A Closer Look at A_{TM}

Recall that $A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input string } w\}$.

Consider *any* TM T that recognizes A_{TM}.

- This means T takes input $\langle M, w\rangle$, where M is a TM and w is a string, and halts with *accept* iff M accepts w.
- The TM simulator Sim_{TM} we described earlier is an example of such a TM.

Define a corresponding TM S, using T as a subprocedure, as follows:

$S =$ "On input $\langle M\rangle$, where M is a TM:
1. Run T on input $\langle M, \langle M\rangle\rangle$.
2. If T accepts, *reject*; if T rejects, *accept*."

Clearly:

- $L(S) = \{\langle M\rangle \mid M \text{ is a TM that rejects } \langle M\rangle\}$
- I.e., S recognizes the language of TM encodings for TMs that reject their own encodings.

What happens when S is run with input $\langle S\rangle$?

- If S accepts $\langle S\rangle$, then:
  - T must reject $\langle S, \langle S\rangle\rangle$, so
  - $\langle S, \langle S\rangle\rangle$ does not belong to A_{TM}, so
  - S does not accept $\langle S\rangle$ – *Contradiction*
- If S rejects $\langle S\rangle$, then:
  - T must accept $\langle S, \langle S\rangle\rangle$, so
  - $\langle S, \langle S\rangle\rangle$ belongs to A_{TM}, so
  - S accepts $\langle S\rangle$ – *Contradiction*
- Thus S neither accepts nor rejects $\langle S\rangle$.
- Therefore S must loop on $\langle S\rangle$.

A Closer Look at A_{TM} (Continued)

So far:

- We assumed that T is an arbitrary recognizer for

$$A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input string } w\}.$$

- We defined a corresponding TM S as follows:

$S =$ "On input $\langle M\rangle$, where M is a TM:
1. Run T on input $\langle M, \langle M\rangle\rangle$.
2. If T accepts, *reject*; if T rejects, *accept*."

- We showed that S loops on $\langle S\rangle$.

Could T be a decider?

- If it is then S is a decider.
- But S loops on some input, namely $\langle S\rangle$.
- Thus S is not a decider.
- Therefore T cannot be a decider.

Since T was assumed to be an arbitrary recognizer for A_{TM}, we conclude that:

- *No* recognizer for A_{TM} can be a decider.
- Therefore A_{TM} is an undecidable language.

Notice the Similarity?

Undecidability of A_{TM}:

- S is a TM that recognizes the language of TM encodings for TMs that reject their own encodings.
- *Does S accept its own encoding?*

Russell's Paradox:

- Let R be the set of all sets that do not contain themselves as members. E.g.:
  - The set of all motorcycles is in R.
  - The set of all non-motorcycles is not in R.
- *Does R contain itself as a member?*

The barber paradox:

- In a certain village there is a man who is a barber. He shaves all and only those men in the village who do not shave themselves.
- *Does this barber shave himself?*

A Non-Turing-Recognizable Language

Definition. A language is *co-Turing-recognizable* if its complement is Turing-recognizable.

Theorem. A language is decidable if and only if it is Turing-recognizable and co-Turing-recognizable.

Proof.

- "Only if" direction:
  - If L is decidable, its complement $\overline{L}$ is decidable. (This was a homework problem.)
  - Since any decidable language is Turing-recognizable, it follows that both L and $\overline{L}$ are Turing-recognizable.
- "If" direction:
  - Suppose both L and $\overline{L}$ are Turing-recognizable.
  - Let M_L be a recognizer for L and let $M_{\overline{L}}$ be a recognizer for $\overline{L}$.
  - Consider the following TM:

    $M =$ "On input $\langle w \rangle$:
    1. Simulate running M_L and $M_{\overline{L}}$ in parallel on w (by using a 2-tape TM and alternately running one step of each at a time)
    2. If M_L accepts, *accept*; if $M_{\overline{L}}$ accepts, *reject*."
  - Every string w is either in L or $\overline{L}$.
  - If $w \in L$, then M_L must halt and accept it.
  - If $w \in \overline{L}$, then $M_{\overline{L}}$ must halt and accept it.
  - Thus this TM halts on any input w.
  - Therefore this TM is a decider.
  - Since it accepts a string w iff $w \in L$, it's a decider for L.
  - Therefore L is decidable.

Corollary. The complement of any undecidable Turing-recognizable language is non-Turing-recognizable.

Proof. Let L be undecidable and Turing-recognizable. If $\overline{L}$ were Turing-recognizable, L would be Turing-recognizable and co-Turing-recognizable, so it would be decidable, contradicting the assumption that it is undecidable. Therefore, $\overline{L}$ cannot be Turing-recognizable.

Corollary. $\overline{A_{\text{TM}}}$ is a non-Turing-recognizable language.

Proof. A_{TM} is Turing-recognizable since Sim_{TM} recognizes it, but, as we have just seen, it is not decidable.

The Halting Problem

The decision problem: *Given a TM M and a string w, does M halt when given input w?*

The corresponding language:

$$HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$$

Theorem. $HALT_{\mathrm{TM}}$ is an undecidable language.
Proof:

- Assume for the sake of contradiction that $HALT_{\mathrm{TM}}$ is decidable, and let H be a decider for it.
- Given any TM M, we could then combine it with H to create a decider M' for the language $L(M)$ as follows:

  $M' =$ "On input w:
  1. Run H on input $\langle M, w\rangle$. If it rejects, *reject*.
  2. Run M on w. If it accepts, *accept*; otherwise *reject*."

- Clearly:
  - Since H is assumed to be a decider, stage 1 terminates.
  - Since stage 2 is only run after H has determined that M would not loop on w, stage 2 also terminates.
  - Therefore M' halts on all inputs.
  - Therefore M' is a decider
- Also:
  - M' accepts w iff M accepts w.
  - Therefore $L(M') = L(M)$.
- Thus the assumption that $HALT_{\mathrm{TM}}$ is decidable allows a recognizer for any Turing-recognizable language to be converted into a decider for that language.
- Thus the assumption that $HALT_{\mathrm{TM}}$ is decidable implies that every Turing-recognizable language is decidable.
- Since A_{TM} is Turing-recognizable but not decidable, the assumption that $HALT_{\mathrm{TM}}$ is decidable must be false.
- Therefore $HALT_{\mathrm{TM}}$ is undecidable.

General Notion of Reducibility

Useful strategy in any problem-solving context:

- *Reduce* a problem to one or more simpler subproblems.
- Then solve the original problem by first solving these simpler subproblems.

Examples in the specific context of algorithm design:

1. Sorting a list can be reduced to the problem of finding the smallest element in any list:[1]
   - Find the smallest element in the original list.
   - Remove this element to obtain a shorter list.
   - Find the smallest element in this list.
   - Etc.
2. The divide-and-conquer strategy amounts to reducing a problem involving a large object (e.g., a list) to subproblems involving objects (e.g., sublists) of about half its size. Examples:
   - quicksort
   - merge sort
3. Our proof of the undecidability of the Halting Problem was based on:
   - assuming there was a decider for it; and
   - showing how we could use such a decider, if it exists, as a subprocedure in the design of a decider for any Turing-recognizable language.

   Thus we showed that the problem of deciding any Turing-recognizable language reduces to the problem of deciding the Halting Problem.

What these all have in common:

If we can reduce a given problem A to solvable problems $B_1, B_2, \ldots, B_k$, we can then design a procedure for solving the original problem by using solvers for $B_1, B_2, \ldots, B_k$ as subprocedures.

Two ways to take advantage of such reductions:

1. Use solvers for the "reduced-to" (i.e., simpler) problem(s) to actually design a solver for the "reduced-from" problem.
2. Assume for the sake of contradiction that the "reduced-to" problem(s) can be solved when we know the "reduced-from" problem can't be. This then proves that the assumption that the "reduced-to" problem(s) can be solved must be false, so the "reduced-to" problem(s) can't be solved either. Our third example above used a reduction for this purpose.

Important: It is this latter use of reductions that makes them such an valuable tool in theoretical computer science – to generate proofs by contradiction showing that certain algorithms *cannot* exist. This is the main use we make of them here.

[1] This particular approach to sorting is called *selection sort*.

Undecidability of E_{TM}

The decision problem: *Given a TM M, is the language M recognizes empty?*

The language: $E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \Phi\}$

Theorem. E_{TM} is undecidable.

Proof Idea:

- We assume for the sake of contradiction that this language is decidable and show that this implies that A_{TM} is decidable, which we know is false. From this contradiction we conclude that E_{TM} must be undecidable.
- The argument involves showing that the problem of deciding A_{TM} instances *reduces* to the problem of deciding E_{TM} instances.
- I.e., we show that the answer to the question of decidability of E_{TM} provides an answer to the question of decidability of A_{TM}. In particular, we show that decidability of E_{TM} implies decidability of A_{TM}.
- The way we do this is to show how a TM can transform any A_{TM} problem instance into a E_{TM} problem instance in such a way that an accept/reject decision by an assumed E_{TM} decider on the transformed instance gives rise to a corresponding decision on the original A_{TM} instance.

Here is a high-level description of the approach:

1. Transform any A_{TM} problem instance into some E_{TM} problem instance.
2. Apply the assumed E_{TM} decider to the transformed problem instance.
3. Use the answer provided by this decider to give an answer for the original A_{TM} problem instance.

This represents a particular way to design an A_{TM} decider using an E_{TM} decider as a subprocedure.

Key Challenge: determining how the transformation in step 1 of this description should be done so that the final answers provided in step 3 are valid. Some basic observations on this transformation:

- A_{TM} problem instances have the form $\langle M, w \rangle$, where M is a TM and w is a string.
- E_{TM} problem instances have the form $\langle M \rangle$, where M is a TM.
- We will use $\langle M' \rangle$ to denote the transformed version of $\langle M, w \rangle$.

Undecidability of E_{TM} (Continued)

What we need our transformation to do:

- Each $\langle M, w\rangle$ must be transformed to its corresponding $\langle M'\rangle$ in such a way that accept/reject decisions made by the E_{TM} decider correspond (one way or the other) to the correct accept/reject decisions for A_{TM}.
- This means that the language of the TM M' whose encoding is the transformed problem instance $\langle M'\rangle$ must be
  - empty whenever $\langle M, w\rangle \in A_{\text{TM}}$ (i.e., whenever M accepts w)
  - non-empty whenever $\langle M, w\rangle \notin A_{\text{TM}}$ (i.e., whenever M does not accept w)
  - or vice-versa

Consider this description of a TM M':

$M' =$ "On input x:
1. Run M on input w.
2. If M accepts, *accept*; if M rejects, *reject*."

Remarks:

- This TM will not actually be run or simulated. Instead, its encoding is all that will be used by the actual TM about to be described.
- M' has M and w built into it and ignores its input x.
- All that matters is what language M' accepts, which we examine below. Another way to design a TM that accepts exactly the same language would be to change line 2 so that if M rejects w, this TM goes into an infinite loop.

What is $L(M')$?

- If M does not accept w, this TM accepts no strings, so $L(M') = \Phi$ in this case.
- If M accepts w, this TM accepts all strings, so $L(M') = \Sigma^*$ in this case.
- That is,
$$L(M') = \begin{cases} \Sigma^* & \text{if } \langle M, w\rangle \in A_{\text{TM}} \\ \Phi & \text{if } \langle M, w\rangle \notin A_{\text{TM}}. \end{cases}$$
- Therefore $L(M')$ is non-empty exactly when M accepts w, i.e., exactly when $\langle M, w\rangle \in A_{\text{TM}}$.

Undecidability of E_{TM} (Continued)

Now that we've identified a way to transform A_{TM} problem instances into E_{TM} problem instances in a way that respects membership/non-membership distinctions, we restate the theorem and give the full proof.

Theorem. E_{TM} is undecidable.

Proof.

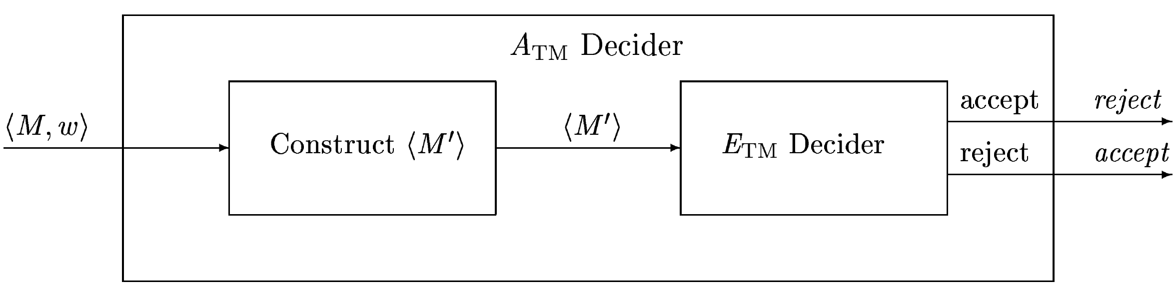
- Assume for the sake of contradiction that E_{TM} is decidable and let $D_{E_{\text{TM}}}$ be a decider for it.
- Consider the following TM:

$D_{A_{\text{TM}}} =$ "On input $\langle M, w\rangle$, where M is a TM and w is a string:

1. Construct $\langle M'\rangle$, the encoding of the following TM:
   "$M' =$ "On input x:
   1. Run M on w.
   2. If M accepts, *accept*; if M rejects, *reject*."
2. Run the emptiness decider $D_{E_{\text{TM}}}$ on input $\langle M'\rangle$.
3. If $D_{E_{\text{TM}}}$ accepts, *reject*; if $D_{E_{\text{TM}}}$ rejects, *accept*."

- Since the construction of $\langle M'\rangle$ from $\langle M, w\rangle$ can be carried out by a TM in a finite number of steps, stage 1 terminates.
- Since $D_{E_{\text{TM}}}$ is assumed to be a decider, stage 2 terminates as well.
- Therefore this TM is a decider.
- As discussed on the previous page, $L(M')$ is empty iff $\langle M, w\rangle \notin A_{\text{TM}}$.
- Therefore this TM is a decider for A_{TM}
- Since A_{TM} is undecidable, the original assumption that E_{TM} is undecidable must be false.

Here is a diagram illustrating the design of the above TM:



Mapping Reductions

Key ingredient in the proof just given that E_{TM} is undecidable:

- showing that the problem of deciding membership in A_{TM} reduces to the problem of deciding membership in E_{TM};
- more precisely, designing the "Construct $\langle M' \rangle$" box in the diagram in such a way that accept/reject decisions for the transformed E_{TM} problem instance $\langle M' \rangle$ yield correct accept/reject decisions for the original A_{TM} problem instance $\langle M, w \rangle$.

We now isolate and formalize this notion.

Suppose that:

1. A and B are languages over an alphabet Σ.
2. There is a function $f : \Sigma^* \longrightarrow \Sigma^*$ such that
   - f can be computed by a TM; and
   - $w \in A$ iff $f(w) \in B$.

Note that this function f assigns to every member of A some member of B and it assigns to every member of $\overline{A}$ some member of $\overline{B}$. Thus, to test whether a given $w \in A$, it is equivalent to test whether $f(w) \in B$. The answer to both questions is the same.

Definition. A function f is *computable* if there is a transducer TM that, when given any input w, halts with only $f(w)$ on its tape.

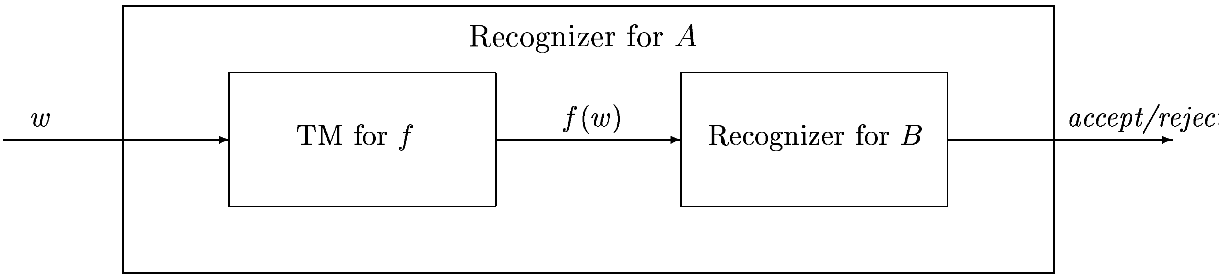
Definition. If A, B, and f satisfy conditions 1 and 2 above, then we say that f is a *mapping reduction* from A to B and that A is *mapping reducible* to B, denoted[2] $A \leq_{\mathrm{m}} B$.

This is clearly a special case of the broader notion of reducibility described earlier. When language A is mapping reducible to language B, i.e., $A \leq_{\mathrm{m}} B$, then the problem of testing membership in A reduces, in the broader sense, to the problem of testing membership in B.

[2] A helpful intuition is to think of the inequality as representing the idea that A problem instances are "no harder than" B problem instances to solve or, equivalently, that B problem instances are "at least as hard as" A problem instances to solve. Here, by "solving a problem instance" we mean determining language membership.

Implications of Mapping Reducibility

Suppose that there is a mapping reduction f from language A to language B. The following diagram depicts how a recognizer for A can be constructed by combining a TM that computes f with a recognizer for B:



Here is a description of the above TM, where F denotes the TM that computes f and M_B denotes the recognizer for B:

$M_A =$ "On input w:
1. Run F on w to compute $f(w)$.
2. Run M_B on $f(w)$. If it accepts, *accept*; if it rejects, *reject*."

Theorem. Suppose $A \leq_{\mathrm{m}} B$. Then:

1. If B is Turing-recognizable, then A is Turing-recognizable.
2. If B is decidable, then A is decidable.
3. If A is non-Turing-recognizable, then B is non-Turing-recognizable.
4. If A is undecidable, then B is undecidable.

Proof. Let f denote the reduction. Recall that this means that it has the property that $f(w) \in B$ iff $w \in A$. For 1 and 2, just consider the diagram and/or description of M_A given above. If $w \in A$, then $f(w) \in B$, so M_B accepts w, so M_A accepts w. If $w \notin A$, then $f(w) \notin B$, so M_B does not accept w, so M_A does not accept w. Therefore M_A is a recognizer for A. Furthermore, step 1 always terminates, so if M_B is a decider then so is M_A. Parts 3 and 4 are each just the contrapositives of parts 1 and 2, respectively, so they follow immediately.

We'll make extensive use of part 4 of this theorem to prove undecidability of several languages.

We'll also use part 3 to prove some languages are not Turing-recognizable.

Observations on Mapping Reducibility

Easily proved facts about $\leq_{\text{m}}$:

- Invariance under complement: $A \leq_{\text{m}} B$ if and only if $\overline{A} \leq_{\text{m}} \overline{B}$.
- Transitivity: If $A \leq_{\text{m}} B$ and $B \leq_{\text{m}} C$, then $A \leq_{\text{m}} C$.

These follow easily from the definition; you may find it a useful exercise to write down their proofs.

Have we already used mapping reductions and not realized it?

Yes. Examine the previous proofs of decidability we've covered or that are given in Chapter 4 of Sipser. Implicit in some of these proofs are the following mapping reductions:

- $A_{\text{NFA}} \leq_{\text{m}} A_{\text{DFA}}$ (using a mapping assigning to any NFA encoding the encoding of its corresponding equivalent DFA)
- $A_{\text{REX}} \leq_{\text{m}} A_{\text{NFA}}$ (using a mapping assigning to any regular expression encoding the encoding of its corresponding equivalent NFA)
- $SUB_{\text{DFA}} \leq_{\text{m}} E_{\text{DFA}}$ (using a mapping assigning to any $\langle D_1, D_2 \rangle$, where D_1 and D_2 are DFAs, the encoding of the DFA C constructed so that $L(C) = L(D_1) - L(D_2)$)
- $L \leq_{\text{m}} A_{\text{CFG}}$ for any CFL L (using a mapping assigning to any string w the string $\langle G, w \rangle$, where G is a CFG that generates L)
- $EQ_{\text{DFA}} \leq_{\text{m}} E_{\text{DFA}}$ (using a mapping assigning to any $\langle D_1, D_2 \rangle$, where D_1 and D_2 are DFAs, the encoding of the DFA C constructed so that its language is the symmetric difference of $L(D_1)$ and $L(D_2)$)[3]

[3]The approach used in the lecture handout, which uses two "calls" to a SUB_{DFA} decider, is *not* based on a mapping reduction; it is, however, an example of a reduction from the problem of testing membership in EQ_{DFA} to the problem of testing membership in E_{DFA} in the broader sense discussed earlier.

Undecidability of E_{TM} Revisited

Theorem. E_{TM} is undecidable.

Proof. We create a mapping reduction by essentially imitating what we did in the earlier proof. But this time we give the description of a transducer TM that transforms any A_{TM} problem instance $\langle M, w\rangle$ to its corresponding E_{TM} problem instance $\langle M'\rangle$:

$F =$ "On input $\langle M, w\rangle$ where M is a TM and w is a string:
1. Construct $\langle M'\rangle$, where M' is the following TM:
   $M' =$ "On input x:
   1. Run M on w.
   2. If M accepts, *accept*; if M rejects, *reject*."
2. Output $\langle M'\rangle$."

However, recall from the earlier proof that

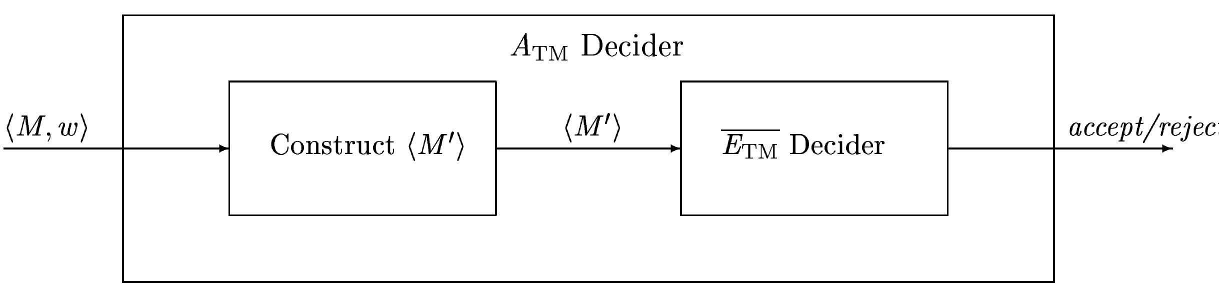
- $L(M')$ is non-empty iff M accepts w, so
- $\langle M'\rangle \notin E_{\text{TM}}$ iff $\langle M, w\rangle \in A_{\text{TM}}$, so
- this transformation is *not* a mapping reduction from A_{TM} to E_{TM}.

However, it *is* a mapping reduction from A_{TM} to $\overline{E_{\text{TM}}}$ since $\langle M'\rangle \in \overline{E_{\text{TM}}}$ iff $\langle M, w\rangle \in A_{\text{TM}}$.

Therefore:

- $A_{\text{TM}} \leq_{\text{m}} \overline{E_{\text{TM}}}$, so
- $\overline{E_{\text{TM}}}$ is undecidable since A_{TM} is (by part 4 of the theorem on reducibility implications), so
- E_{TM} is undecidable since the complement of a undecidable language is undecidable (which follows from the fact that the complement of a decidable language is decidable).

If we had not simply cited the theorem on reducibility implications we could have gone through a few additional steps to obtain a self-contained proof by contradiction that $\overline{E_{\text{TM}}}$ is undecidable since A_{TM} is undecidable. Here is a diagram that essentially illustrates that full argument:



Undecidability of $REGULAR_{\text{TM}}$

The decision problem: *Given TM M, is the language recognized by M regular?*

The language: $REGULAR_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$

Theorem. $REGULAR_{\text{TM}}$ is undecidable.

Proof. We show that $A_{\text{TM}} \leq_{\text{m}} REGULAR_{\text{TM}}$ and the result follows immediately from part 4 of the theorem on reducibility implications since A_{TM} is undecidable.

Here is the description of a transducer TM that transforms any A_{TM} problem instance $\langle M, w \rangle$ to its corresponding $REGULAR_{\text{TM}}$ problem instance $\langle M' \rangle$.

$F =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string:

1. Construct $\langle M' \rangle$, where M' is the following TM:

   $M' =$ "On input x:

   1. If x has the form 0^n1^n for some $n \geq 0$ *accept*.
   2. Run M on input w.
   3. If M accepts, *accept*; if M rejects, *reject*."

2. Output $\langle M' \rangle$."

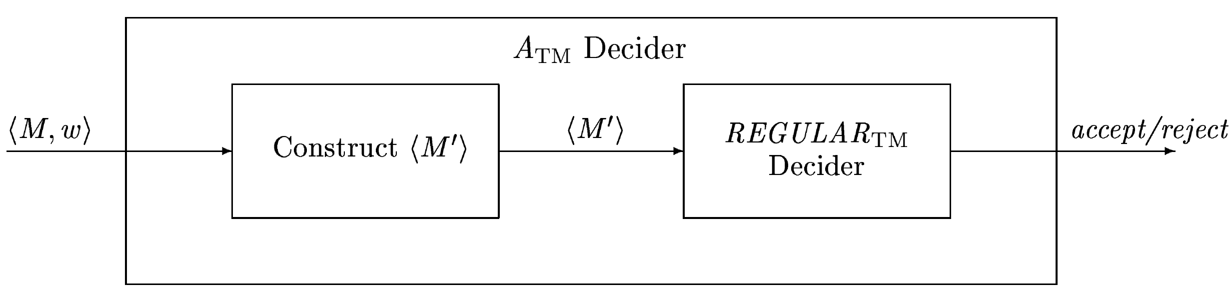
What is $L(M')$?

- In its stage 1, it always accepts any string in $\{0^n1^n \mid n \geq 0\}$.
- In addition, whenever M accepts w it accepts all other strings in its stage 2.
- Thus

$$L(M') = \begin{cases} \text{the regular language } \Sigma^* & \text{if } \langle M, w \rangle \in A_{\text{TM}} \\ \text{the non-regular language } \{0^n1^n \mid n \geq 0\} & \text{if } \langle M, w \rangle \notin A_{\text{TM}}. \end{cases}$$

Therefore $M' \in REGULAR_{\text{TM}}$ iff $\langle M, w \rangle \in A_{\text{TM}}$, proving that $A_{\text{TM}} \leq_{\text{m}} REGULAR_{\text{TM}}$. Since A_{TM} is undecidable, $REGULAR_{\text{TM}}$ must also be undecidable.

Here is a diagram that summarizes the full argument by contradiction proving that $REGULAR_{\text{TM}}$ is undecidable since A_{TM} is undecidable:



Undecidability of EQ_{TM}

The decision problem: *Given two TMs M_1 and M_2, are they equivalent?*

The language: $EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Theorem. EQ_{TM} is undecidable.

Proof. We show that $E_{\text{TM}} \leq_{\text{m}} EQ_{\text{TM}}$ and the result follows immediately from part 4 of the theorem on reducibility implications since E_{TM} is undecidable.

Here is the description of a transducer TM that transforms any E_{TM} problem instance $\langle M\rangle$ to its corresponding EQ_{TM} problem instance $\langle M_1, M_2\rangle$.

$F =$ "On input $\langle M\rangle$, where M is a TM:

1. Construct $\langle M, M_\Phi\rangle$, where M_Φ is the following TM:

   $M_\Phi =$ "On input x:

   1. *reject.*"

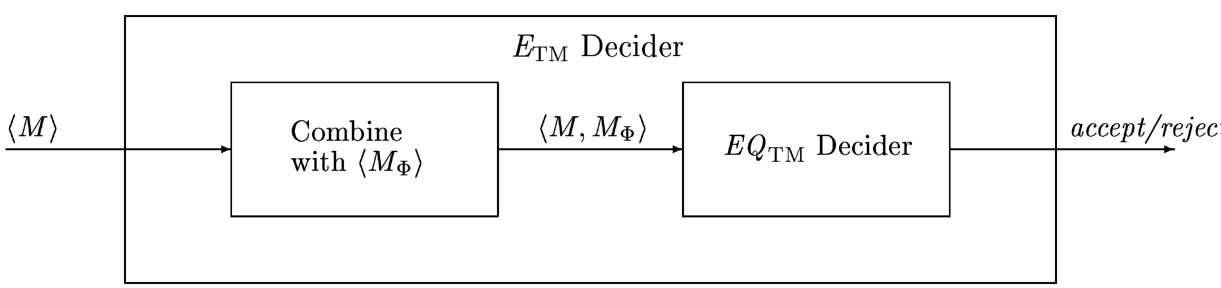
2. Output $\langle M, M_\Phi\rangle$."

M_Φ is just a trivial TM that rejects all inputs.

Clearly:

- $\langle M\rangle \in E_{\text{TM}}$ iff $L(M) = \Phi = L(M_\Phi)$.
- Therefore $\langle M\rangle \in E_{\text{TM}}$ iff $\langle M, M_\Phi\rangle \in EQ_{\text{TM}}$.
- Thus $E_{\text{TM}} \leq_{\text{m}} EQ_{\text{TM}}$.
- Therefore EQ_{TM} is undecidable since E_{TM} is.

Here is a diagram that summarizes the full argument by contradiction proving that EQ_{TM} is undecidable since E_{TM} is undecidable.



Every Turing-Recognizable Language Reduces to $HALT_{\mathrm{TM}}$

Recall that $HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

The proof we gave earlier for the undecidability of $HALT_{\mathrm{TM}}$ was not based on a mapping reduction. Now we prove the following theorem, from which it follows immediately that $HALT_{\mathrm{TM}}$ is undecidable by choosing for L any undecidable Turing-recognizable language.

Theorem. Let L be any Turing-recognizable language. Then $L \leq_{\mathrm{m}} HALT_{\mathrm{TM}}$.

Proof. Let M be a recognizer for L. Here is the description of a transducer TM that transforms any string w in L to a string in $HALT_{\mathrm{TM}}$:

$F =$ "On input w:
1. Construct $\langle M', w\rangle$, where M' is the following TM:
   $M' =$ "On input x:
   1. Run M on input x.
   2. If M accepts, *accept*; if M rejects, loop forever."
2. Output $\langle M', w\rangle$."

Observe that:

- If $w \in L$:
  - M accepts w, so
  - M' halts and accepts w, so
  - $\langle M', w\rangle \in HALT_{\mathrm{TM}}$.
- If $w \notin L$:
  - M does not accept w (either by rejecting or looping), so
  - M' loops on w, so
  - $\langle M', w\rangle \notin HALT_{\mathrm{TM}}$.

Therefore $L \leq_{\mathrm{m}} HALT_{\mathrm{TM}}$.

A Non-Turing-Recognizable, Non-Co-Turing-Recognizable Language

Recall that $EQ_{\mathrm{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$.

Theorem. EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

Proof. We break this into two parts, first proving that EQ_{TM} is not Turing-recognizable, then proving that its complement is not Turing-recognizable.

Lemma 1. EQ_{TM} is not Turing-recognizable.

Proof. We show that $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$. Since $\overline{A_{\mathrm{TM}}}$ is not Turing-recognizable, it will then follow from part 3 of the theorem on reducibility implications that EQ_{TM} is not Turing-recognizable.

Consider this transducer TM mapping $\overline{A_{\mathrm{TM}}}$ problem instances $\langle M, w\rangle$ to EQ_{TM} problem instances, which have the form $\langle M_1, M_2\rangle$:

$F =$ "On input $\langle M, w\rangle$ where M is a TM and w is a string:

0. If the input is not a valid encoding $\langle M, w\rangle$, output $\langle T, T\rangle$, where T is any convenient TM (e.g., M_Φ, defined below).
1. Construct $\langle M', M_\Phi\rangle$, where M' and M_Φ are the following TMs:
   $M' =$ "On input x:
   1. Run M on input w.
   2. If M accepts, *accept*; if M rejects, *reject*."

   $M_\Phi =$ "On input x:
   1. *reject*."
2. Output $\langle M', M_\Phi\rangle$."

Note:

- For completeness we have included a stage 0 just to handle the case when the input string is not a valid encoding of any TM/string combination. Generally, even when such a stage is necessary it is ignored in other TM descriptions, with the tacit understanding that there is a simple way to deal with invalid input strings like this without spelling it out explicitly.
- In this case, the mapping needs to produce a string that belongs to EQ_{TM}.
- If the input is in $\overline{A_{\mathrm{TM}}}$ because it fails to be a valid encoding of any $\langle M, w\rangle$, then stage 0 guarantees that the corresponding output string $\langle T, T\rangle$ belongs to EQ_{TM}, as desired.

Continuing with the proof, we first examine $L(M')$. Clearly,

$$L(M') = \begin{cases} \Sigma^* & \text{if } \langle M, w\rangle \in A_{\mathrm{TM}} \\ \Phi & \text{if } \langle M, w\rangle \notin A_{\mathrm{TM}}. \end{cases}$$

A Non-Turing-Recognizable, Non-Co-Turing-Recognizable Language (Continued)

Thus (restricting attention to valid encodings $\langle M, w\rangle$) we see that

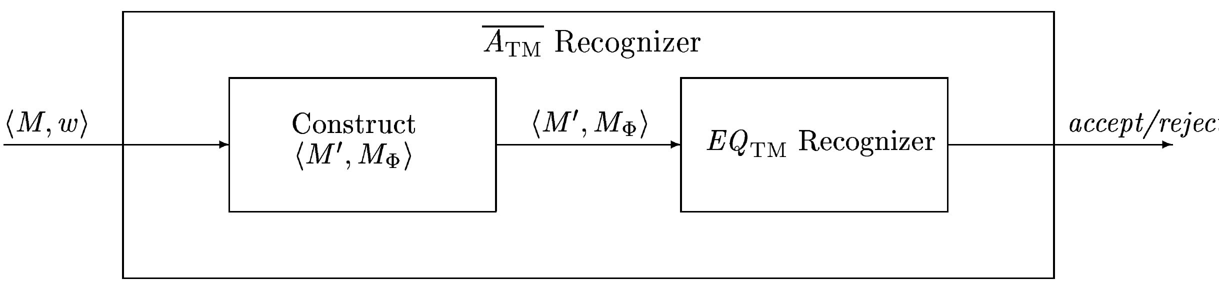
$$\begin{aligned}\langle M, w\rangle \in \overline{A_{\mathrm{TM}}} &\Rightarrow \langle M, w\rangle \notin A_{\mathrm{TM}} \\ &\Rightarrow L(M') = \Phi = L(M_\Phi) \\ &\Rightarrow \langle M', M_\Phi\rangle \in EQ_{\mathrm{TM}}\end{aligned}$$

and

$$\begin{aligned}\langle M, w\rangle \notin \overline{A_{\mathrm{TM}}} &\Rightarrow \langle M, w\rangle \in A_{\mathrm{TM}} \\ &\Rightarrow L(M') = \Sigma^* \neq \Phi = L(M_\Phi) \\ &\Rightarrow \langle M', M_\Phi\rangle \notin EQ_{\mathrm{TM}}.\end{aligned}$$

Therefore, in all cases, the input string to the TM F belongs to $\overline{A_{\mathrm{TM}}}$ iff the output string from F belongs to EQ_{TM}, so F is a mapping reduction $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$. Since $\overline{A_{\mathrm{TM}}}$ is not Turing-recognizable, it follows that EQ_{TM} is not Turing-recognizable.

Here is a diagram that summarizes the full argument by contradiction proving that EQ_{TM} cannot be Turing-recognizable since a recognizer for it could be used as a subprocedure to construct a recognizer for $\overline{A_{\mathrm{TM}}}$. (This diagram ignores the invalid-encoding case handled by stage 0).



A Non-Turing-Recognizable, Non-Co-Turing-Recognizable Language (Continued)

Lemma 2. $\overline{EQ_{\mathrm{TM}}}$ is not Turing-recognizable.

Proof. As in the proof of Lemma 1 we could construct from scratch a mapping reduction from some non-Turing-recognizable language (we now know of two: $\overline{A_{\mathrm{TM}}}$ and EQ_{TM}) to $\overline{EQ_{\mathrm{TM}}}$. Instead we will take advantage of mapping reductions already derived.

Recall that:

- we proved the undecidability of E_{TM} by constructing a mapping reduction $A_{\mathrm{TM}} \leq_{\mathrm{m}} E_{\mathrm{TM}}$; and
- we proved the undecidability of EQ_{TM} by constructing a mapping reduction $E_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$.

Therefore:

- by transitivity, $A_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$,
- which is equivalent to $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} \overline{EQ_{\mathrm{TM}}}$,
- so it follows that $\overline{EQ_{\mathrm{TM}}}$ is not Turing-recognizable since $\overline{A_{\mathrm{TM}}}$ is not Turing-recognizable.

Summary of Mapping Reductions Explicitly Described in This Handout

- $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} E_{\mathrm{TM}}$
- $A_{\mathrm{TM}} \leq_{\mathrm{m}} REGULAR_{\mathrm{TM}}$
- $E_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$
- $L \leq_{\mathrm{m}} HALT_{\mathrm{TM}}$ for any Turing-recognizable language L
- $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$

Designing Mapping Reductions: Two Examples

Consider the language

$$L = \{\langle M \rangle \mid M \text{ is a TM and } |L(M)| = 5\}\,.$$

Try to design mapping reductions

- $A_{\mathrm{TM}} \leq_{\mathrm{m}} L$ and
- $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} L$.

Need to fill in this template, where F is the TM implementing the desired mapping reduction:[4]

$F =$ "On input $\langle M, w \rangle$ where M is a TM and w is a string:
1. Construct $\langle M' \rangle$, for the following TM:
   $M' =$ "On input x:
   .
   .
   .   "
2. Output $\langle M' \rangle$."

To prove $A_{\mathrm{TM}} \leq_{\mathrm{m}} L$:

- Want $\langle M, w \rangle \in A_{\mathrm{TM}}$ iff $\langle M' \rangle \in L$.
- Equivalently, want M' to accept exactly 5 strings exactly when M accepts w.

Can we design such an M'?

To prove $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} L$:

- Want $\langle M, w \rangle \in \overline{A_{\mathrm{TM}}}$ iff $\langle M' \rangle \in L$.
- Equivalently, want M' to accept exactly 5 strings exactly when M does not accept w.

Can we design such an M'?

In addition, if either or both of these mapping reductions can be shown to exist, what can we conclude about L?

[4] For simplicity, we ignore the invalid-input case.