Summary: NFA and DFA recognize the same languages

We now return to the question:

- Suppose $A$, $B$ are regular languages, what about
- $\text{not } A := \{ w : w \text{ is not in } A \}$ \hspace{1cm} \text{REGULAR}
- $A \cup B := \{ w : w \text{ in } A \text{ or } w \text{ in } B \}$ \hspace{1cm} \text{REGULAR}
- $A \circ B := \{ w_1 w_2 : w_1 \text{ in } A \text{ and } w_2 \text{ in } B \}$
- $A^* := \{ w_1 w_2 \ldots w_k : k \geq 0, w_i \text{ in } A \text{ for every } i \}$
Theorem: If $A$, $B$ are regular languages, then so is
\[ A \cup B := \{ w : w \text{ in } A \text{ or } w \text{ in } B \} \]

Proof idea: Given DFA $M_A : L(M_A) = A$,
\[ \text{DFA } M_B : L(M_B) = B, \]

Construct NFA $N : L(N) = A \cup B$
Construction:

• Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$: $L(M_A) = A$,
  
  $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$: $L(M_B) = B$,

• Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

• $Q := ?$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ : $L(M_A) = A$,
- DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$ : $L(M_B) = B$,
- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  - $Q := \{q\} \cup Q_A \cup Q_B$ ,  $F := ?$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$
- DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B$

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  - $Q := \{q\} \cup Q_A \cup Q_B$, $F := F_A \cup F_B$
  - $\delta(r,x) := \{ \delta_A(r,x) \}$ if $r$ in $Q_A$ and $x \neq \varepsilon$
  - $\delta(r,x) := ?$ if $r$ in $Q_B$ and $x \neq \varepsilon$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ : $L(M_A) = A$
- DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$ : $L(M_B) = B$
- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  - $Q := \{q\} \cup Q_A \cup Q_B$, $F := F_A \cup F_B$
  - $\delta(r,x) := \{ \delta_A(r,x) \}$ if $r$ in $Q_A$ and $x \neq \varepsilon$
  - $\delta(r,x) := \{ \delta_B(r,x) \}$ if $r$ in $Q_B$ and $x \neq \varepsilon$
  - $\delta(q,\varepsilon) := ?$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A,$
- DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B,$

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  - $Q := \{q\} \cup Q_A \cup Q_B \quad , \quad F := F_A \cup F_B$
  - $\delta(r, x) := \{ \delta_A(r, x) \}$ if $r \in Q_A$ and $x \neq \varepsilon$
  - $\delta(r, x) := \{ \delta_B(r, x) \}$ if $r \in Q_B$ and $x \neq \varepsilon$
  - $\delta(q, \varepsilon) := \{q_A, q_B\}$

- We have $L(N) = A \cup B$
Example

Is \( L = \{ w \in \{0,1\}^* : |w| \text{ is divisible by 3 OR} \) \n\quad \text{w starts with a 1}\} \) regular?
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OR is like \( U \), so try to write \( L = L_1 \cup L_2 \)
where \( L_1, L_2 \) are regular
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\( L_1 = \{w : |w| \text{ is div. by 3}\} \quad \quad L_2 = \{w : w \text{ starts with a 1}\} \)

\[
M_1 = \begin{array}{c}
0,1 \\
\downarrow \\
0,1 \\
\downarrow \\
0,1 \\
\end{array}
\]

\[ L(M_1) = L_1 \]
Example

Is \( L = \{ w \in \{0,1\}^* : |w| \text{ is divisible by 3 OR } w \text{ starts with a 1} \} \) regular?

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\( L_1 = \{ w : |w| \text{ is div. by 3} \} \) \hspace{1cm} \( L_2 = \{ w : w \text{ starts with a 1} \} \)

\[ M_1 = \]

\[ M_2 = \]

\[ L(M_1) = L_1 \]

\[ L(M_2) = L_2 \]
Example

Is \( L = \{ w \in \{0,1\}^* : |w| \text{ is divisible by 3 OR } \) w starts with a 1\} regular?

OR is like U, so try to write \( L = L_1 \cup L_2 \)
where \( L_1, L_2 \) are regular

\( L_1 = \{ w : |w| \text{ is div. by 3} \} \)
\( L_2 = \{ w : w \text{ starts with a 1} \} \)

\( \Rightarrow L \) is regular.

\( L(M) = L(M_1) \cup L(M_2) = L_1 \cup L_2 = L \)
We now return to the question:

- Suppose $A$, $B$ are regular languages, then

  - $\text{not } A := \{ w : w \text{ is not in } A \}$ \text{ REGULAR}
  - $A \cup B := \{ w : w \in A \text{ or } w \in B \}$ \text{ REGULAR}
  - $A \cdot B := \{ w_1 w_2 : w_1 \in A \text{ and } w_2 \in B \}$
  - $A^* := \{ w_1 w_2 \ldots w_k : k \geq 0 , w_i \in A \text{ for every } i \}$
Theorem: If A, B are regular languages, then so is
\[ A \circ B := \{ w : w = xy \text{ for some } x \text{ in } A \text{ and } y \text{ in } B \}. \]

Proof idea: Given DFAs \( M_A, M_B \) for A, B

construct NFA \( N : L(N) = A \circ B \).
Construction:

• Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$: $L(M_A) = A$,
  
  DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$: $L(M_B) = B$,

• Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

• $Q := ?$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,
  $DFA \: M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B$,
- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  - $Q := Q_A \cup Q_B$, $q := ?$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A,$
  
  $DFA \; M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B,$

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  
  $Q := Q_A \cup Q_B \; , \; q := q_A \; , \; F := ?$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$, DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B$,

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  - $Q := Q_A \cup Q_B$ ,  $q := q_A$ ,  $F := F_B$
  - $\delta(r,x) := ?$ if $r \in Q_A$ and $x \neq \varepsilon$
Construction:

• Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ : $L(M_A) = A$,
  
  DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$ : $L(M_B) = B$,

• Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  
  $Q := Q_A \cup Q_B$ ,  $q := q_A$ ,  $F := F_B$

• $\delta(r,x) := \{ \delta_A(r,x) \}$ if $r$ in $Q_A$ and $x \neq \epsilon$

• $\delta(r,\epsilon) := ?$ if $r$ in $F_A$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,
- DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B$,
- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  - $Q := Q_A \cup Q_B$ ,  $q := q_A$ ,  $F := F_B$
  - $\delta(r,x) := \{ \delta_A(r,x) \}$ if $r \in Q_A$ and $x \neq \varepsilon$
  - $\delta(r,\varepsilon) := \{ q_B \}$ if $r \in F_A$
  - $\delta(r,x) := ?$ if $r \in Q_B$ and $x \neq \varepsilon$
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,
  
  $DFA M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B$,

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
  
  $Q := Q_A \cup Q_B$,  \( q := q_A \),  \( F := F_B \)

- $\delta(r,x) := \{ \delta_A(r,x) \}$ if $r$ in $Q_A$ and $x \neq \varepsilon$

- $\delta(r,\varepsilon) := \{ q_B \}$ if $r$ in $F_A$

- $\delta(r,x) := \{ \delta_B(r,x) \}$ if $r$ in $Q_B$ and $x \neq \varepsilon$

- We have $L(N) = A \circ B$
Example

Is \( L = \{ w \in \{0,1\}^* : w \text{ contains a } 1 \text{ after a } 0 \} \) regular?

Note: \( L = \{01, 0001001, 111001, \ldots \} \)
Example

Is \( L = \{ w \in \{0,1\}^* : w \text{ contains a } 1 \text{ after a } 0 \} \) regular?

Let \( L_0 = \{ w : w \text{ contains a } 0 \} \), \( L_1 = \{ w : w \text{ contains a } 1 \} \). Then \( L = L_0 \circ L_1 \).
Example

Is \( L = \{ w \in \{0,1\}^* : \text{w contains a 1 after a 0} \} \) regular?

Let \( L_0 = \{ w : \text{w contains a 0} \} \)
\( L_1 = \{ w : \text{w contains a 1} \} \). Then \( L = L_0 \circ L_1 \).

\[
M_0 = \begin{array}{c}
1 & 0,1 \\
\circ & \circ
\end{array}
\]

\( L(M_0) = L_0 \)
Example

Is \( L = \{ w \in \{0,1\}^* : w \text{ contains a 1 after a 0} \} \) regular?

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\[
M_0 = \begin{array}{ccc}
1 & 0,1 \\
\bullet & 0 & \bullet
\end{array}
\]

\( L(M_0) = L_0 \)

\[
M_1 = \begin{array}{ccc}
0 & 0,1 \\
\bullet & 1 & \bullet
\end{array}
\]

\( L(M_1) = L_1 \)
Example

Is \( L = \{w \in \{0,1\}^* : w \text{ contains a } 1 \text{ after a } 0\} \) regular?

Let \( L_0 = \{w : w \text{ contains a } 0\} \)
\( L_1 = \{w : w \text{ contains a } 1\} \).
Then \( L = L_0 \circ L_1 \).

\[
L(M) = L(M_0) \circ L(M_1) = L_0 \circ L_1 = L
\]

\( \Rightarrow \) \( L \) is regular.
We now return to the question:

- Suppose $A$, $B$ are regular languages, then

- $\text{not } A := \{ w : w \text{ is not in } A \}$ \hspace{1cm} \text{REGULAR}
- $A \cup B := \{ w : w \text{ in } A \text{ or } w \text{ in } B \}$ \hspace{1cm} \text{REGULAR}
- $A \circ B := \{ w_1 \, w_2 : w_1 \in A \text{ and } w_2 \in B \}$ \hspace{1cm} \text{REGULAR}
- $A^* := \{ w_1 \, w_2 \ldots \, w_k : k \geq 0 , w_i \text{ in } A \text{ for every } i \}$
Theorem: If $A$ is a regular language, then so is
\[ A^* := \{ w : w = w_1 \ldots w_k, \ w_i \ \text{in} \ A \ \text{for} \ i=1,\ldots,k \} \]

- Proof idea: Given DFA $M_A : L(M_A) = A$,
  Construct NFA $N : L(N) = A^*$

![Diagram of DFA and NFA]
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,

Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

- $Q := \text{?}$
Construction:

• Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,

  Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

• $Q := \{q\} \cup Q_A$, $F := \ ?$
Construction:
• Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,

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• $\delta(r,x) := \varepsilon$ if $r$ in $Q_A$ and $x \neq \varepsilon$
Construction:

Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A,$

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- $\delta(r,x) := \{ \delta_A(r,x) \}$ if $r \in Q_A$ and $x \neq \varepsilon$
- $\delta(r,\varepsilon) := ?$ if $r \in \{q\} \cup F_A$
Construction:

Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,

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- $\delta(r,\varepsilon) := \{q_A\}$ if $r \in \{q\} \cup F_A$
- We have $L(N) = A^*$
Example

Is \( L = \{ w \in \{0,1\}^* : w \text{ has even length} \} \) regular?
Example

Is \( L = \{w \in \{0,1\}^* : w \text{ has even length}\} \) regular?

Let \( L_0 = \{w : w \text{ has length } 2\} \). Then \( L = L_0^* \).
Example

Is \( L = \{ w \in \{0,1\}^* : w \text{ has even length} \} \) regular?

Let \( L_0 = \{ w : w \text{ has length } = 2 \} \). Then \( L = L_0^* \).

\[
\begin{align*}
M_0 &= \\
&= \\
&= \\
&= \\
L(M_0) &= L_0
\end{align*}
\]
Example

Is \( L = \{ w \in \{0,1\}^* : w \text{ has even length} \} \) regular?

Let \( L_0 = \{ w : w \text{ has length } = 2 \} \). Then \( L = L_0^* \).

\[ M = \]

\[ L(M) = L(M_0)^* = L_0^* = L \]

\( \Rightarrow \) \( L \) is regular.
We now return to the question:

• Suppose A, B are regular languages, then

• \( \text{not } A := \{ w : w \text{ is not in } A \} \)

• \( A \cup B := \{ w : w \text{ in } A \text{ or } w \text{ in } B \} \)

• \( A \circ B := \{ w_1 \; w_2 : w_1 \text{ in } A \text{ and } w_2 \text{ in } B \} \)

• \( A^* := \{ w_1 \; w_2 \; \ldots \; w_k : k \geq 0, \; w_i \text{ in } A \text{ for every } i \} \)

are all regular!
We now return to the question:

- Suppose A, B are regular languages, then
- \( \text{not } A := \{ w : w \text{ is not in } A \} \)
- \( A \cup B := \{ w : w \text{ in } A \text{ or } w \text{ in } B \} \)
- \( A \circ B := \{ w_1 w_2 : w_1 \text{ in } A \text{ and } w_2 \text{ in } B \} \)
- \( A^* := \{ w_1 w_2 \ldots w_k : k \geq 0 , w_i \text{ in } A \text{ for every } i \} \)

What about \( A \cap B := \{ w : w \text{ in } A \text{ and } w \text{ in } B \} \)?
We now return to the question:
• Suppose A, B are regular languages, then
• \( \text{not } A := \{ w : w \text{ is not in } A \} \)
• \( A \cup B := \{ w : w \text{ in } A \text{ or } w \text{ in } B \} \)
• \( A \circ B := \{ w_1 \, w_2 : w_1 \text{ in } A \text{ and } w_2 \text{ in } B \} \)
• \( A^* := \{ w_1 \, w_2 \ldots w_k : k \geq 0, w_i \text{ in } A \text{ for every } i \} \)

De Morgan's laws: \( A \cap B = \text{not } ( (\text{not } A) \cup (\text{not } B) ) \)
By above, (not A) is regular, (not B) is regular, (not A) U (not B) is regular, not ( (not A) U (not B) ) = A \cap B regular
We now return to the question:

- Suppose \( A, B \) are regular languages, then

  - \( \text{not } A := \{ w : w \text{ is not in } A \} \)
  - \( A \cup B := \{ w : w \text{ in } A \text{ or } w \text{ in } B \} \)
  - \( A \circ B := \{ w_1 w_2 : w_1 \text{ in } A \text{ and } w_2 \text{ in } B \} \)
  - \( A^* := \{ w_1 w_2 \ldots w_k : k \geq 0, w_i \text{ in } A \text{ for every } i \} \)
  - \( A \cap B := \{ w : w \text{ in } A \text{ and } w \text{ in } B \} \)

are all regular