The k-Induction Principle

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Consider the following standard induction principle over the natural numbers (including 0):

$$P(0) \wedge \forall n\, (P(n) \Rightarrow P(n+1)) \quad \Rightarrow \quad \forall n P(n)\,. \tag{1}$$

An alternative is the 2-induction principle:

$$P(0) \wedge P(1) \wedge \forall n\, ((P(n) \wedge P(n+1)) \Rightarrow P(n+2)) \quad \Rightarrow \quad \forall n P(n)\,. \tag{2}$$

We can generalize these principles to *k-induction*, for $k \geq 1$, as follows. Let

$$A_k \quad := \quad \left(\bigwedge_{i=0}^{k-1} P(i) \right) \wedge \forall n \left(\left(\bigwedge_{i=0}^{k-1} P(n+i) \right) \Rightarrow P(n+k) \right)\,. \tag{3}$$

The k-induction principle now states:

$$I_k \quad :: \quad A_k \;\Rightarrow\; \forall n P(n)\,. \tag{4}$$

Note that I_1 simplifies to the standard induction principle (1), which is hence also called 1-induction. Similarly, I_2 simplifies to 2-induction (2).

In the rest of this document, we discuss the following questions:

1. Is k-induction a valid proof method?
2. Can it provide an advantage over standard induction?

Correctness of k-induction

We justify the k-induction principle using *strong induction* on n. The strong induction principle states that the following is valid:

$$\forall n\, ((\forall m < n\, P(m)) \Rightarrow P(n)) \quad \Rightarrow \quad \forall n P(n)\,. \tag{5}$$

To prove k-induction correct, i.e. the validity of $A_k \Rightarrow \forall n P(n)$, for $k \geq 1$, assume A_k holds. We prove $\forall n P(n)$ using (5) by proving its left-hand side. We summarize all facts we have: given n,

$$\begin{aligned} \forall m < n\, P(m) & \qquad \text{from left-hand side of (5)} && (6)\\ \textstyle\bigwedge_{i=0}^{k-1} P(i) & \qquad \text{from } A_k && (7)\\ \forall n' ((\textstyle\bigwedge_{i=0}^{k-1} P(n'+i)) \Rightarrow P(n'+k)) & \qquad \text{from } A_k \ (n \text{ renamed to } n') && (8) \end{aligned}$$

The proof obligation is $P(n)$, the consequent of the implication in the left-hand side of (5). We distinguish two cases:

1. $k-1 \geq n$: in that case $P(n)$ follows from (7).
2. $k-1 < n$, i.e. $k \leq n$: in that case we prove $P(n)$ using (8). Let $n' = n-k \geq 0$, then $P(n'+k) = P(n)$; it remains to prove that $\bigwedge_{i=0}^{k-1} P(n'+i)$, which reduces to proving $P(n-k) \wedge P(n-k+1) \wedge \ldots \wedge P(n-1)$. Since $n-1 \geq k-1$, this follows from (7). $\square$

Is k-induction "better" than standard induction?

Suppose A_k holds, for some fixed k. By (4), therefore, $P(n)$ is valid for any n. This in turn means that A_k in fact holds for *every* k , as is immediately obvious from the definition (3). The proof obligations A_k for k-induction, for various k, are therefore all logically equivalent. How, then, can "true" k-induction ($k > 1$) be more useful than standard (1-)induction?

The answer is purely pragmatic: A_k may in practice be easier to prove than A_1: the second conjunct of A_k, the implication, has an antecedent that gets stronger as k increases, so we have more to work with. In contrast, the consequent, $P(n+k)$, is always a single instance of P that needs to be proved. The fact that the first conjunct of A_k, the base cases, also gets stronger as k increases and thus requires "more proof", is of little consequence: the arguments to predicate P are constants.

Let us look at an example. Consider the Fibonacci sequence, defined by

$$fib(n) = \begin{cases} n & \text{if } n \leq 1 \\ fib(n-1) + fib(n-2) & \text{otherwise.} \end{cases}$$

Suppose we want to prove $fib(n) \geq n$ for $n \geq 5$. Induction seems to lend itself! In classical (1-)induction, one would show that $fib(5) = 5 \geq 5$, and would then try to prove that $fib(n) \geq n$ implies $fib(n{+}1) \geq n{+}1$. The term $fib(n{+}1)$ reduces to $fib(n) + fib(n-1)$, at which point we are stuck: the induction hypothesis does not tell us anything about $fib(n-1)$.

The solution is 2-induction: we first show that $fib(5) = 5 \geq 5$ and $fib(6) = 8 \geq 6$. This is the first conjunct of Equation (3) for $k = 2$, the base cases. The second conjunct requires us to prove that $fib(n) \geq n \wedge fib(n+1) \geq n+1$ implies $fib(n+2) \geq n+2$. This follows immediately from $fib(n+2) = fib(n+1) + fib(n)$ (and the prerequisite $n \geq 5$).