Derandomization: New Results and Applications

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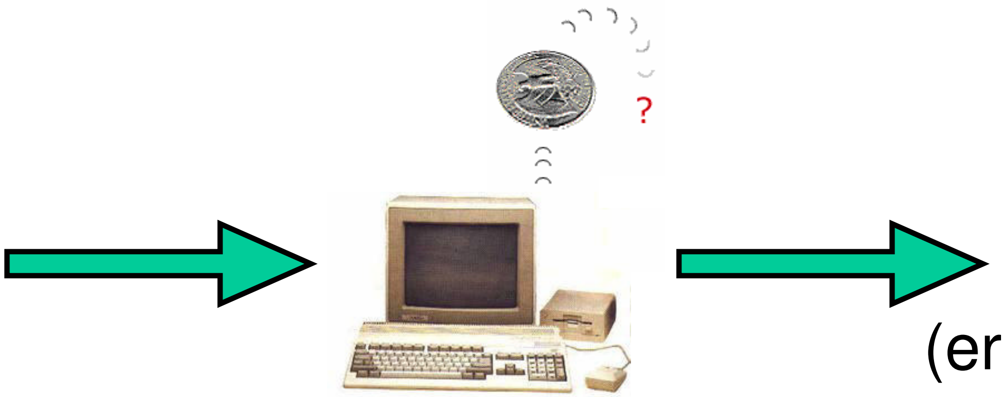
March 2006

Randomness in Computation



Input

Answer
(error probability 1%)

- Useful throughout Computer Science
  - Algorithms
  - Learning Theory
  - Complexity Theory

- Question: Is Randomness necessary?

Derandomization

- Goal: remove randomness
- Why study derandomization?
- Breakthrough [R '04]:
  Connectivity in logarithmic space (SL = L)
- Breakthrough [AKS '02]:
  Primality in polynomial time (PRIMES $\in$ P)

Randomness vs. Time

- Goal:
  simulate randomized computation deterministically

- Trivial Derandomization:
  If A uses n random bits, enumerate all 2^n possibilities

  Probabilistic polynomial-time $\subseteq$ exponential time

  $\text{BPP} \subseteq \text{Time}(2^{\text{poly}(n)})$

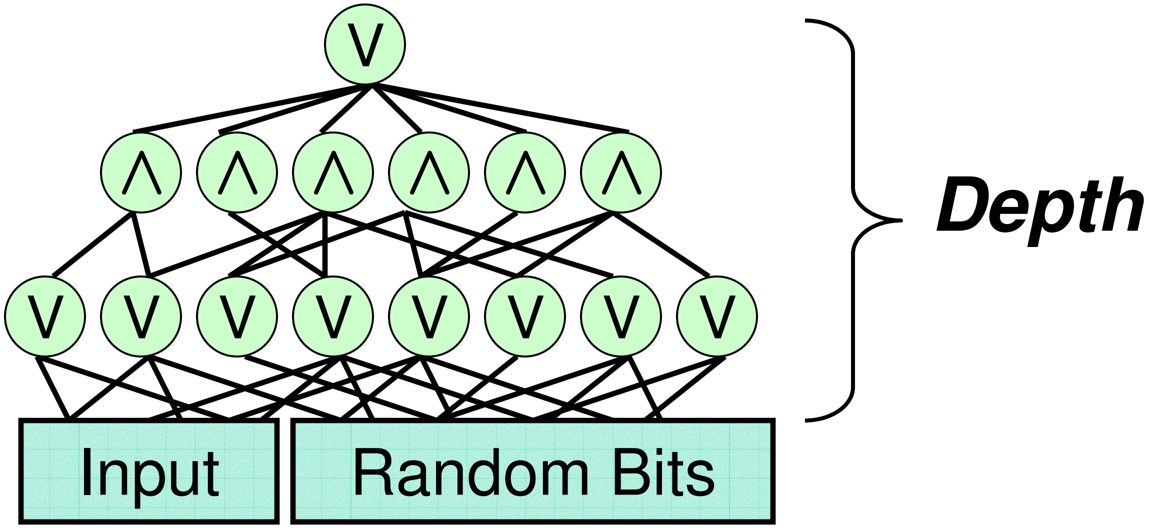
- Strong Belief: BPP = P $\big($ Time(poly(n)) $\big)$
  Complexity Assumptions $\Rightarrow$ BPP = P [BFNW,NW,IW,…]

Outline

- Overview of derandomization
- Derandomization of restricted models
  - Application: Hardness Amplification in NP
  - New derandomization
- Derandomization of general models
  - BPP vs. PH
  - Proof of Lower Bound

Constant-Depth Circuits

- Probabilistic constant-depth circuit (BP AC^0)



- Theorem [N ‘91]: BP AC^0 $\subseteq$ Time($n^{\text{polylog } n}$)
  - Compare to BP P $\subseteq$ Time($2^{\text{poly}(n)}$)

Application: Avg-Case Hardness of NP

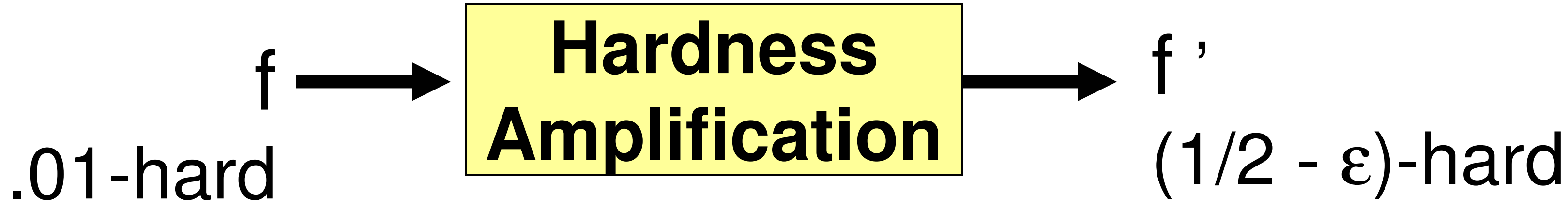
- Study hardness of NP on random instances
  - Natural question, essential for cryptography

- Currently cannot relate to P ≠ NP [FF,BT,V]

- Hardness amplification

  Definition: $f : \{0,1\}^n \rightarrow \{0,1\}$ is δ-hard if

  for every efficient algorithm M : $\Pr_x[M(x) \neq f(x)] \geq \delta$



f

.01-hard

Hardness Amplification

f ’

(1/2 - ε)-hard

Previous Results

- Yao's XOR Lemma: $f'(x_1,\ldots, x_n) := f(x_1) \oplus \cdots \oplus f(x_n)$
  $f' \approx (1/2 - 2^{-n})$-hard, almost optimal

- Cannot use XOR in NP: $f \in NP \not\Rightarrow f' \in NP$

- Idea: $f'(x_1,\ldots, x_n) = C(f(x_1),\ldots, f(x_n))$, C monotone
  - e.g. $f(x_1) \wedge (f(x_2) \vee f(x_3))$. $f \in NP \Rightarrow f' \in NP$

- Theorem [O'D]: There is C s.t. $f' \approx (1/2 - 1/n)$-hard

- Barrier: No monotone C can do better!

Our Result on Hardness Amplification

- Theorem [HVV]: Amplification in NP up to $\approx 1/2 - 2^{-n}$
  - Matches the XOR Lemma

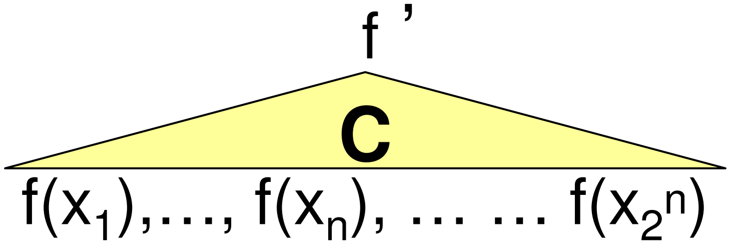
- Technique: Derandomize!

  Intuitively, $f' := C(f(x_1),\ldots, f(x_n), \ldots \ldots f(x_{2^n}))$

  f' $(1/2 - 1/2^n)$-hard by previous result

  Problem: Input length = 2^n

  Note C is constant-depth



  Derandomize: input length → n, keep hardness

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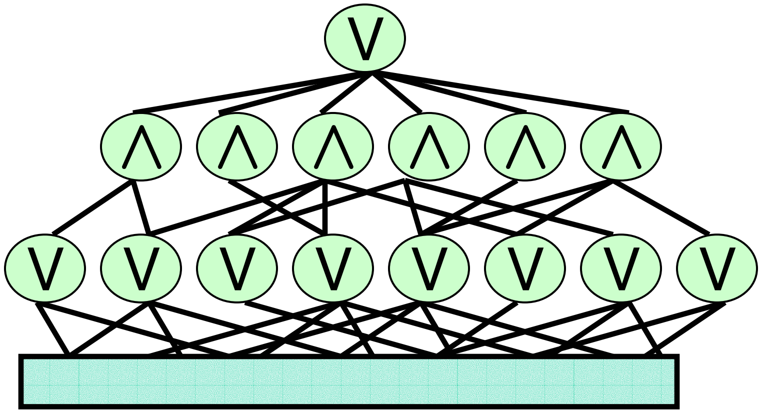
Previous Results

- Recall Theorem [N]:

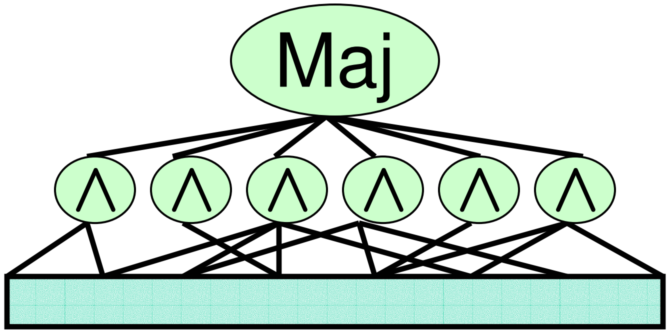
  BP $AC^0 \subseteq \mathrm{Time}(n^{\mathrm{polylog}\ n})$



- But AC^0 is weak: Majority $\notin AC^0$
  - Majority$(x_1,\ldots,x_n)$:= $\sum_i x_i > n/2$?

- Theorem [LVW]:

  BP Maj AND $\subseteq \mathrm{Time}(2^{n^{\varepsilon}})$

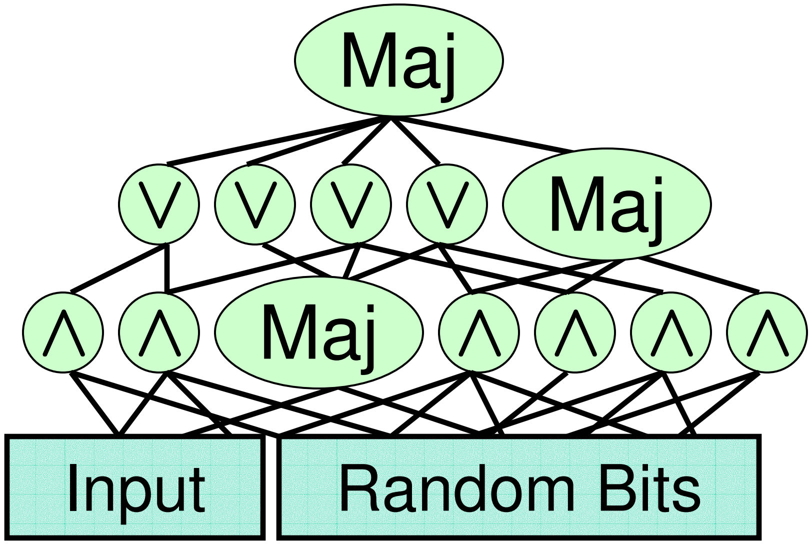


- Derandomize incomparable classes

Our New Derandomization

- Theorem **[V] :** BP Maj $AC^0 \subseteq$ Time($2^{n^{\varepsilon}}$)

  Derandomize
  constant-depth circuits
  with few Majority gates =



- Improves on [LVW]. Slower than [N] but richer richest probabilistic circuit class in Time($2^{n^{\varepsilon}}$)

- Techniques: Communication complexity +
  switching lemma [BNS,HG,H,HM,CH]

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BPP vs. POLY-TIME HIERARCHY

- Probabilistic Polynomial Time (BPP):

  for every x, Pr $\left[\text{ M(x) errs }\right] \leq$ 1%

- Strong belief: BPP = P [NW,BFNW,IW,…]

  Still open: BPP $\subseteq$ NP ?

- Theorem [SG,L; '83]: BPP $\subseteq \Sigma_2$ P

- Recall

  NP = Σ_1 P → $\exists$ y M(x,y)

  Σ_2 P → $\exists$ y $\forall$ z M(x,y,z)

The Problem we Study

- More precisely [SG,L] give

  BPTime(t) $\subseteq \Sigma_2$Time(t^2)

- Question[Rest of this Talk]:

  Is quadratic slow-down necessary?

- Motivation: Lower bounds

  Know NTime ≠ Time on some models [P+,F+,…]

  Technique: *speed-up* computation with quantifiers

  To prove NTime ≠ BPTime cannot afford Σ_2Time(t^2)

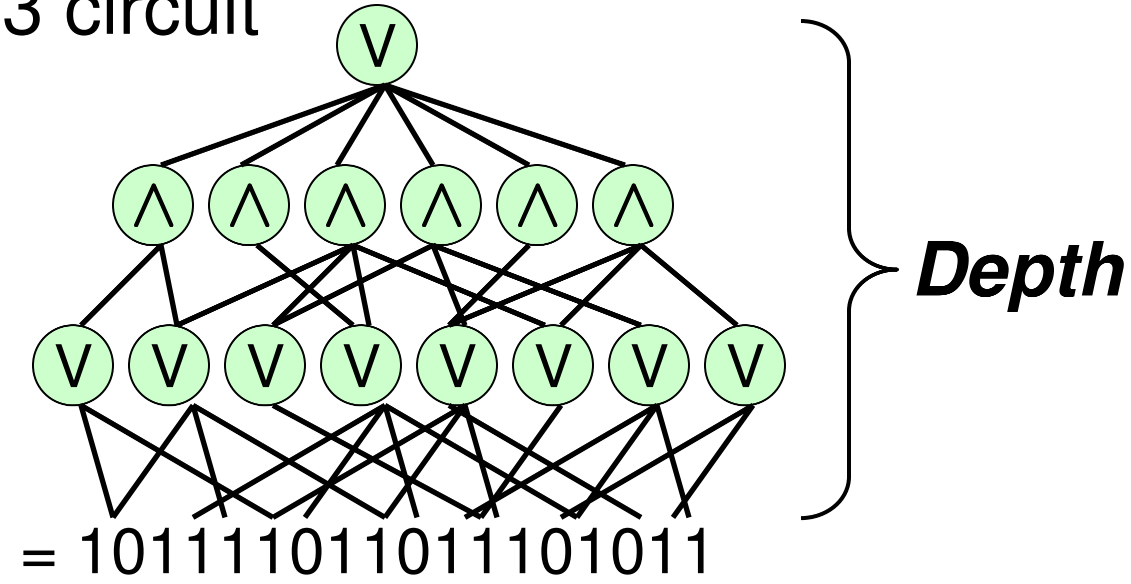
Approximate Majority

- Input: R = 101111011011101011

- Task: Tell $\Pr_i [R_i = 1] \geq 99\%$ from $\Pr_i [R_i = 1] \leq 1\%$

  Do not care if $\Pr_i [R_i = 1] \sim 50\%$ (approximate)

- Model: Depth-3 circuit



The connection [FSS]

$M(x;u) \in \mathrm{BPTime}(t)$ ⇒ $R = 11011011101011$

$|R| = 2^t$ $\quad \hookrightarrow R_i = M(x;i)$

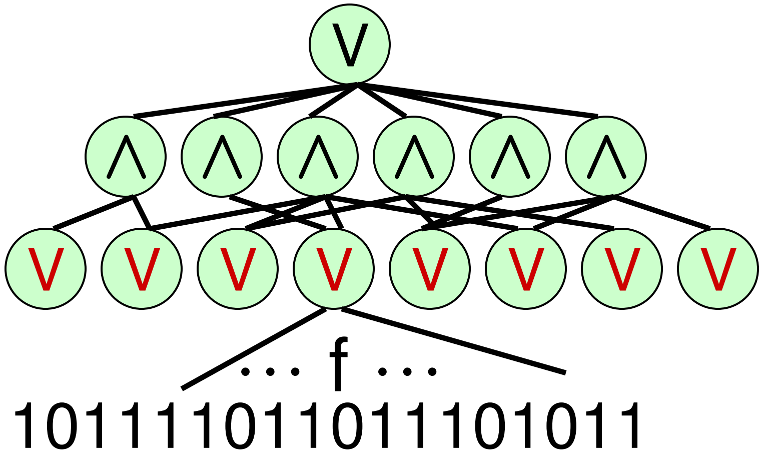
Compute M(x):

Tell $\Pr_u[M(x) = 1] \geq 99\%$
from $\Pr_u[M(x) = 1] \leq 1\%$

⇒ Compute Appr-Maj

$\mathrm{BPTime}(t) \subseteq \Sigma_2 \mathrm{Time}(t')$
$= \exists \forall \mathrm{Time}(t')$

⇒



Running time t'
– run M at most t'/t times

⇒ **Bottom fan-in f = t' / t**

Our Results

- **Theorem[V] :** Small depth-3 circuits for Approximate Majority on N bits have bottom fan-in $\Omega(\log N)$

- **Corollary**: Quadratic slow-down necessary for relativizing techniques:

$$\mathrm{BPTime}^{A}(t) \cancel{\subseteq} \Sigma_2\mathrm{Time}^{A}(t^{1.99})$$

- **Theorem**[DvM,V]: $\mathrm{BPTime}(t) \subseteq \Sigma_3\mathrm{Time}(t \cdot \log^5 t)$
  - Previous result [A]: $\mathrm{BPTime}(t) \subseteq \Sigma_{O(1)}\mathrm{Time}(t)$

- For time, the level is the third!

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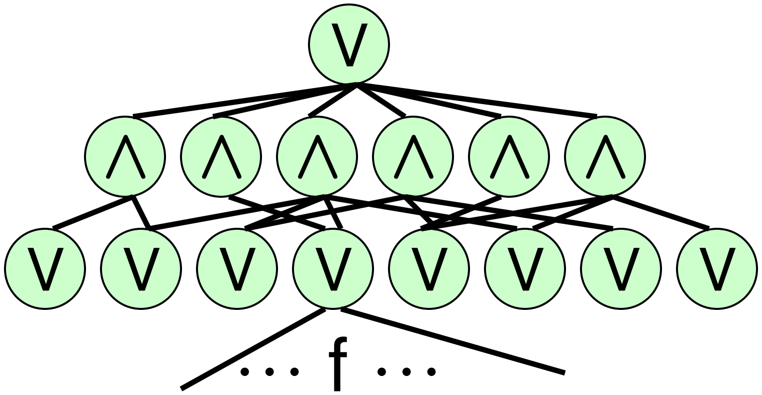
Our Negative Result

- **Theorem[V]:** $2^{N^{\varepsilon}}$-size depth-3 circuits for Approximate Majority on N bits have bottom fan-in $f = \Omega(\log N)$

- Recall:



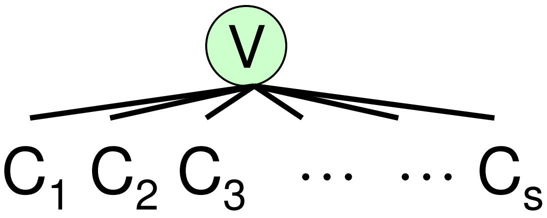
$R = 101111011011101011 \qquad |R| = N$

Tells $R \in \text{YES} := \{ R : \Pr_i [R_i = 1] \geq 99\% \}$

from $R \in \text{NO} := \{ R : \Pr_i [R_i = 1] \leq 1\% \}$

Proof

- Circuit is OR of s depth-2 circuits



- By definition of OR :

$$R \in \text{YES} \Rightarrow \text{some } C_i\,(R) = 1$$
$$R \in \text{NO} \Rightarrow \text{all } C_i\,(R) = 0$$

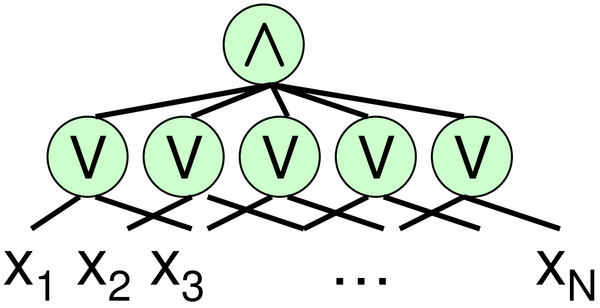
- By averaging, fix $C = C_i$ s.t.

$$\boxed{\begin{array}{l} \Pr_{R \in \text{YES}}\,[C\,(x) = 1\,] \geq 1/s \\ \forall\, R \in \text{NO} \quad \Rightarrow \quad C\,(R) = 0 \end{array}}$$

- Claim: Impossible if C has bottom fan-in $\leq \varepsilon \log N$

CNF Claim

- Depth-2 circuit ⇒ CNF



$(x_1 \vee x_2 \vee \neg x_3\ ) \wedge (\neg x_4) \wedge (x_5 \vee x_3)$

bottom fan-in ⇒ **clause** size

- **Claim:** All CNF C with clauses of size $\varepsilon \cdot \log N$

  $\boxed{\begin{array}{l}\text{Either } \Pr_{R \in YES} [C\ (x) = 1\ ] \leq 1 / 2^{N^{\varepsilon}} \\ \text{or there is } R \in NO : C(x) = 1\end{array}}$

- Note: Claim ⇒ Theorem

Either $\Pr_{R \in YES}[C(x)=1] \leq 1/2^{N^{\varepsilon}}$ or $\exists R \in NO : C(x) = 1$

Proof Outline

- Definition: $S \subseteq \{x_1,x_2,\ldots,x_N\}$ is a covering if every clause has a variable in S

E.g.: $S = \{x_3,x_4\}$ $C = (x_1 V x_2 V \neg x_3) \wedge (\neg x_4) \wedge (x_5 V x_3)$

- Proof idea: Consider **smallest** covering S

Case $|S|$ BIG : $\Pr_{R \in YES}[C(x) = 1] \leq 1 / 2^{N^{\varepsilon}}$

Case $|S|$ tiny : Fix few variables and repeat

$$\boxed{\text{Either } \Pr_{R \in \text{YES}}[C(x)=1] \leq 1/2^{N^{\varepsilon}} \text{ or } \exists\, R \in \text{NO} : C(x) = 1}$$

Case |S| BIG

- $|S| \geq N^{\delta} \;\Rightarrow$ have $N^{\delta}/(\varepsilon \cdot \log N)$ disjoint clauses Γ_i
  - Can find Γ_i greedily

- $\Pr_{R \in \text{YES}}\left[C(R) = 1\right] \leq \Pr\left[\,\forall\, i,\ \Gamma_i(R) = 1\,\right]$

$$= \prod_i \Pr[\,\Gamma_i(R) = 1] \qquad \text{(independence)}$$

$$\leq \prod_i (1 - 1/100^{\varepsilon \log N}) = \prod_i \left(1 - 1/N^{O(\varepsilon)}\right)$$

$$= \left(1 - 1/N^{O(\varepsilon)}\right)^{|S|} \leq e^{-N^{\Omega(1)}} \qquad ✓$$

$$\boxed{\text{Either } \Pr_{R \in YES}[C(x)=1] \leq 1/2^{N^{\varepsilon}} \text{ or } \exists\, R \in NO : C(x) = 1}$$

Case |S| tiny

- $|S| < N^{\delta} \quad \Rightarrow \quad$ Fix variables in S
  - Maximize $\Pr_{R \in YES}[C(x)=1]$

- Note: S covering ⇒ clauses shrink

  Example

  $$(x_1 V x_2 V x_3\,) \wedge (\neg x_3) \wedge (x_5 V \neg x_4) \;\xrightarrow[x_4 \leftarrow 1]{x_3 \leftarrow 0}\; (x_1 V x_2\,) \wedge (x_5)$$

- Repeat

  Consider smallest covering S’, etc.

Either $\Pr_{R \in \mathrm{YES}}[C(x)=1] \le 1/2^{N^{\varepsilon}}$ or $\exists\, R \in \mathrm{NO} : C(x) = 1$

Finish up

- Recall: Repeat ⇒ shrink clauses

  So repeat at most $\varepsilon \cdot \log N$ times

- When you stop:

  Either smallest covering size $\ge N^{\delta}$ ✓

  Or $C = 1$

  Fixed $\le (\varepsilon \cdot \log N)\, N^{\delta} \ll N$ vars.

  Set rest to 0 ⇒ $R \in \mathrm{NO} : C(R) = 1$ ✓

Q.E.D.

Conclusion

- Derandomization: powerful technique

- Restricted models: Constant-depth circuits (AC^0)
  - – Derandomization of AC^0 [N]
  - – Application: Hardness Amplification in NP [HVV]
  - – Derandomization of AC^0 with few Maj gates [V]

- General models: BPP vs. PH
  - – $\mathrm{BPTime}(t) \subseteq \Sigma_2\mathrm{Time}(t^2)$ [SG,L]
  - – $\mathrm{BPTime}(t) \cancel{\subseteq} \Sigma_2\mathrm{Time}(t^{1.99})$ (w.r.t. oracle) [V]

    Lower Bound for Approximate Majority

Thank you!