The \( k \)-Induction Principle

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Consider the following standard induction principle over the natural numbers (including 0):

\[
P(0) \land \forall n (P(n) \Rightarrow P(n + 1)) \Rightarrow \forall n P(n).
\]  

(1)

An alternative is the 2-induction principle:

\[
P(0) \land P(1) \land \forall n ((P(n) \land P(n + 1)) \Rightarrow P(n + 2)) \Rightarrow \forall n P(n).
\]  

(2)

We can generalize these principles to \( k \)-induction, for \( k \geq 1 \), as follows. Let

\[
A_k := \left( \bigwedge_{i=0}^{k-1} P(i) \right) \land \forall n \left( \left( \bigwedge_{i=0}^{k-1} P(n+i) \right) \Rightarrow P(n+k) \right).
\]  

(3)

The \( k \)-induction principle now states:

\[
I_k :: A_k \Rightarrow \forall n P(n).
\]  

(4)

Note that \( I_1 \) simplifies to the standard induction principle (1), which is hence also called 1-induction. Similarly, \( I_2 \) simplifies to 2-induction (2).

In the rest of this document, we discuss the following questions:

1. Is \( k \)-induction a valid proof method?
2. Can it provide an advantage over standard induction?

Correctness of \( k \)-induction

We justify the \( k \)-induction principle using strong induction on \( n \). The strong induction principle states that the following is valid:

\[
\forall n ( (\forall m < n P(m)) \Rightarrow P(n)) \Rightarrow \forall n P(n).
\]  

(5)

To prove \( k \)-induction correct, i.e. the validity of \( A_k \Rightarrow \forall n P(n) \), for \( k \geq 1 \), assume \( A_k \) holds. We prove \( \forall n P(n) \) using (5) by proving its left-hand side. We summarize all facts we have: given \( n \),

\[
\forall m < n P(m) \quad \text{from left-hand side of (5)} \quad (6)
\]

\[
\bigwedge_{i=0}^{k-1} P(i) \quad \text{from } A_k \quad (7)
\]

\[
\forall n' ( (\bigwedge_{i=0}^{k-1} P(n'+i)) \Rightarrow P(n'+k)) \quad \text{from } A_k \text{ (} n \text{ renamed to } n' \) \quad (8)
\]

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The proof obligation is $P(n)$, the consequent of the implication in the left-hand side of (5). We distinguish two cases:

1. $k - 1 \geq n$: in that case $P(n)$ follows from (7).

2. $k - 1 < n$, i.e. $k \leq n$: in that case we prove $P(n)$ using (8). Let $n' = n - k \geq 0$, then $P(n' + k) = P(n)$; it remains to prove that $\bigwedge_{i=0}^{k-1} P(n' + i)$, which reduces to proving $P(n) \land P(n - 1) \land \ldots \land P(n - 1)$. Since $n - 1 \geq k - 1$, this follows from (7). □

Is $k$-induction “better” than standard induction?

Suppose $A_k$ holds, for some fixed $k$. By (4), therefore, $P(n)$ is valid for any $n$. This in turn means that $A_k$ in fact holds for every $k$, as is immediately obvious from the definition (3). The proof obligations $A_k$ for $k$-induction, for various $k$, are therefore all logically equivalent. How, then, can “true” $k$-induction ($k > 1$) be more useful than standard (1-)induction?

The answer is purely pragmatic: $A_k$ may in practice be easier to prove than $A_1$: the second conjunct of $A_k$, the implication, has an antecedent that gets stronger as $k$ increases, so we have more to work with. In contrast, the consequent, $P(n + k)$, is always a single instance of $P$ that needs to be proved. The fact that the first conjunct of $A_k$, the base cases, also gets stronger as $k$ increases and thus requires “more proof”, is of little consequence: the arguments to predicate $P$ are constants.

Let us look at an example. Consider the Fibonacci sequence, defined by

$$fib(n) = \begin{cases} n & \text{if } n \leq 1 \\ fib(n - 1) + fib(n - 2) & \text{otherwise.} \end{cases}$$

Suppose we want to prove $fib(n) \geq n$ for $n \geq 5$. Induction seems to lend itself! In classical (1-)induction, one would show that $fib(5) = 5 \geq 5$, and would then try to prove that $fib(n) \geq n$ implies $fib(n + 1) \geq n + 1$. The term $fib(n + 1)$ reduces to $fib(n) + fib(n - 1)$, at which point we are stuck: the induction hypothesis does not tell us anything about $fib(n - 1)$.

The solution is 2-induction: we first show that $fib(5) = 5 \geq 5$ and $fib(6) = 8 \geq 6$. This is the first conjunct of Equation (3) for $k = 2$, the base cases. The second conjunct requires us to prove that $fib(n) \geq n \land fib(n + 1) \geq n + 1$ implies $fib(n + 2) \geq n + 2$. This follows immediately from $fib(n + 2) = fib(n + 1) + fib(n)$ (and the prerequisite $n \geq 5$).