Exam 2 — CSU 390 Theory of Computation — Fall 2007

Instructions

- The exam is open book: You may use your notes, homeworks, any handouts and solutions provided toyou in this class, the text (Sipser), and any other paper-based references. You may not use any electronic devices such as laptops, cell phones, PDAs, etc.
- Please, write your answers in the blue books provided. **Show all your work** and indicate your final answers clearly.

Problem 1 [20 points]

Consider the following context-free grammar:

$S \rightarrow xS|xStS|a$

where $\{a, t, x\}$ are terminals.

Prove that this grammar is ambiguous by providing the following:

- List all words of length less than 6 in the language L generated by this grammar.

  Solution

  (a) length 1: a

  (b) length 2: $S \rightarrow xS \rightarrow xa$

  (c) length 3: $S \rightarrow xS \rightarrow xxS \rightarrow xxa$

  (d) length 4: $S \rightarrow xS \rightarrow xxS \rightarrow xxxS \rightarrow xxxa$

  (e) length 4: $S \rightarrow xStS \rightarrow xatS \rightarrow xata$

  (f) length 5: $S \rightarrow xS \rightarrow xxS \rightarrow xxxS \rightarrow xxxxS \rightarrow xxxxa$

  (g) length 5: $S \rightarrow xS \rightarrow xxStS \rightarrow xxatS \rightarrow xxata$

  (h) length 5: $S \rightarrow xStS \rightarrow xxStS \rightarrow xxatS \rightarrow xxata$

  (i) length 5: $S \rightarrow xStS \rightarrow xStxS \rightarrow xatxS \rightarrow xatxa$

- Show a string w in the language L that has (at least) two leftmost derivations in this language.

  Solution

  String $xxata$ has two leftmost derivations shown in (g) and (h).

- Show the two derivations for the string w.

  Solution

  (g) length 5: $S \rightarrow xS \rightarrow xxStS \rightarrow xxatS \rightarrow xxata$

  (h) length 5: $S \rightarrow xStS \rightarrow xxStS \rightarrow xxatS \rightarrow xxata$

- Show the parse trees corresponding to these derivations.

```
        S                       S
      / |                     / | | \
     /  |                    /  | |  \
    x   S                   x   S t   S
      / | | \                 / |     |
     /  | |  \               x  S     a
    x   S t   S                 |
        |     |                 a
        a     a
```

- Give a concise description of the language generated by this grammar.

  Solution

  Each string starts with x. One option is a finite number of xs followed by a, by using teh rule $S \rightarrow xS$ exclusively.The rule $S \rightarrow xStS$ introduces ambiguity, because we can first apply the rule $S \rightarrow xS$ then apply the rule $S \rightarrow xStS$ to the first S, or start with $S \rightarrow xStS$ and apply the rule $S \rightarrow xS$ to the first S. This ambiguity persists for an arbitrary number of levels in the parse tree.

Problem 2 [20 points]

Give context-free grammars that generate the following languages over $\{a, b\}$. In each case annotate the rules to indicate what each rule generates.

- $L_1 = \{a^n b^{n+2} | n \geq 0\}$

  Solution

  $S \rightarrow aSb|bb$

  The first rule generated an equal number of as and bs, actually arranged as $a^n b^n$, with the variable S remaining in the middle. The second rule then replaces S by bb, thus adding two more b to the previously balanced string.

- $L_2 = \{w||w| \text{ is odd and the middle symbol is } a\}$

  Solution

  $S \rightarrow aSa|aSb|bSa|bSb|a$

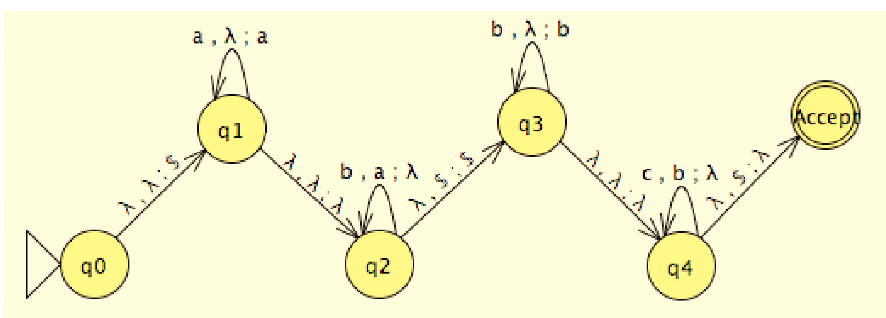
  The first four rules add one terminal on each side of the middle, covering the four possible combinations of the two letters a and b. The fifth rule converts the variable in the middle to the terminal a, assuring both of the requirements — that the string length is odd, and that it is a.

Problem 3 [15 points]

Give a state transition diagram and a brief informal description for a pushdown automaton that recognizes the following language:

$L_3 = \{a^m b^{m+n} c^n | m, n \geq 0\}$

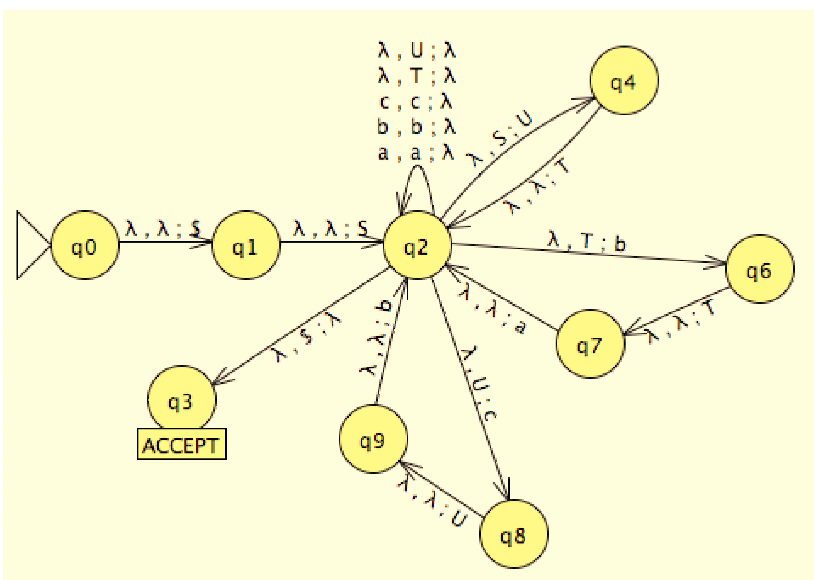
Solution



- Start by pushing \$ onto the stack to mark the bottom of the stack and move to the state 1.
- In the state 1 push all as onto the stack.
- When the first b is about to be read move to the state 2.
- Pop as and match them with input bs until the bottom of the stack is reached.

- Pop and push back the bottom of the stack marker \$ and move to the state 3.
- Push the remaining bs in the input onto the stack.
- When the first c is about to be read move to the state 4.
- Pop bs and match them with input cs until the bottom of the stack is reached.
- Pop the the bottom of the stack marker \$ and move to the *Accept* state.
- Add the transition that rejects any additional symbol after the bottom of the stack has been reached in the state 4.

Alternate Solution
Start with the CFG that recognizes the language L_3:
$S \rightarrow TU$
$T \rightarrow aTb|\epsilon$
$U \rightarrow bUc|\epsilon$
and construct the PDA following the technique described in the book on page 118:



Problem 4 [15 points]

Prove that the language $L_4 = \{a^n b^j c^k | k > n, k > j\}$ is not context-free.

Solution

Suppose the pumping length is p. Choose the following string
$w = a^p b^p c^{p+1}$

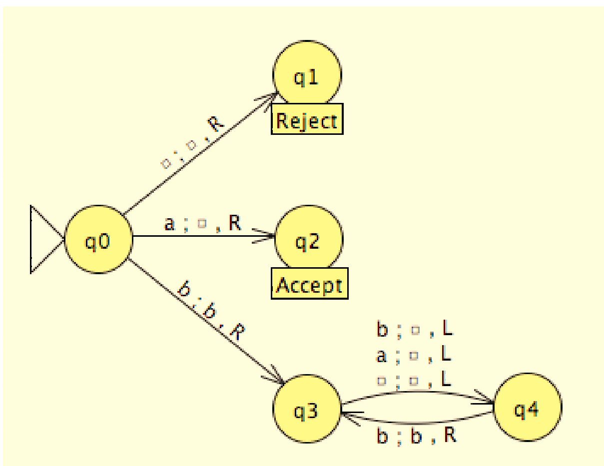
Then any substring $s = vxy$ of w of length $|s| \leq p$ must have one of the following forms:

- $s = a^m$ where $m \leq p$. Then pumping up would produce a string $a^{m'} b^p c^{p+1}$ with $m' > (p+1)$ and so the pumped up string would not be in L_4.
- $s = b^m$ where $m \leq p$. Then pumping up would produce a string $a^p b^{m'} c^{p+1}$ with $m' > (p+1)$ and so the pumped up string would not be in L_4.
- $s = c^m$ where $m \leq p$. This cannot be pumped down, as in the resulting string $a^p b^p c^{m'}$ we have $m' \leq p$ and so the pumped down string would not be in L_4.
- $s = a^{m_1} b^{m_2}$. This string cannot be pumped up, as it would either produce as and bs out of order, or produce a string with more as and bs than cs and so the pumped up string would not be in L_4.
- $s = b^{m_1} c^{m_2}$. We can divide s in one of the following ways:
  - $s = vxy = b^{m1'} \circ x \circ c^{m2'}$, but then the string cannot be pumped down, as there would no longer be more cs than aa.
  - $|y| = 0$ and $v = b^m$, or $y = b^m$ and $|v| = 0$ — but then the string cannot be pumped up, as there would be more bs than cs.
  - $|y| = 0$ and $v = c^m$, or $y = c^m$ and $|v| = 0$ — but then the string cannot be pumped down, as there would no longer be more cs than bs or as.
  - Finally, if either $v = b^{m_1} c^{m_2}$ or $y = b^{m_1} c^{m_2}$ then pumping up the string would produce bs and cs out of order.

Problem 5 [15 points]

- Give a state transition diagram for a TM M with input alphabet $\{a, b\}$ that accepts all strings starting with a, rejects the empty string, and loops on all other strings. Your diagram may use the implicit-reject convention.

  Solution



- Is $L(M)$, the language recognized by your TM M, a decidable language? Prove your answer.

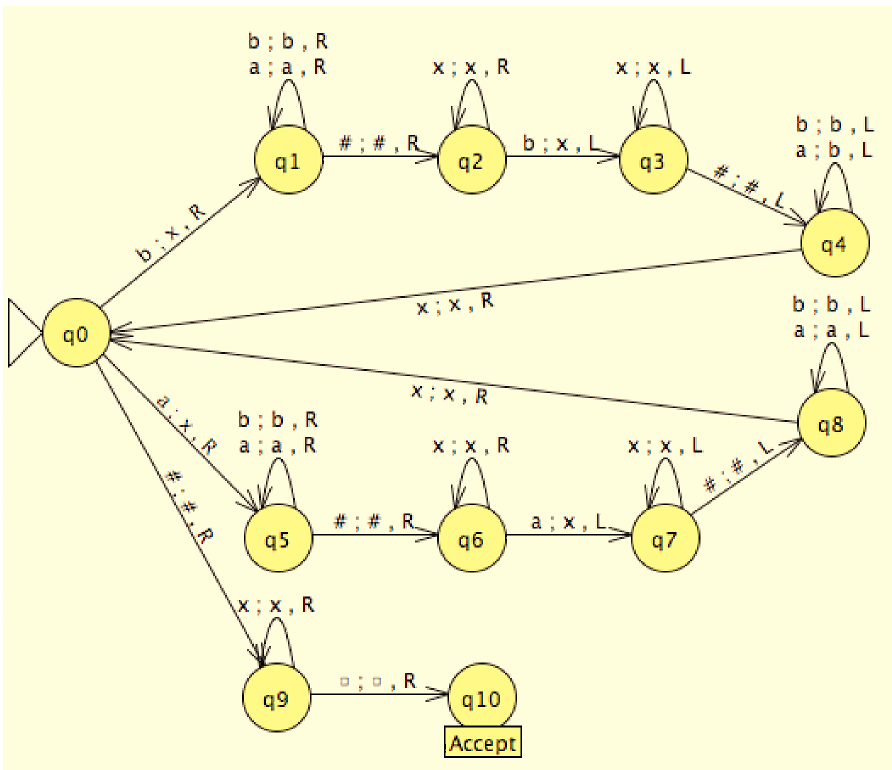
  Solution

  This is a decidable language. We can build a Turing machine that would just eliminate the transition to the state 3 and replace it with an implicit rejection.

Problem 6 [15 points]

- Construct a TM to test for equality of two strings over the alphabet $\{a, b\}$, where the strings are separated by a cell containing #.

  Solution



- Describe in English the actions of your TM.
  - In state 0 read a character and write x. Follow to the state 1 if the input was b and to state 5 if the input was a.
  - From state 1 keep reading all the letters of the first string without writing anything new and moving to the right each time.
  - When # appears on the input, move right and transition to state 2.
  - Keep reading over the input that has been marked with xs.
  - The first letter after xs should match the b that we have read in the first string. If it does, write x on the tape and move to the left, and transition to the state 3.
  - Keep reading over all marked letters of the second string, until you find #.
  - Read # and transition to the state 4.
  - Read over the letters of the first string moving left until you find the rightmost marked one.
  - When you find x move to the right to look at the first un-matched letter of the first string and transition back to state 0.
  - For matching letter a we follow the same process through states $5, 6, 7, 8$ and back to 0.
  - If all the letters in the first string have been matched (or the first string was empty) read the # in the state 0 and transition to the state 9.
  - In the state 9 read all the marked letters of the second string, moving to the right.
  - If the letter after all marked ones is a blank, then all the letters in the second string have been matched with corresponding letters in the first string and we move to the state 10, the $ACCEPT$ state.
  - In all states, encountering a letter not explicitly expected leads to the $REJECT$ state.