Instructions

- The exam is open book: You may use your notes, homeworks, any handouts and solutions provided to you in this class, the text (Sipser), and any other paper-based references. You may not use any electronic devices such as laptops, cell phones, PDAs, etc.

- Please, write your answers in the blue books provided. Show all your work and indicate your final answers clearly.

Problem 1 [20 points]

Recall that the reverse $L^R$ of a language $L$ is defined by

$$L^R = \{ w | w^R \in L \},$$

where $w^R$ means the reverse of the string $w$.

1. Is the class of decidable languages closed under the reverse operation?

2. Is the class of Turing-recognizable languages closed under the reverse operation?

Give a proof of your answers. Hint: The same approach can be used to provide an answer to both questions. If it involves constructing a Turing machine, give only an informal implementation-level description.

Problem 2 [20 points]

Given a string $w = a_1a_2...a_n$, where each $a_i \in \{0,1\}$, define $ECHO(w) = a_1a_1a_2a_2...a_na_n$. Then, for any language $L \subseteq \{0,1\}^*$, define $ECHO(L) = \{ECHO(w) | w \in L \}$. For example, if

$$L = \{\epsilon, 0, 0110, 01010\},$$

then

$$ECHO(L) = \{\epsilon, 00, 00111100, 0011001100\}.$$  

Prove that each of the following language classes is closed under application of the $ECHO$ operation to its members:

1. The class $P$.

2. The class $NP$. 

For the remaining problems you will need to create appropriate mapping reductions. In all cases, the reduction is simple and straightforward. No complicated constructions are required for any of these.

**Problem 3 [40 points - (20, 20)]**

Prove that the following languages are undecidable:

1. $\text{SUB}_{\text{TM}} = \{ < M_1, M_2 > | M_1 \text{ and } M_2 \text{ are Turing machines and } L(M_1) \subseteq L(M_2) \}$.

2. $\text{FINITE}_{\text{TM}} = \{ < M > | M \text{ is a TM and } |L(M)| \text{ is finite} \}$.

In other words, (the encoding of) a Turing machine $M$ is in $\text{FINITE}_{\text{TM}}$ if the number of strings accepted by $M$ is finite.

*Hint: Reduce from $\text{A}_{\text{TM}}$ or from $\overline{\text{A}_{\text{TM}}}$, whichever you are able to do. (One may not be possible.)*

**Problem 4 [20 points]**

A tautology is a Boolean formula that always evaluates to 1 (i.e. $\text{TRUE}$) no matter what values are assigned to its variables. For example, the formula

$$\phi = (x_1 \vee \overline{x_1}) \wedge (x_2 \vee \overline{x_2})$$

is a tautology.

Define

$$\text{TAUT} = \{ < \phi > | \phi \text{ is a tautology} \}.$$ 

At least one of $\text{TAUT}$ or its set complement $\overline{\text{TAUT}}$ is NP-complete. Give an NP-completeness proof for one of them.

*Hint: Reduce from $\text{SAT}$. 