The $k$-Induction Principle

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Consider the following standard induction principle over the natural numbers (including 0):

\[ P(0) \land \forall n (P(n) \Rightarrow P(n+1)) \Rightarrow \forall n P(n). \]  

(1)

An alternative is the 2-induction principle:

\[ P(0) \land P(1) \land \forall n ((P(n) \land P(n+1)) \Rightarrow P(n+2)) \Rightarrow \forall n P(n). \]  

(2)

We can generalize these principles to $k$-induction, for $k \geq 1$, as follows. Let

\[ A_k := \left( \bigwedge_{i=0}^{k-1} P(i) \right) \land \forall n \left( \left( \bigwedge_{i=0}^{k-1} P(n+i) \right) \Rightarrow P(n+k) \right). \]  

(3)

The $k$-induction principle now states:

\[ I_k ::= A_k \Rightarrow \forall n P(n). \]  

(4)

Note that $I_1$ simplifies to the standard induction principle (1), which is hence also called 1-induction. Similarly, $I_2$ simplifies to 2-induction (2).

In the rest of this document, we discuss the following questions:

1. Is $k$-induction a valid proof method?
2. Can it provide an advantage over standard induction?

Correctness of $k$-induction

We justify the $k$-induction principle using strong induction on $n$. The strong induction principle states that the following is valid:

\[ \forall n ((\forall m < n P(m)) \Rightarrow P(n)) \Rightarrow \forall n P(n). \]  

(5)

To prove $k$-induction correct, i.e. the validity of $A_k \Rightarrow \forall n P(n)$, for $k \geq 1$, assume $A_k$ holds. We prove $\forall n P(n)$ using (5) by proving its left-hand side. We summarize all facts we have: given $n$,

\begin{align*}
\forall m < n P(m) & \quad \text{from left-hand side of (5)} \quad (6) \\
\bigwedge_{i=0}^{k-1} P(i) & \quad \text{from } A_k \quad (7) \\
\forall n'((\bigwedge_{i=0}^{k-1} P(n' + i)) \Rightarrow P(n' + k)) & \quad \text{from } A_k \quad (n \text{ renamed to } n') \quad (8)
\end{align*}
The proof obligation is $P(n)$, the consequent of the implication in the left-hand side of (5). We distinguish two cases:

1. $k - 1 \geq n$: in that case $P(n)$ follows from (7).

2. $k - 1 < n$, i.e. $k \leq n$: in that case we prove $P(n)$ using (8). Let $n' = n - k \geq 0$, then $P(n' + k) = P(n)$; it remains to prove that $\bigwedge_{i=0}^{k-1} P(n' + i)$, which reduces to proving $P(n - k) \land P(n - k + 1) \land \ldots \land P(n - 1)$. Since $n - 1 \geq k - 1$, this follows from (7).

\[\square\]

Is $k$-induction “better” than standard induction?

Suppose $A_k$ holds, for some fixed $k$. By (4), therefore, $P(n)$ is valid for any $n$. This in turn means that $A_k$ in fact holds for every $k$, as is immediately obvious from the definition (3). The proof obligations $A_k$ for $k$-induction, for various $k$, are therefore all logically equivalent. How, then, can “true” $k$-induction ($k > 1$) be more useful than standard (1-)induction?

The answer is purely pragmatic: $A_k$ may in practice be easier to prove than $A_1$: the second conjunct of $A_k$, the implication, has an antecedent that gets stronger as $k$ increases, so we have more to work with. In contrast, the consequent, $P(n + k)$, is always a single instance of $P$ that needs to be proved. The fact that the first conjunct of $A_k$, the base cases, also gets stronger as $k$ increases and thus requires “more proof”, is of little consequence: the arguments to predicate $P$ are constants.

Let us look at an example. Consider the Fibonacci sequence, defined by

$$
\text{fib}(n) = \begin{cases} 
  n & \text{if } n \leq 1 \\
  \text{fib}(n - 1) + \text{fib}(n - 2) & \text{otherwise}
\end{cases}
$$

Suppose we want to prove $\text{fib}(n) \geq n$ for $n \geq 5$. Induction seems to lend itself! In classical (1-)induction, one would show that $\text{fib}(5) = 5 \geq 5$, and would then try to prove that $\text{fib}(n) \geq n$ implies $\text{fib}(n+1) \geq n+1$. The term $\text{fib}(n+1)$ reduces to $\text{fib}(n) + \text{fib}(n-1)$, at which point we are stuck: the induction hypothesis does not tell us anything about $\text{fib}(n-1)$.

The solution is 2-induction: we first show that $\text{fib}(5) = 5 \geq 5$ and $\text{fib}(6) = 8 \geq 6$. This is the first conjunct of Equation (3) for $k = 2$, the bases cases. The second conjunction requires us to prove that $\text{fib}(n) \geq n \land \text{fib}(n + 1) \geq n + 1$ implies $\text{fib}(n + 2) \geq n + 2$. This follows immediately from the definition (and the prerequisite $n \geq 5$).

\[\square\]