The complexity of distributions

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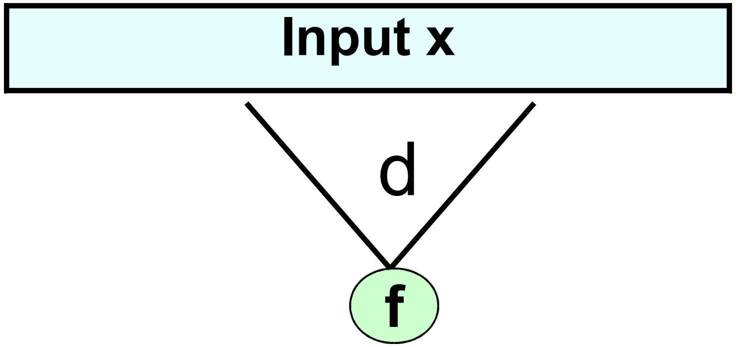
Local functions

- $f : \{0,1\}^n \rightarrow \{0,1\}$ d-local : output depends on d input bits



- Fact: Parity(x) = 1 $\Leftrightarrow \sum x_i = 1 \bmod 2$ is not n-1 local

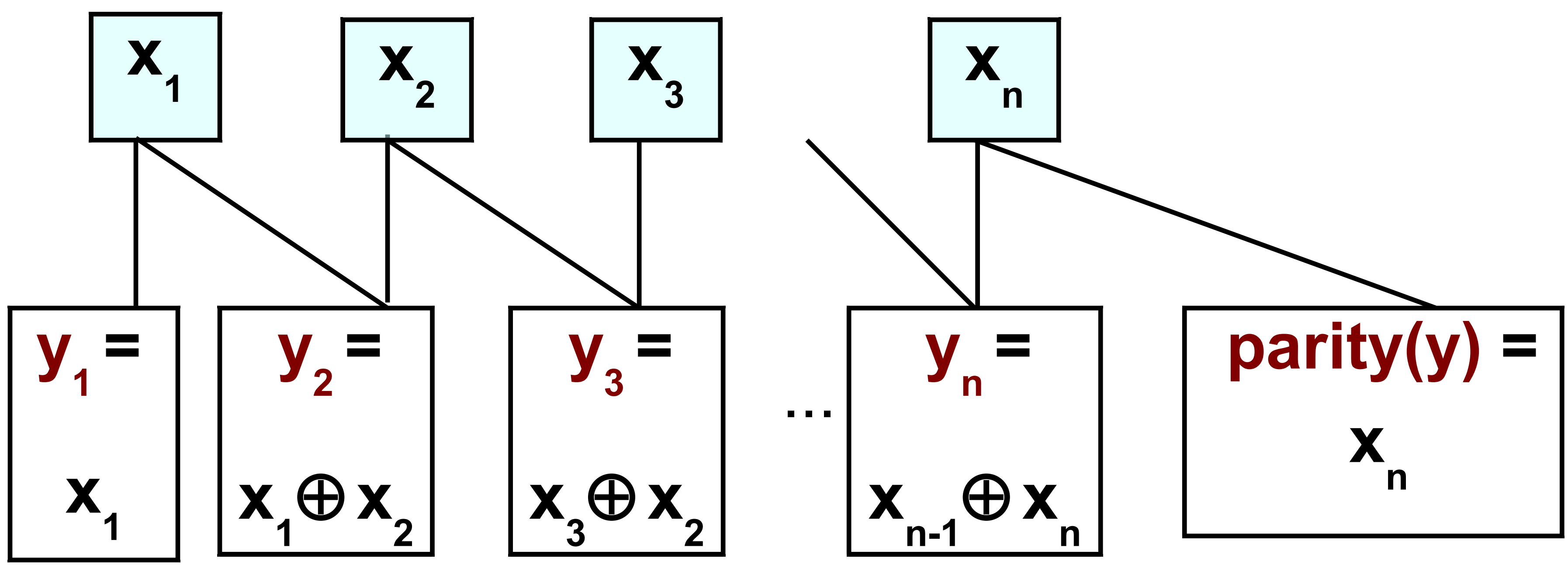
- Proof: Flip any input bit $\Rightarrow$ output flips ♦

Local generation of (Y, parity(Y))

- Theorem [Babai '87; Boppana Lagarias '87]

  There is $f : \{0,1\}^n \to \{0,1\}^{n+1}$, each bit is 2-local

  Distribution $f(X) \equiv$ (Y, parity(Y))   ($X, Y \in \{0,1\}^n$ uniform)



Message

- Complexity theory of distributions (as opposed to functions)

  How hard is it to generate distribution D given random bits ?

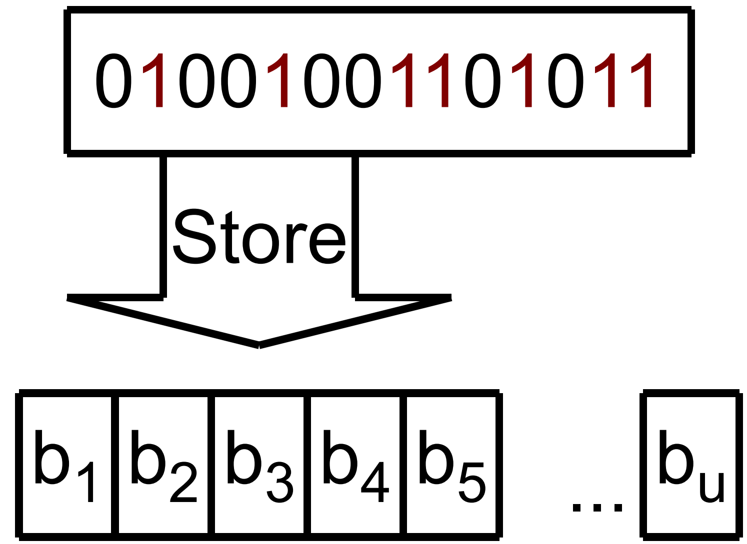
  E.g., D = (Y, parity(Y)), D = W_k := uniform n-bit with k 1's

Rest of this talk

- Connection with succinct data structures
- Lower bound for locally generating $W_{n/2}$ = n-bit with n/2 1's
- Decision tree model
- Bounded-depth circuit model (with Shachar Lovett)

Succinct data structures for sets

- Store $S \subseteq \{1, 2, \ldots, n\}$ of size $|S| = k$



In u bits $b_1, \ldots, b_u \in \{0,1\}$

- Want:

Small space u (optimal = $\lceil \lg_2 (n \text{ choose } k) \rceil$)

Answer "$i \in S$?" by probing few bits (optimal = 1)

- In combinatorics: Nešetřil Pultr, …, Körner Monti

Previous results

- Store $S \subseteq \{1, 2, \dots, n\}$, $|S| = k$, in bits, answer "$i \in S$?"

- [Minsky Papert '69, Buhrman Miltersen Radhakrishnan Venkatesh; Pagh; ...; Pătraşcu; V. '09]

- Surprising upper bounds
  space = optimal + o(n), probe O(log n)

- No lower bounds for $k = n / 2^a$

General connection

- Claim: If store $S \subseteq \{1, 2, \ldots, n\}$, $|S| = k$ in u = optimal + r bits answer "$i \in S$?" by (non-adaptively) probing d bits.

Then $\exists\, f : \{0,1\}^u \to \{0,1\}^n$, d-local

Distance($f(X)$, W_k = uniform set of size k) $< 1 - 2^{-r}$

$$\left(\text{Distance}(A, B) := \max_T \left| \Pr[A \in T] - \Pr[B \in T] \right| \right)$$

- Proof: f_i := "$i \in S$?"

$f(X) = W_k$ with probability $(n \text{ choose } k) / 2^u = 2^{-r}$ ♦

Our result

- Theorem[V.] $f : \{0,1\}^{\text{optimal} + n^{o(1)}} \to \{0,1\}^n$, $(d < \varepsilon \log n)$-local.

  Distance($f(X)$, W_k = uniform set of size $k = \Theta(n)$) $> 1 - n^{-\Omega(1)}$

- Tight up to $\Omega()$ if $k = n/2$: $f(x) = x$, (n choose $n/2$) $= O(2^n/\sqrt{n})$

- Corollary: To store $S \subseteq \{1, 2, \ldots, n\}$, $|S| = k = n / 2^a$

  answer "$i \in S$?" probing $d < \varepsilon \log(n)$ bits:

  Need space > optimal + $\Omega(\log n)$

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Our result

- Theorem[V.]: Let $f : \{0,1\}^n \to \{0,1\}^n$: ($d=O(1)$)-local.

  There is $T \subseteq \{0,1\}^n$: $\left| \Pr[f(x) \in T] - \Pr[W_{n/2} \in T] \right| > 1 - n^{-\Omega(1)}$

- Warm-up scenarios:

- $f(x) = 000111$ Low-entropy $T := \{ 000111 \}$

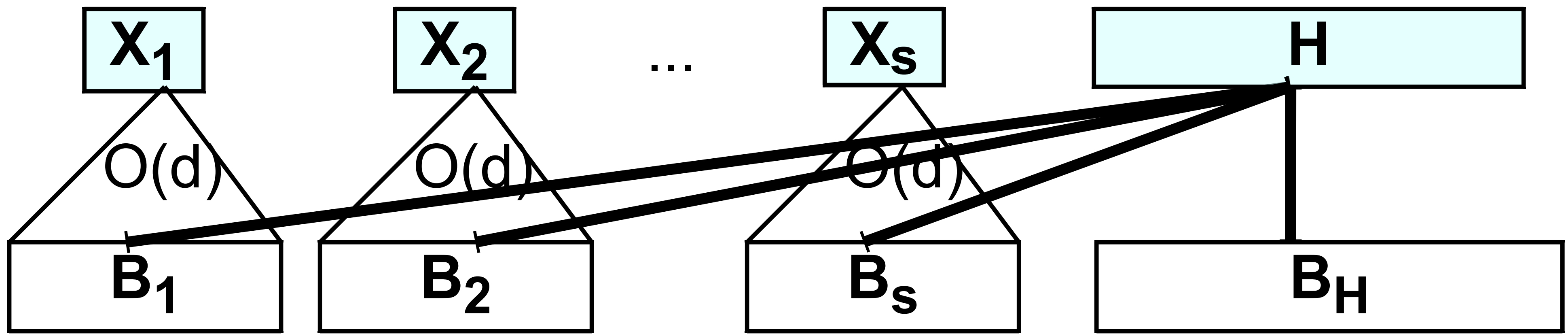
  $\left| \Pr[f(x) \in T] - \Pr[W_{n/2} \in T] \right| = \left| 1 - |T| / (n \text{ choose } n/2) \right|$

- $f(x) = x$ "Anti-concentration" $T := \{ z : \sum_i z_i = n/2 \}$

  $\left| \Pr[f(x) \in T] - \Pr[W_{n/2} \in T] \right| = \left| 1/\sqrt{n} - 1 \right|$

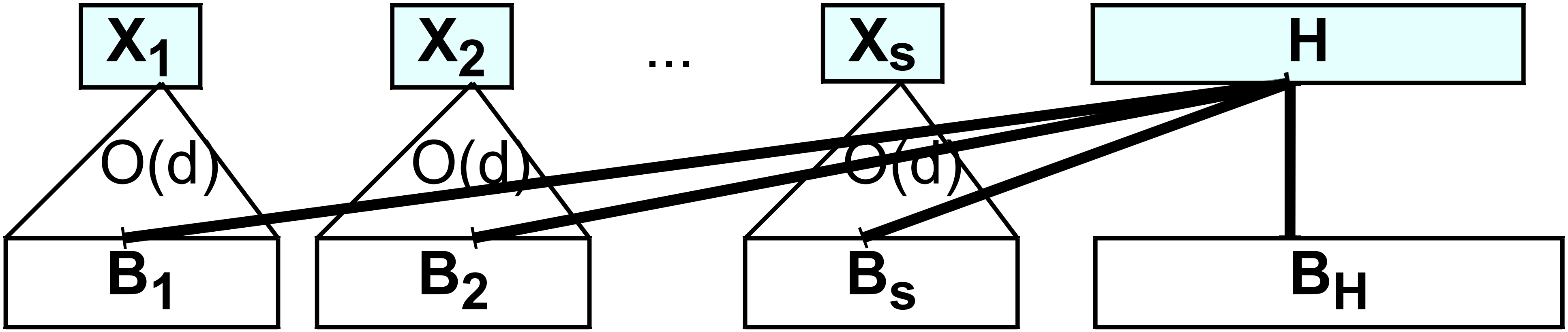
Proof

- Partition input bits $X = (X_1, X_2, \ldots, X_s, H)$



- Fix H. Output block B_i depends only on bit X_i
- Many B_i constant ($B_i(0,H) = B_i(1,H)$) $\Rightarrow$ low-entropy
- Many B_i depend on X_i ($B_i(0,H) \neq B_i(1,H)$)

  Idea: Independent $\Rightarrow$ anti-concentration: can't sum to n/2

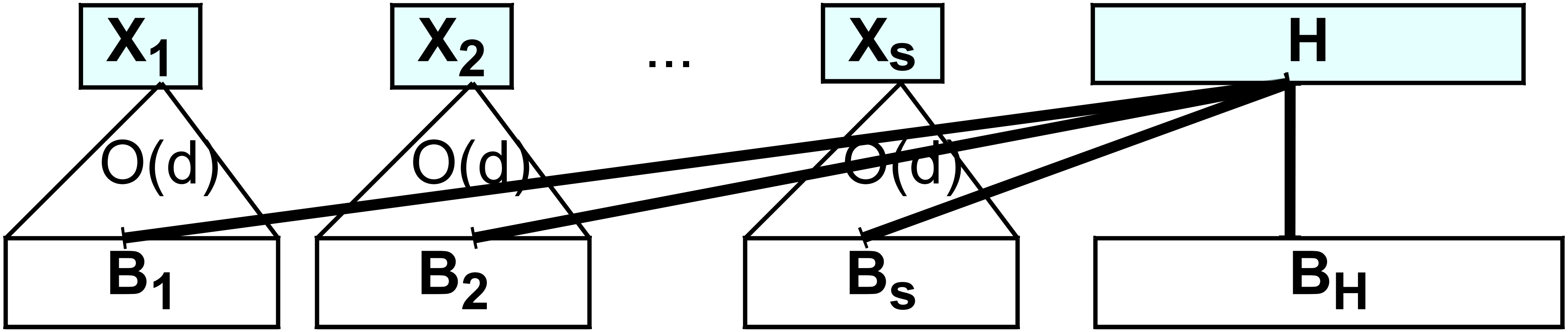


- If many $B_i(0,H)$, $B_i(1,H)$ have different sum of bits, use

Anti-concentration Lemma [Littlewood Offord] For $a_1, a_2, \ldots, a_s \neq 0$, any c, $\Pr_{X \in \{0,1\}^s} [\sum_i a_i X_i = c] < 1/\sqrt{n}$

- Problem: $B_i(0,H) = 100$, $B_i(1,H) = 010$
  high entropy but no anti-concentration

- Fix: want many blocks 000, so high entropy $\Rightarrow$ different sum



- Test $T \subseteq \{0,1\}^n$: $\Pr[f(X_1,\ldots,X_s,H) \in T] \approx 1$; $\Pr[W_{n/2} \in T] \approx 0$

$z \in T \Leftrightarrow$

$\exists$ H : $\exists$ $X_1,\ldots,X_s$ w/ many blocks B_i fixed : $f(X_1,\ldots,X_s,H) = z$

OR

Few blocks $z|_{B_i}$ are 000

OR

$\sum_i z_i \neq n/2$

Rest of this talk

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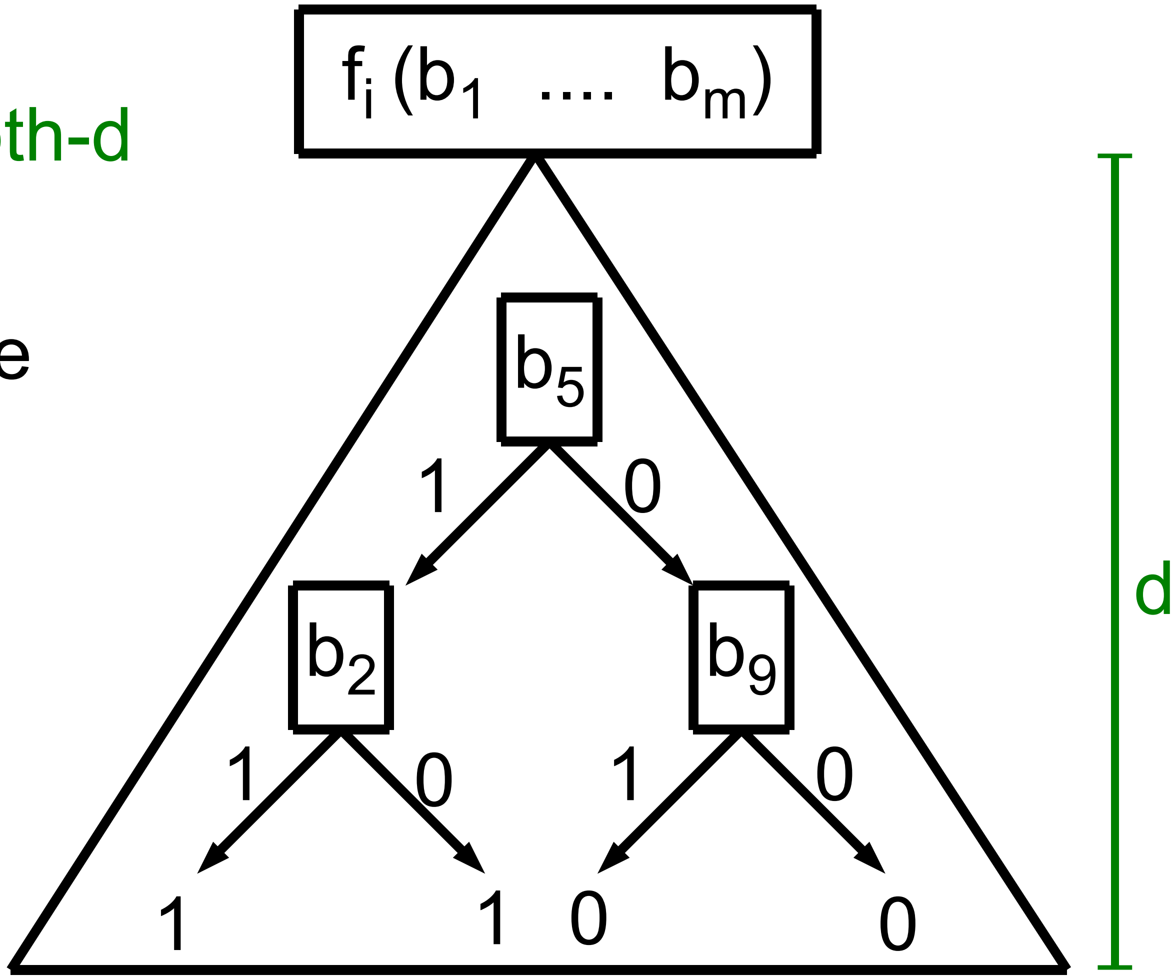
Decision tree model

- $f : \{0,1\}^m \to \{0,1\}^n$ depth-d
  each output bit f_i
  is depth-d decision tree



- Depth d $\subseteq$ 2^d local

Our result for decision trees

- Theorem[V.] $f : \{0,1\}^* \rightarrow \{0,1\}^n$: each bit depth $< 0.1 \log n$

  Distance($f(X)$, $W_{n/2}$)$> n^{-\Omega(1)}$

- Worse than $1 - n^{-\Omega(1)}$ bound for O(1)-local functions

- Theorem[Czumaj Kanarek Lorys Kutyłowski, V.]

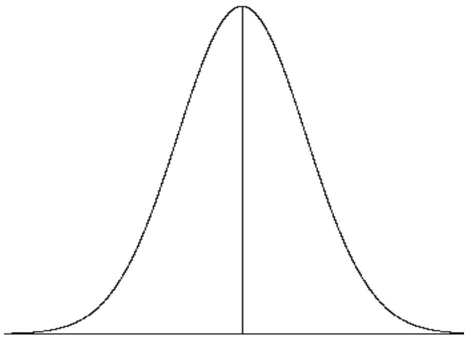
  $\exists\, f : \{0,1\}^* \rightarrow \{0,1\}^n$: each bit depth $O(\log n)$

  Distance($f(X)$, $W_{n/2}$) $< 1/n$

Tool for lower bound proof

- Central limit theorem:

  $x_1, x_2, \ldots, x_n$ independent $\Rightarrow \sum x_i \approx$ normal



- Bounded-independence central limit theorem [Diakonikolas Gopalan Jaiswal Servedio V.]

  $x_1, x_2, \ldots, x_n$ k-wise independent $\Rightarrow \sum x_i \approx$ normal

- Note: For next result, Paley–Zygmund inequality enough

Proof

- Theorem[V.] $f : \{0,1\}^* \to \{0,1\}^n$: each bit depth $< 0.1 \log n$

  $$\text{Distance}(\ f(X),\ W_{n/2}\ ) > n^{-\Omega(1)}$$

- Proof: Is output distribution $f(X)$ ($k = 10$)-wise independent?

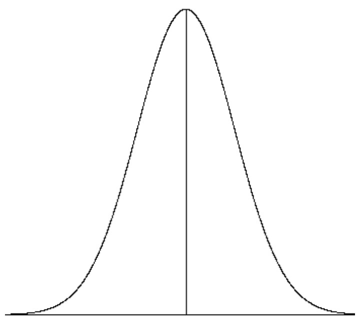
  NO $\Rightarrow W_{n/2} \approx$ k-wise independent

  Distance(those k bits, uniform on $\{0,1\}^k$) $> 2^{-k(0.1 \log n)}$

  (granularity of decision tree probability)

  YES $\Rightarrow$ by prev. theorem $\sum f(X)_i \approx$ normal

  so often $\sum f(X)_i \neq n/2$ ♦



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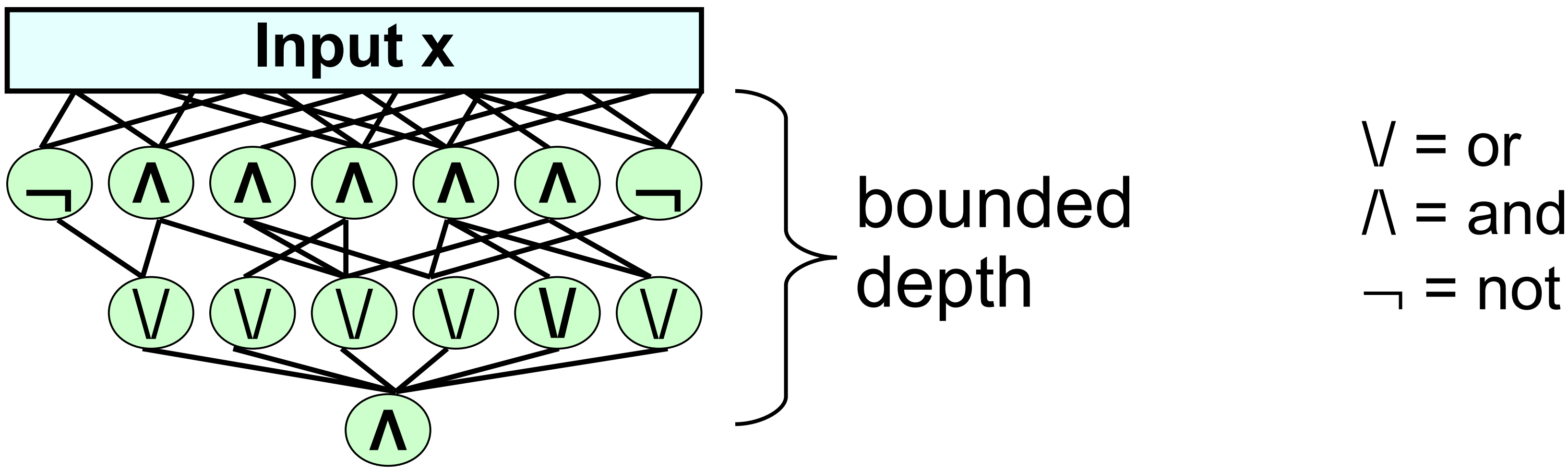
Bounded-depth circuits

- More general model: small bounded-depth circuits (AC^0)



- Challenge: $\exists$ explicit boolean f : cannot generate (Y, f(Y)) ?

- Theorem[Matias Vishkin, Hagerup, Czumaj Kanarek Lorys Kutyłowski, V.]
  Can generate (Y, majority(Y)) (exp. small error)

- Theorem [Lovett V.] Cannot generate error-correcting code

Lower bound for codes

- Code $C \subseteq \{0,1\}^n$ of size $|C| = 2^{k = \Omega(n)}$

  $x \neq y \in C \Rightarrow x, y$ far : hamming distance $\Omega(n)$

- Theorem [Lovett V.] $f : \{0,1\}^* \to \{0,1\}^n$, $f \in AC^0$

  Distance(f(X), uniform over C) $> 1 - n^{-\Omega(1)}$

- Consequences for data structures for codewords, complexity of pseudorand. generators against AC^0 [Nisan]

Warm-up

- Fact: $f : \{0,1\}^k \to \{0,1\}^n$, $f \in AC^0$
  f cannot compute encoding function of C,
  mapping message $m \in \{0,1\}^k$ to codeword

- Proof:

- [Linial Mansour Nisan, Boppana] low sensitivity of AC^0:
  m, m' random at hamming distance 1
  $\Rightarrow$ f(m), f(m') close in hamming distance.

- But $f(m) \neq f(m') \in C \Rightarrow$ far in hamming distance ♦

Lower bound for codes

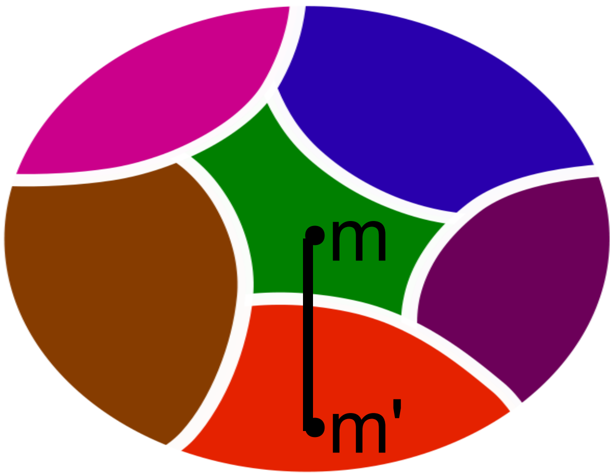
- Theorem [Lovett V.] $f : \{0,1\}^{L >> k} \to \{0,1\}^n$, $f \in AC^0$

  Distance(f(X), uniform over C) $> 1 - n^{-\Omega(1)}$

  Problem: f needs not compute encoding function.

  Input length >> message length

- Idea: Input $\{0,1\}^L$ to f partitioned in |C| sets



- Isoperimetric inequality [Harper, Hart]:

  Random m, m' at distance 1 often in ≠ sets ⇒ low sensitivity

Lower bound for codes

- Theorem [Lovett V.] $f : \{0,1\}^{L >> k} \to \{0,1\}^n$, $f \in AC^0$
  Distance(f(X), uniform over C) $> 1 - n^{-\Omega(1)}$
- Note: to get
  Need isoperimetric inequality for m, m' at distance >> 1

  Fact[thanks to Samorodnitsky] $\forall A \subseteq \{0,1\}^L$ of density α
  random m, m' obtained flipping bits w/ probability p :
  $\alpha^2 \leq \Pr[\text{both } m \in A \text{ and } m' \in A] \leq \alpha^{1/(1-p)}$

Complexity of generators against AC^0

- Pseudorandom generator against circuit of depth d
  (want: reduce randomness w/ minimum overhead)

- Direct implementation of Nisan's generator takes depth $\geq$ d
  (circuit + generator $\rightarrow$ depth 2d)

- [Lovett V.] Generating output distribution of Nisan's generator takes depth $\geq$ d
  (for some choice of designs)

- [V.] Generator in depth 2 (circuit + generator $\rightarrow$ depth d+1)
  [Braverman] + [Guruswami Umans Vadhan]

Conclusion

- Complexity of distributions = uncharted territory

- Lower bound for generating W_k locally
  $\Rightarrow$ lower bound for succinct data structures for storing sets of size $n / 2^a$

- Lower bound for decision trees

- Lower bound for bounded-depth circuits (AC^0)

- $\sum\prod\surd\cap\notin\cup\supset\supseteq\not\subset\subset\subseteq\in\Downarrow\Rightarrow\Uparrow\Leftarrow\Leftrightarrow\vee\wedge\geq\leq\forall\exists\Omega\alpha\beta\varepsilon\gamma\delta\rightarrow$
- $\neq\approx$
-
- Recall: edit style changes ALL settings.
- Click on “line” for just the one you highlight

More connections

- More uses of generating W_k := uniform n-bit string with k 1's
- McEliece cryptosystem
- Switching networks, …

Previous results

- Store S $\subseteq$ {1, 2, … , n}, |S| = k, in bits, answer "i $\in$ S?"
- [Minsky Papert '69] Average-case study
- [Buhrman Miltersen Radhakrishnan Venkatesh; Pagh '00]

  Space O(optimal), probe O(1) when $k = \Theta(n)$

  Lower bounds for $k < n^{1-\varepsilon}$
- [..., Pagh, Pătraşcu] space = optimal + o(n), probe O(log n)
- [V. '09] lower bounds for $k = \Omega(n)$, except $k = n / 2^a$