Solving Recurrences via Iteration

Consider the recurrence $T(n) = 4T(n/2) + n^2/\lg n$. In order to solve the recurrence, I would first suggest rewriting the recurrence with the recursive component *last* and using a generic parameter not to be confused with n. We may think of the following equation as our general pattern, which holds for any value of $\square$.

$$T(\square) = \frac{\square^2}{\lg \square} + 4T(\square/2) \tag{1}$$

Since our pattern (Equation 1) is valid for any value of $\square$, we may use it to "iterate" the recurrence as follows.

$$\begin{aligned} T(n) &= \frac{n^2}{\lg n} + 4T(n/2) \\ &= \frac{n^2}{\lg n} + 4\left(\frac{(n/2)^2}{\lg(n/2)} + 4T(n/2^2)\right) \\ &= \frac{n^2}{\lg n} + \frac{n^2}{\lg(n/2)} + 4^2T(n/2^2) \end{aligned} \tag{2}$$

Always simplify the expression, eliminating parentheses as in Equation 2, before expanding further. Continuing...

$$\begin{aligned} T(n) &= \frac{n^2}{\lg n} + \frac{n^2}{\lg(n/2)} + 4^2\left(\frac{(n/2^2)^2}{\lg(n/2^2)} + 4T(n/2^3)\right) \\ &= \frac{n^2}{\lg n} + \frac{n^2}{\lg(n/2)} + \frac{n^2}{\log(n/2^2)} + 4^3T(n/2^3) \\ &\vdots \\ &= \frac{n^2}{\lg n} + \frac{n^2}{\lg(n/2)} + \frac{n^2}{\lg(n/2^2)} + \ldots + \frac{n^2}{\lg(n/2^{k-1})} + 4^kT(n/2^k) \\ &= \sum_{j=0}^{k-1} \frac{n^2}{\lg(n/2^j)} + 4^kT(n/2^k) \end{aligned}$$

We will next show that the pattern we have established is correct, by induction.

Claim 1 *For all* $k \geq 1$, $T(n) = \sum_{j=0}^{k-1} \frac{n^2}{\lg(n/2^j)} + 4^kT(n/2^k)$.

Proof: The proof is by induction on k. The base case, $k = 1$, is trivially true since the resulting equation matches the original recurrence. For the inductive step, assume that the statement is true for $k = i - 1$; i.e.,

$$T(n) = \sum_{j=0}^{i-2} \frac{n^2}{\lg(n/2^j)} + 4^{i-1}T(n/2^{i-1}).$$

Our task is then to show that the statement is true for $k = i$; i.e.,

$$T(n) = \sum_{j=0}^{i-1} \frac{n^2}{\lg(n/2^j)} + 4^i T(n/2^i).$$

This may be accomplished by starting with the inductive hypothesis and applying the definition of the recurrence, as follows.

$$\begin{aligned}
T(n) &= \sum_{j=0}^{i-2} \frac{n^2}{\lg(n/2^j)} + 4^{i-1} T(n/2^{i-1}) \\
&= \sum_{j=0}^{i-2} \frac{n^2}{\lg(n/2^j)} + 4^{i-1} \left[\frac{(n/2^{i-1})^2}{\lg(n/2^{i-1})} + 4T(n/2^i) \right] \\
&= \sum_{j=0}^{i-2} \frac{n^2}{\lg(n/2^j)} + 4^{i-1} \frac{n^2/4^{i-1}}{\lg(n/2^{i-1})} + 4^i T(n/2^i) \\
&= \sum_{j=0}^{i-2} \frac{n^2}{\lg(n/2^j)} + \frac{n^2}{\lg(n/2^{i-1})} + 4^i T(n/2^i) \\
&= \sum_{j=0}^{i-1} \frac{n^2}{\lg(n/2^j)} + 4^i T(n/2^i)
\end{aligned}$$

□

We thus have that $T(n) = \sum_{j=0}^{k-1} \frac{n^2}{\lg(n/2^j)} + 4^k T(n/2^k)$ for all $k \geq 1$. We next choose a value of k which causes our recurrence to reach a known base case. Since $n/2^k = 1$ when $k = \lg n$, and $T(1) = \Theta(1)$, we have

$$\begin{aligned}
T(n) &= \sum_{j=0}^{\lg n - 1} \frac{n^2}{\lg(n/2^j)} + 4^{\lg n} T(1) \\
&= n^2 \sum_{j=0}^{\lg n - 1} \frac{1}{\lg n - j} + n^{\lg 4} \Theta(1) \\
&= n^2 \sum_{\ell=1}^{\lg n} \frac{1}{\ell} + \Theta(n^2) \\
&= n^2\, \Theta(\ln \lg n) + \Theta(n^2) \\
&= \Theta(n^2 \log \log n).
\end{aligned}$$