Life can only be understood backwards; but it must be lived forwards.

Soren Kierkegaard
Dynamic programming

An interesting question is, "Where did the name, dynamic programming, come from?"
The 1950's were not good years for mathematical research. We had a very interesting gentleman in Washington named Wilson. He was Secretary of Defence, and he actually had a pathological fear and hatred of the word, research. I'm not using the term lightly; I'm using it precisely. His face would suffuse, he would turn red, and he would get violent if people used the term, research, in his presence. You can imagine how he felt, then, about the term, mathematical. The RAND Corporation was employed by the Air Force, and the Air Force had Wilson as its boss, essentially. Hence, I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics inside the RAND Corporation. What title, what name, could I choose? In the first place, I was interested in planning, in decision-making, in thinking. But planning, is not a good word for various reasons. I decided therefore to use the word, "programming". I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying- I thought, let's kill two birds with one stone. Let's take a word which has an absolutely precise meaning, namely dynamic, in the classical physical sense. It also has a very interesting property as an adjective, and that is it's impossible to use the word, dynamic, in the pejorative sense. Try thinking of some combination which will possibly give it a pejorative meaning. It's impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.

Richard Bellman, Eye of the Hurricane an autobiography, p. 159
Problem: Input $w_1 \ w_2 \ldots \ w_n$, $t$ each $0 \leq w_i \leq k$

Output: Number of $x \in \{0,1\}^n : \sum w_i \ x_i = t$

Let's try a recursive approach...
• Problem: Input $w_1 w_2 \ldots w_n, t$ each $0 \leq w_i \leq k$

• Output: Number of $x \in \{0,1\}^n : \sum w_i x_i = t$

• $S(i,s) :=$ number of $x \in \{0,1\}^i$ such that $\sum_{j \leq i} w_j x_j = s$

• Structure of solutions: $S(i,s) = S(i-1,s) + S(i-1,s-w_i)$ $i = n,\ldots$

• Gives recursive approach: $T(n) =$ ?
Problem: Input \( w_1, w_2, \ldots, w_n, t \) each \( 0 \leq w_i \leq k \)

Output: Number of \( x \in \{0,1\}^n : \sum w_i x_i = t \)

\[ S(i,s) := \text{number of } x \in \{0,1\}^i \text{ such that } \sum_{j \leq i} w_j x_j = s \]

Structure of solutions: \( S(i,s) = S(i-1,s) + S(i-1,s-w_i) \) \( i = n, \ldots \)

Gives recursive approach: \( T(n) = 2 \ T(n-1) \ \Rightarrow \ T(n) \geq 2^n \)

How can we do faster when \( k = n \)?
Problem: Input \(w_1, w_2, \ldots, w_n, t\) each \(0 \leq w_i \leq k\)

Output: Number of \(x \in \{0,1\}^n: \sum w_i x_i = t\)

\(S(i,s) := \) number of \(x \in \{0,1\}^i\) such that \(\sum_{j \leq i} w_j x_j = s\)

Structure of solutions: \(S(i,s) = S(i-1,s) + S(i-1,s-w_i)\) \(i = n, \ldots\)

Gives recursive approach: \(T(n) = 2 \ T(n-1) \Rightarrow T(n) \geq 2^n\)

How can we do faster when \(k = n\)?

Stop solving the same problems over and over again!

Total sum is \(\leq kn\), so there really are only \(?\) different \(S(i,t)\)
Problem: Input \( w_1 \ w_2 \ldots \ w_n \), \( t \) each \( 0 \leq w_i \leq k \)

Output: Number of \( x \in \{0,1\}^n : \sum w_i \ x_i = t \)

\[
S(i,s) := \text{number of } x \in \{0,1\}^i \text{ such that } \sum_{j \leq i} w_j \ x_j = s
\]

Structure of solutions: \( S(i,s) = S(i-1,s) + S(i-1,s-w_i) \) \( i = n, \ldots \)

Gives recursive approach: \( T(n) = 2 \ T(n-1) \Rightarrow T(n) \geq 2^n \)

How can we do faster when \( k = n \) ?

Stop solving the same problems over and over again!

Total sum is \( \leq kn \), so there really are only \( kn^2 \) different \( S(i,t) \) Just solve all of those!
Problem: Input \( w_1 \ w_2 \ldots \ w_n \), \( t \) each \( 0 \leq w_i \leq k \)

Output: Number of \( x \in \{0,1\}^n : \sum w_i \ x_i = t \)

- Fill first column
- (for \( i = 2 \ldots n \))
  (for \( s = 0 \ldots kn \))

Algorithm
Problem: Input \( w_1 \ w_2 \ldots \ w_n \), \( t \) each \( 0 \leq w_i \leq k \)

Output: Number of \( x \in \{0,1\}^n : \sum w_i x_i = t \)

- Fill first column
- (for \( i = 2 \ldots n \))
  - (for \( s = 0 \ldots kn \))
    - \( S(i,s) = S(i-1,s) + S(i-1,s-w_i) \)

\[ T(n) = ? \]
Problem: Input \( w_1 \ w_2 \ldots \ w_n , t \) each \( 0 \leq w_i \leq k \)

Output: Number of \( x \in \{0,1\}^n : \sum w_i x_i = t \)

- Fill first column
  - (for \( i = 2 \ldots n \))
    - (for \( s = 0 \ldots kn \))
      \[ S(i,s) = S(i-1,s) + S(i-1,s-w_i) \]

- \( T(n) = O(kn^2) \)
- Problem: Input \( w_1 \ w_2 \ ... \ w_n \), \( t \) each \( 0 \leq w_i \leq k \)
- Output: Number of \( x \in \{0,1\}^n : \sum w_i x_i = t \)

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- Fill first column
- (for \( i = 2 \ldots n \))
  - (for \( s = 0 \ldots kn \))
    \[
    S(i,s) = S(i-1,s) + S(i-1,s-w_i)
    \]
- Space: Trivial: \( kn^2 \) Better: ??
Problem: Input $w_1 w_2 \ldots w_n$, $t$ each $0 \leq w_i \leq k$

Output: Number of $x \in \{0,1\}^n : \sum w_i x_i = t$

Fill first column

(for $i = 2 \ldots n$)

(for $s = 0 \ldots kn$)

$S(i,s) = S(i-1,s) + S(i-1,s-w_i)$

Space: Trivial: $kn^2$ Better: $O(kn)$, just keep two columns
Steps for dynamic programming approach:

- Identify subproblems (here $S(i,s)$)
- Count subproblems (here $kn^2$)
- Obtain recursion (here $S(i,s) = S(i-1,s) + S(i-1,s-w_i)$)

(aka structure of solutions, optimal substructure property)

- The algorithm solves all the subproblems

- Running time = Number of subproblems (here $kn^2$) x Time to compute recursion (here $O(1)$)
Problem: Have $t$ and $\infty$ piles of coins of values $d_1, \ldots, d_k$

Want to use minimum number of coins to obtain target $t$

Example: $d = (5,4,1)$  $t = 8$  $t = 5+1+1+1, \quad t = 4+4$

Subproblems: $\text{Cost}[c] := \text{number of coins to obtain } c$

Number of subproblems: $t$
Problem: Have $t$ and $\infty$ piles of coins of values $d_1, \ldots, d_k$

Want to use minimum number of coins to obtain target $t$

Example: $d = (5, 4, 1)$  $t = 8$  $t = 5+1+1+1$,  $t = 4+4$

Structure of solution: $\text{Cost}[c] = ?$

Suppose you obtain $c$ with some optimal number of coins. If coin of type $d_i$ was used, then your coins without $d_i$ must be optimal for $c – d_i$
• Problem: Have $t$ and $\infty$ piles of coins of values $d_1, \ldots, d_k$
• Want to use minimum number of coins to obtain target $t$
• Example: $d = (5,4,1)$ $t = 8$ $t = 5+1+1+1$, $t = 4+4$

• Structure of solution: $\text{Cost}[c] = 1 + \min_{i \leq k} \text{Cost}[c - d_i ]$

Algorithm:
• Initialize vector Cost to $\infty$
• For ($c = 1..t$) ???
Problem: Have \( t \) and \( \infty \) piles of coins of values \( d_1, \ldots, d_k \).

Want to use minimum number of coins to obtain target \( t \).

Example: \( d = (5,4,1) \), \( t = 8 \) \( \rightarrow \) \( t = 5+1+1+1 \), \( t = 4+4 \).

Structure of solution: \( \text{Cost}[c] = 1 + \min_{i \leq k} \text{Cost}[c - d_i] \).

Algorithm:
- Initialize vector \( \text{Cost} \) to \( \infty \).
- For \((c = 1..t)\) \( \text{Cost}[c] = 1 + \min_{i \leq k} \text{Cost}[c - d_i] \).

\( T(n) = kn \)

To know which coins to use, store \( \text{argmin} \)

Can reconstruct solution backwards.
• Dynamic programming in economics

• Let us plan Bob's next L years.

• He has $w$, and every year makes $w$

• At the beginning of each year, he must decide how much to consume, rest is saved.
  Savings earn interest $(1+\rho)$ (round to integer)
  Consuming $C$ yields utility $\log(C)$
  $(\$10K \text{ vs. } \$20K \text{ is different from } \$1M \text{ vs. } \$1M+\$10K)$

• He wants to maximize sum of utility throughout his lifetime
Life can only be understood backwards; but it must be lived forwards.

Soren Kierkegaard
Formulate as problem

$U[k,i] :=$ utility for years $i, i+1, \ldots, L$ if at beginning of year $i$ have $k$. Note $k$ integer $\leq M := wL (1 + \rho)^L$

$U[k,L] := ?$

How much should Bob consume in his last year of life?
• Formulate as problem

• $U[k,i] := \text{utility for years } i, i+1, \ldots, L \text{ if at beginning of year } i \text{ have } $k. \text{ Note } k \text{ integer } \leq M := wL \left(1 + \rho \right)^L$

• $U[k,L] := \log(k)$
  Consumption = $k$, because at last year $L$ he spends all

• $U[k,i] := \text{What recursion for } i < L$?
• Formulate as problem

• $U[k,i] := \text{utility for years } i, i+1, \ldots, L \text{ if at beginning of year } i \text{ have }$ $\$k. \text{ Note } k \text{ integer } \leq M := wL (1 + \rho)^L$

• $U[k,L] := \log(k)$
  
  Consumption = $k$, because at last year $L$ he spends all

• $U[k,i] := \max_c : 0 \leq c \leq M \log(c) + U[(k - c)(1+\rho) + w, i+1]$

  Consumption = $\text{argmax}$

• Time = $O(LM^2)$
• Slightly more realism

• With probability q Bob earns interest rate \((1+\rho)\)

• With probability \(1-q\) Bob loses money rate \((1- \rho )\)

• \(U[k,i] :=\) expected utility for years i, i+1, \ldots, L if at beginning of year i has $k

• \(U[k,L] := \log(k)\)

• \(U[k,i] := \max_{c: 0 \leq c \leq M} \log(c) + ?\)
• Slightly more realism

• With probability $q$ Bob earns interest rate $(1+\rho)$

• With probability $1-q$ Bob loses money rate $(1-\rho)$

• $U[k,i] := \text{expected utility for years } i, i+1, \ldots, L \text{ if at beginning of year } i \text{ has } $k$

• $U[k,L] := \log(k)$

• $U[k,i] := \max_{c: 0 \leq c \leq M} \log(c) + q U[(k - c)(1+\rho) + w, i+1] + (1-q) U[(k - c)(1-\rho) + w, i+1]$
• Longest common subsequence

• Given two strings X and Y want to find a longest subsequence Z,
  i.e. string Z whose symbols appear in X, Y in the same order, but not necessarily consecutively

• Example: X = XMJYAUZ
  Y = MZJAWXU
  Z = MJAU
Definition: For a string $X = (x_1, x_2, \ldots, x_n)$, we denote by $X_i$ the prefix $(x_1, x_2, \ldots, x_i)$.

For two strings $X$ and $Y$ define

$L(i,j) = \text{length longest subsequence of } X_i \text{ and } Y_j$

- $= ?$ if $i = 0$ or $j = 0$
- $= ?$ if $x_i = y_j$
- $= ?$ if $x_i \neq y_j$
• **Definition:** For a string $X = (x_1, x_2, \ldots, x_n)$, we denote by $X_i$ the prefix $(x_1, x_2, \ldots, x_i)$.

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• For two strings \( X \) and \( Y \) define

\[
L(i,j) = \text{length longest subsequence of } X_i \text{ and } Y_j
\]

\[
= 0 \quad \text{if } i = 0 \text{ or } j = 0
\]

\[
= L(i-1,j-1) + 1 \quad \text{if } x_i = y_j
\]

\[
= ? \quad \text{if } x_i \neq y_j
\]
• **Definition:** For a string $X = (x_1, x_2, \ldots, x_n)$, we denote by $X_i$ the prefix $(x_1, x_2, \ldots, x_i)$.

• For two strings $X$ and $Y$ define

$$L(i,j) = \text{length longest subsequence of } X_i \text{ and } Y_j$$

- $= 0$ if $i = 0$ or $j = 0$
- $= L(i-1,j-1) + 1$ if $x_i = y_j$
- $= \max \{L(i-1,j), L(i,j-1) \}$ if $x_i \neq y_j$
- \textbf{LCSLength}(X[1..m], Y[1..n])
  
  \[ L = \text{zero array}(0..m, 0..n) \]

  \[
  \text{for } i := 1..m \\
  \text{for } j := 1..n \\
  \begin{align*}
  &\text{if } X[i] = Y[j] \\
  &\quad L[i,j] := L[i-1,j-1] + 1 \\
  &\text{else} \\
  &\quad L[i,j] := \text{max}(L[i,j-1], L[i-1,j])
  \end{align*}
  \]

  return \( L[m,n] \)

- \text{Running time} = \mathcal{O}(mn)
• As before, if we want to output the sequence, we record which rule was used at each point

<table>
<thead>
<tr>
<th></th>
<th>Ø</th>
<th>A</th>
<th>G</th>
<th>C</th>
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</thead>
<tbody>
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<td>C</td>
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<td>↑(A) (G)</td>
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<td>←(AC) &amp; (GC) &amp; (GA)</td>
<td>←(AC) &amp; (GC) &amp; (GA)</td>
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</tbody>
</table>

• Then we can reconstruct the sequence backwards.
We have described dynamic programming in an iterative “bottom-up” fashion, i.e., solve all the problems from the smallest to the biggest.

Alternatively, dynamic programming may be implemented in a “top-down” recursive fashion. You keep a list of the subproblems solved, and at the beginning you check if the current subproblem was already solved, if so you just read off the solution and return.

This is called Memoization.

Recall even divide-and-conquer may be implemented either in a recursive “top-down” fashion, or in an iterative “bottom-up” fashion.
• Dynamic programming requires solving all subproblems, leads to algorithms with running time usually $n^2$ or $n^3$.

• Sometimes, greedy is faster.
Greedy Algorithms

A greedy algorithm always makes the choice that looks best at the moment.

That is, it keeps making locally optimal decision in the hope that this will lead to a globally optimal solution.
Activity Selection problem

Input: Set of $n$ activities that need the same resource.
$A:= \{ a_1, a_2, \ldots, a_n \}$, $\forall i \in [n]$ $a_i = [s_i, f_i)$.

Activity $a_i$ takes time $[s_i, f_i)$. Activities $a_i$, $a_j$ are compatible if $s_j \geq f_i$.

Output:
Maximum-size subset of mutually compatible activities.
Example:

<table>
<thead>
<tr>
<th>(a_i)</th>
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<tr>
<td>(s_i)</td>
<td>1</td>
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<td>6</td>
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<tr>
<td>(f_i)</td>
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A set of compatible activities = ?
Example:

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<tr>
<th>a_i</th>
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A set of compatible activities = (a_3, a_9, a_{11}).
Example:

A set of compatible activities = \( (a_3, a_9, a_{11}) \).

A maximal set of compatible activities = ?
Example:

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<tr>
<td>$a_i$</td>
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</tbody>
</table>

A set of compatible activities = \((a_3,a_9,a_{11})\).

A maximal set of compatible activities = \((a_1,a_4,a_8,a_{11})\).
Example:

<table>
<thead>
<tr>
<th>$a_i$</th>
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A set of compatible activities = $(a_3, a_9, a_{11})$.

A maximal set of compatible activities = $(a_1, a_4, a_8, a_{11})$.

Is there another maximal set?
Example:

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<tr>
<td>$s_i$</td>
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<td>$f_i$</td>
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</table>

A set of compatible activities = $(a_3, a_9, a_{11})$.

A maximal set of compatible activities = $(a_1, a_4, a_8, a_{11})$.

Is there another maximal set? Yes. $(a_2, a_4, a_9, a_{11})$
• **Claim:** some optimal solution contains activity with earliest finish time

• **Proof:**
  Let \([s^*, f^*]\) be activity with earliest finish time \(f^*\)

  Let \(S\) be an optimal solution
  Write \(S = S' \cup [s,f)\) where \([s,f)\) has earliest finish time among activities in \(S\)

  Then \(S' \cup [s^*, f^*]\) is also an optimal solution, because every activity in \(S'\) has start time > \(f > f^*\).
• **Greedy Algorithm:**
  Pick activity with earliest finish time,
  that does not overlap with activities already picked
  Repeat

• **Claim:** The algorithm is correct
• **Proof:** Follows from applying previous claim iteratively.

• Let us see the algorithm in more detail
Greedy activity selection algorithm

activity-selection(A) { 
  sort A increasingly according to f[i];
  n:= length[A];
  S:={a[i]};
  i:=1;
  for (m=2; m ≤ n; m++)
    if (s[m] ≥ f[i] ) {
      Add a[i] to S;
      i :=m;
    }
  return S
}
activity-selection(A) {  
    sort A increasingly according to f[i];  
    n:= length[A];  
    S:={a[i]};  
    i:=1;  
    for (m=2; m ≤ n; m++) {  
        if (s[m] ≥ f[i] ) {  
            Add a[i] to S;  
            i :=m;  
        }  
    }  
    return S;  
}
activity-selection(A) { 
    sort A increasingly according to f[i]; 
    n:= length[A]; 
    S:=\{a[i]\}; 
    i:=1; 
    for (m=2; m ≤ n; m++) 
        if (s[m] ≥ f[i] ) { 
            Add a[i] to S; 
            i :=m; 
        }
    return S;
}
activity-selection(A) {
    sort A increasingly according to f[i];
    n := length[A];
    S := {a[i]};
    i := 1;
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activity-selection(A) {
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    S := {a[i]};
    i := 1;
    for (m = 2; m ≤ n; m++)
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
    return S;
}

Example:
S := \{a_1\}
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:=\{a[i]\};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;
        }
    return S;
}

Example:
S:=\{a_1\}

<table>
<thead>
<tr>
<th>i</th>
<th>a_i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>1</td>
<td>s_i</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>5</td>
<td>3</td>
<td>5</td>
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<td>2</td>
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</tr>
<tr>
<td>2</td>
<td>f_i</td>
<td>4</td>
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<td>12</td>
<td>13</td>
<td>14</td>
</tr>
</tbody>
</table>

Example:

\[ \begin{array}{c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
1 & 2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14
\end{array} \]
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:={a[i]};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
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    i := 1;
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        if (s[m] ≥ f[i]) {
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}
activity-selection(A) {
    sort A increasingly according to f[i];
    n := length[A];
    S := {a[i]};
    i := 1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
    return S;
}

Example:
S := {a_1}

\[
\begin{array}{c|cccccccccccc}
  a_i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
  s_i & 1 & 3 & 0 & 5 & 3 & 5 & 6 & 8 & 8 & 2 & 12 \\
  f_i & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\end{array}
\]

\[
\begin{array}{c|cccccccccccc}
  a_i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
  s_i & 1 & 3 & 0 & 5 & 3 & 5 & 6 & 8 & 8 & 2 & 12 \\
  f_i & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\end{array}
\]
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:={a[i]};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;}
    return S;
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    n:= length[A];
    S:=\{a[i]\};
i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i])
            Add a[i] to S;
        i := m;
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  S:=\{a[i]\};
  i:=1;
  for (m=2; m ≤ n; m++)
    if (s[m] ≥ f[i] ) {
      Add a[i] to S;
      i :=m;}
  return S;
}

Example:
S:=\{a_1\}

<table>
<thead>
<tr>
<th>i</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
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<td>2</td>
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</tbody>
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<table>
<thead>
<tr>
<th>a_i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tr>
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<td>11</td>
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<td>13</td>
<td>14</td>
</tr>
</tbody>
</table>
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:=\{a[i]\};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;}
    return S;
}

Example:
S:=\{a_1,a_4\}

\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\(a_i\) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
\(s_i\) & 1 & 3 & 0 & 5 & 3 & 5 & 6 & 8 & 8 & 2 & 12 \\
\hline
\(f_i\) & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
\end{tabular}
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:={a[i]};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;
        }
    return S;
}
activity-selection(A) {
    sort A increasingly according to f[i];
    n := length[A];
    S := {a[i]};
    i := 1;
    for (m = 2; m ≤ n; m++) {
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
    }
    return S;
}

Example:
S := {a₁, a₄}
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:=\{a[i]\};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;
        }
    return S;
}
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:=\{a[i]\};
    i:=1;
    for (m=2; m \leq n; m++)
        if (s[m] \geq f[i] ) {
            Add a[i] to S;
            i :=m;}
    return S;
}

Example:
S:=\{a_1 ,a_4\}

<table>
<thead>
<tr>
<th>i</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>(a_i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>8</th>
<th>9</th>
<th>10</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(s_i)</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>5</td>
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<td>5</td>
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</table>

Example:
\begin{center}
\begin{tabular}{cccccccccccc}
\hline
i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
A   & 1 & 6 & 5 & 7 & 8 & 9 & 11 & 12 & 13 & 14 &  \\
\hline
\end{tabular}
\end{center}
Example:
$S := \{ a_1, a_4 \}$

```plaintext
activity-selection(A) {
    sort A increasingly according to $f[i]$;
    n := length[A];
    S := \{ a[i] \};
    i := 1;
    for (m = 2; m \leq n; m++)
        if ($s[m] \geq f[i]$)
            Add a[i] to S;
            i := m;
}
```

```
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
$i$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
$a_i$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
$s_i$ & 1 & 3 & 0 & 5 & 3 & 5 & 6 & 8 & 2 & 12 & \\
\hline
$f_i$ & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
\end{tabular}
```

```plaintext
Example:
$S := \{ a_1, a_4 \}$

```
```
activity-selection(A) {
    sort A increasingly according to f[i];
    n := length[A];
    S := {a[i]};
    i := 1;
    for (m = 2; m ≤ n; m++)
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
    return S;
}

Example:
S := \{a_1, a_4\}

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<tr>
<th>a_i</th>
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<tbody>
<tr>
<td>s_i</td>
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<td>10</td>
<td>11</td>
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</tr>
</tbody>
</table>

Example:

\[\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 5 & 6 & 7 & 2 & 4 & 8 & 11 & 3 & 9 & 7 & 10 \\
\end{array}\]
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:={a[i]};
    i:=1;
    for (m=2; m ≤ n; m++)
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    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i := m;
        }
    return S;
}

Example:
S:= {a_1, a_4}

<table>
<thead>
<tr>
<th>a_i</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</tbody>
</table>

Graphical representation of activities with start times (s) and finish times (f):
Example:
\[ S := \{a_1, a_4\} \]
\[ s[8] \geq f[4] \] ?

```
activity-selection(A) {
    sort A increasingly according to \( f[i] \);
    n := length[A];
    S := \{a[i]\};
    i := 1;
    for (m=2; m \leq n; m++)
        if (s[m] \geq f[i] ) {
            Add a[i] to S;
            i := m;
        }
    return S;
}
```
activity-selection(A) {
    sort A increasingly according to f[i];
    n := length[A];
    S := {a[i]};
    i := 1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
} return S;

Example:
S := \{a_1, a_4, a_8\}
activity-selection(A) {
    sort A increasingly according to f[i];
    n := length[A];
    S := {a[i]};
    i := 1;
    for (m = 2; m ≤ n; m++)
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
    return S;
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activity-selection(A) {
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    S := {a[i]};
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    for (m = 2; m ≤ n; m++)
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
    return S;
}

Example:
S := \{a_1, a_4, a_8\}
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:={a[i]};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
    return S;
}
Example:
\[ S := \{ a_1, a_4, a_8 \} \]
\[ s[9] < f[8] \]

```
activity-selection(A) {
    sort A increasingly according to f[i];
    n := length[A];
    S := \{ a[i] \};
    i := 1;
    for (m = 2; m ≤ n; m++)
        if (s[m] ≥ f[i])
            Add a[i] to S;
            i := m;
} return S;
```
activity-selection(A) {
    sort A increasingly according to f [i];
    n:= length[A];
    S:=a[i];
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;}
    return S;
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activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:={a[i]};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;
        }
    return S;
}

Example:
S:={a_1 ,a_4 ,a_8}
s[10] < f[8]
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:={a[i]};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;}
    return S;
}

Example:
S:={a_1 ,a_4 ,a_8 }
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:=a[i];
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;
        }
    return S;
}
activity-selection(A) {
    sort A increasingly according to f[i];
    n:= length[A];
    S:={a[i]};
    i:=1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i] ) {
            Add a[i] to S;
            i :=m;
        }
    return S;
}
activity-selection(A) {
    sort A increasingly according to f[i];
    n := length[A];
    S := {a[i]};
    i := 1;
    for (m=2; m ≤ n; m++)
        if (s[m] ≥ f[i]) {
            Add a[i] to S;
            i := m;
        }
    return S;
}

Example:
S := \{a_1, a_4, a_8, a_{11}\}

m = 12,
n = 11

<table>
<thead>
<tr>
<th>a_i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
<th>10</th>
<th>11</th>
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</thead>
<tbody>
<tr>
<td>s_i</td>
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</tr>
</tbody>
</table>
Example:
S:={a_1 ,a_4 ,a_8 ,a_{11}}
m = 12,
n = 11

activity-selection(A) {
sort A increasingly according to f [i];
n:= length[A];
S:={a[i]};
i:=1;
for (m=2; m ≤ n; m++)
  if (s[m] ≥ f[i] ) {
    Add a[i] to S;
    i :=m;}
}
activity-selection(A) {
sort A increasingly according to f[i];
n:= length[A];
S:=\{a[i]\};
i:=1;
for (m=2; m ≤ n; m++)
  if (s[m] ≥ f[i] ) {
    Add a[i] to S;
    i := m;
  } return S;
}
Deleted scenes
Knapsack

Input:
- \( S := \{(v_1, w_1), (v_2, w_2), \ldots, (v_n, w_n)\} \).
  - \((v_i, w_i)\) means item \(i\) is worth \(v_i\) and weighs \(w_i\).
- \(W\), weight-capacity of knapsack.

Output:
- Items that maximize value in knapsack.

Can we take a fraction of an item?
Fractional Knapsack : Yes
0-1 Knapsack : No
Fractional Knapsack

- Compute $v_i / w_i$ for each item.
- Sort $S$ according to $v_i / w_i$ decreasingly.
- Take as much as possible of the item with the most $v_i / w_i$. 
Fractional knapsack\((W, S)\)

Sort \(S\), decreasingly according to \(v_i / w_i\);

\(x[1..n] = 0; //x[i] = amount of i to be taken\)

\(weight = 0; i = 1;\)

\[\text{while (weight < W and i ≤ n)}\]
\[\quad \text{if weight + w[i] ≤ W} \{\]
\[\qquad x[i] = 1;\]
\[\qquad weight += w[i];\]
\[\qquad i++;\]
\[\} \quad \text{else}\{\]
\[\qquad x[i] = (W - weight) / w[i];\]
\[\qquad weight = W;\]
\[\}\]

return \(x\)
Fractional knapsack \((W,S)\)

Sort \(S\), decreasingly according to \(v_i / w_i\);

\[x[1..n] = 0; //x[i] = \text{amount of } i \text{ to be taken}\]

\[\text{weight} = 0; \text{ } i = 1;\]

\[
\text{while (weight < } W \text{ and } i \leq n) \\
\quad \text{if weight} + w[i] \leq W \{ \\
\quad \quad x[i] = 1; \\
\quad \quad \text{weight} += w[i]; \\
\quad \quad i++ \\
\quad \} \text{ else } \{ \\
\quad \quad x[i] = (W - \text{weight}) / w[i]; \\
\quad \quad \text{weight} = W; \\
\quad \} \\
\]

\[
\text{return } x
\]
Fractional knapsack \((W, S)\)

Sort \(S\), decreasingly according to \(v_i / w_i\);

for \((i = 1; i \leq n; i++)\)

\(x[i] = 0;\)

Weight = 0; \(i = 1;\)

while (weight < \(W\) and \(i \leq n\))

if weight + \(w[i]\) \(\leq W\)

then \(x[i] = 1;\)

\{weight = weight + \(w[i]\);

\(i = i + 1;\}\}

else

\{\(x[i] = (w - weight) / w[i];\)

weight = \(W;\}\}

return \(x\)

Running time:

\(T(n) = O(n \log n).\)