)\cdot T(\langle f_k,g_k\rangle)\\&\sqsubseteq T(\langle g_{k+1},f_{k+1}\rangle)\cdot T(\langle f_{k+1},g_{k+1}\rangle)\\&=\langle g'_{k+1}f'_{k+1},g'_{k+1}f'_{k+1}\rangle.\end{aligned}$$

So $g'_kf'_k\sqsubseteq g'_{k+1}f'_{k+1}$. For the limit, we calculate similarly:

$$\begin{aligned}\bigsqcup_k\langle g'_gf'_k,g'_kf'_k\rangle&=\bigsqcup_k T(\langle g_k,f_k\rangle)\cdot T(\langle f_k,g_k\rangle)\\&=\bigsqcup_k T(\langle g_kf_k,g_kf_k\rangle)\\&=T\Big(\bigsqcup_k\langle g_kf_k,g_kf_k\rangle\Big)\\&=T(1)\\&=1.\end{aligned}$$

Our major tool for constructing ρ-preserving functors is the following:

Theorem 4.12. *Let $K_1, \ldots K_n$, $\mathbf{K}$ be order-enriched categories and let $T: \mathbf{K}_1 \times \cdots \times \mathbf{K}_n \to \mathbf{K}$ be a functor continuous on the morphism sets and covariant in some arguments and contravariant in the others. Then we can construct a covariant ρ-preserving functor $T': (\mathbf{K}_1 \times \cdots \times \mathbf{K}_n)\mathbf{R} \to \mathbf{KR}$ with the same object function as T and which is given on morphisms by*

$$T'(((f_1, \ldots, f_n), (g_1, \ldots, g_n))) =$$
$$(T(k_1, \ldots, k_n), T(l_1, \ldots, l_n))$$

where

$$k_i = \begin{cases} f_i & \textit{if } T \textit{ is covariant in its } i\text{th } \textit{argument,} \\ g_i & \textit{otherwise} \end{cases}$$

and

$$l_i = \begin{cases} g_i & \textit{if } T \textit{ is covariant in its } i\text{th } \textit{argument,} \\ f_i & \textit{otherwise.} \end{cases}$$

Proof. As defined, T' is evidently a covariant functor $(\mathbf{K}_1 \times \cdots \times \mathbf{K}_n)\mathbf{P} \to \mathbf{KP}$, continuous on the morphism sets, with the symmetry property of Proposition 4.11.

We can now list examples of functors T, continuous on the morphism sets, to which Theorem 4.12 may be applied. In each case, $\mathbf{K}$ may be any of the categories of Examples 4.1–4.3.

(i) The Cartesian product functor $\times : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$.

(ii) The coproduct functor (or any of the related "union" functors)$+ : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ (See Fig. 2).

(iii) The internal hom-functor $\mathrm{Hom} : \mathbf{K}^{\mathrm{op}} \times \mathbf{K} \to \mathbf{K}$ given by $\mathrm{Hom}(L, M) = [L \to M]$; if $f \in \mathbf{K}(L, M)$ and $g \in \mathbf{K}(N, P)$, then $\mathrm{Hom}(f, g) \in \mathbf{K}([M \to N], [L \to P])$ is given by $\mathrm{Hom}(f, g)(h) = fhg$.

(iv) The diagonal functor $\Delta : \mathbf{K} \to \mathbf{K} \times \mathbf{K}$ given by $\Delta(x) = (x, x)$, $\Delta(f) = (f, f)$.

(v) All of the functors $\mathbf{K}^n \to \mathbf{K}^m$ obtained as products of projections $\mathbf{K}^n \to \mathbf{K}$ (this includes Δ as a special case).

We may now display the functors associated with some typical data structures. In each case, we may realize the structure in any category $\mathbf{K}$ to which the given functor and Theorem 3.6 apply. Unless otherwise noted, we choose $x = \{1\}$.

(*a*) Let A be an object of "atoms". Let $T(L) = \{1\} + (A \times L)$. L^* is the object of stacks of A's. The image of $\{1\}$ is the empty stack.

(b) Let A be an object of "atoms". Let $T(L) = A + (L \times L)$. L^* is the object of lists accessed by "car" and "cdr".

(c) If we wish the null list to be distinguishable, then we may set $T(L) = \{1\} + A + (L \times L)$. The choice of T depends on the use to be made of the data type, the operations desired, and the type of partial information needed. Note that

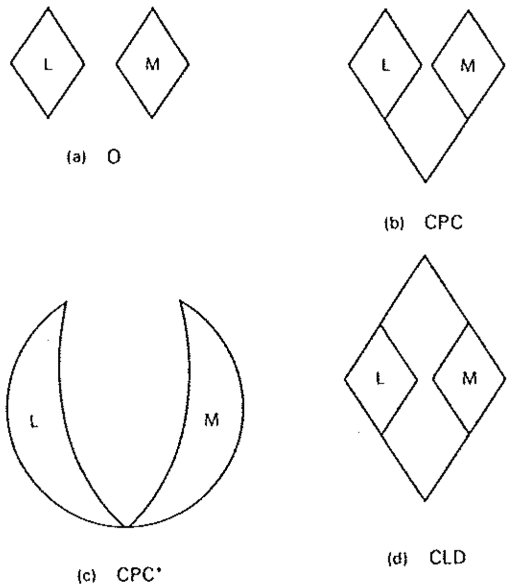

Fig. 2. Coproducts in several categories of orders ((b) and (d) are weak coproducts).

$\{1\}+A+(L\times L)$, $(\{1\}+A)+(L\times L)$, and $\{1\}+(A+(L\times L))$ are distinct, non-isomorphic lattices [1].

(d) Let $\langle\Omega, r\rangle$ be a ranked set [4]. Let $T(L)=\Sigma\{L^{r(s)}: s\in\Omega\}$. Then L^* is the object of ranked Ω-trees [23, 24]. In this case there is a compact representation of L^* as a set of trees [6, 26].

(e) Let $T=\mathrm{Hom}\circ\Delta$; thus $T(L)=[L\to L]$ and $T(\langle f, g\rangle)=\langle\mathrm{Hom}(g,f), \mathrm{Hom}(f,g)\rangle$. Choose $x=\{\perp, \perp\}$ and $\theta_0\in\mathbf{KR}(x, Tx)$, and use Proposition 4.8. If $\mathbf{K}=\mathbf{CONTL}$, then L^* is one of Scott's original models of the lambda-calculus [19].

(f) Let D be an object of $\mathbf{K}$, let $T(L)=D+[L\to L]$, $T(\langle f, g\rangle)=\langle 1_D+\mathrm{Hom}(g,f), 1_D+\mathrm{Hom}(f,g)\rangle$. Then L^* is a model for a typed lambda-calculus based on the primitive data type D.

(g) Hierarchical graphs (similar to [15]). Let G be a fixed set of unlabelled graphs. A hierarchical graph graph is to be a graph from G whose nodes are labelled with atoms A or other hierarchical graphs. For $g\in G$, let $|g|$ be the number of nodes in g. So a hierarchical graph is either an atom or a graph g with $|g|$ other hierarchical graphs as the node labels. So we have $T(L)=A+\Sigma\{L^{|g|}: g\in G\}$. This gives a representation of these objects as trees.

5. Conclusions and open problems

We extend Scott's fixed-point construction to categories enriched by an ordering on the morphism sets. This allows data structures to be realized in an assortment of categories of orders.

This construction corresponds to the construction of domains at language-definition time; by contrast, Scott's construction of domains via projections of a 'universal' domain [22] seems to correspond to the construction of domains at run-time via simulation in a fixed underlying type.

An open problem is an adequate account of the various limit-colimit coincidences that arise in these constructions.

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