2-8-06: Strong normalization for System F

Review of the LR

$\mathcal{C}[\![\tau]\!]\delta = \{e \mid e \mapsto^* v \wedge v \in \mathcal{V}[\![\tau]\!]\delta\}$
$\mathcal{V}[\![\mathbf{T}]\!]\delta = \{c\}$
$\mathcal{V}[\![\tau_1 \to \tau_2]\!]\delta = \{v \mid \forall v' \in \mathcal{V}[\![\tau_1]\!]\delta . v(v') \in \mathcal{C}[\![\tau_2]\!]\delta\}$
$\mathcal{V}[\![\forall \alpha.\tau]\!]\delta = \{v \mid \forall \sigma . \emptyset \vdash \sigma \text{ type} \wedge \forall R \in \text{Cand}. v[\sigma] \in \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\sigma, R)\}$
$\mathcal{V}[\![\alpha]\!]\delta = \delta_S(\alpha)$

Lemma [Type substitution] $\quad \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\delta_T(\sigma), \mathcal{V}[\![\sigma]\!]\delta) = \mathcal{C}[\![\tau[\sigma/\alpha]]\!]\delta$. Equivalent to show $\mathcal{V}[\![\tau]\!]\delta, \alpha \mapsto (\delta_T(\sigma), \mathcal{V}[\![\sigma]\!]\delta) = \mathcal{V}[\![\tau[\sigma/\alpha]]\!]\delta$.

Proof: cases and induction on τ. Let $\delta' = \delta, \alpha \mapsto (\delta_T(\sigma), \mathcal{V}[\![\sigma]\!]\delta)$.

Case: $\tau = \mathbf{T}$. Trivial because $\mathbf{T}[\sigma/\alpha] = \mathbf{T}$.

Case: $\tau = \tau' \to \tau''$. ($\subseteq$) Suppose $v \in \mathcal{V}[\![\tau' \to \tau'']\!]\delta'$. To show: $v \in \mathcal{V}[\![\tau[\sigma/\alpha]]\!]\delta$. Note $\tau[\sigma/\alpha] = \tau'[\sigma/\alpha] \to \tau''[\sigma/\alpha]$. Suppose $v' \in \mathcal{V}[\![\tau'[\sigma/\alpha]]\!]\delta$. Suffices to show $v(v') \in \mathcal{C}[\![\tau''[\sigma/\alpha]]\!]\delta$. By induction, $\mathcal{V}[\![\tau'[\sigma/\alpha]]\!]\delta = \mathcal{V}[\![\tau']\!]\delta'$, so $v' \in \mathcal{V}[\![\tau']\!]\delta'$. We can apply the definition of the logical relation to deduce: $v(v') \in \mathcal{C}[\![\tau'']\!]\delta'$. By induction, $v(v') \in \mathcal{C}[\![\tau''[\sigma/\alpha]]\!]\delta$.

Other direction is analogous.

Case: $\tau = \alpha$. Need to show $\mathcal{V}[\![\alpha]\!]\delta' = \mathcal{V}[\![\alpha[\sigma/\alpha]]\!]\delta$. By definition, $\mathcal{V}[\![\alpha]\!]\delta' = \mathcal{V}[\![\sigma]\!]\delta$. Likewise $\mathcal{V}[\![\alpha[\sigma/\alpha]]\!]\delta = \mathcal{V}[\![\sigma]\!]\delta$.

Case: $\tau = \beta \neq \alpha$. $\mathcal{V}[\![\beta]\!]\delta' = \mathcal{V}[\![\beta[\sigma/\alpha]]\!]\delta$. But $\mathcal{V}[\![\beta[\sigma/\alpha]]\!]\delta = \mathcal{V}[\![\beta]\!]\delta = \delta_S(\beta) = \mathcal{V}[\![\beta]\!]\delta'$.

DO: last case.

Strong normalization

New reduction rules.

$$\frac{e \mapsto e'}{e[\sigma] \mapsto e'[\sigma]} \qquad \frac{}{(\Lambda\alpha.e)[\sigma] \mapsto e[\sigma/\alpha]} \qquad \frac{e \mapsto e'}{\Lambda\alpha.e \mapsto \Lambda\alpha.e'}$$

Logical relation:

$\mathcal{C}[\![\mathbf{T}]\!]\delta = \text{SN}$
$\mathcal{C}[\![\tau' \to \tau'']\!]\delta = \{e \mid \forall e' \in \mathcal{C}[\![\tau']\!]\delta . e(e') \in \mathcal{C}[\![\tau'']\!]\delta\}$
$\mathcal{C}[\![\forall\alpha.\tau]\!]\delta = \{e \mid \forall\sigma.\emptyset \vdash \sigma \text{ type}. \forall R \in \text{Cand}. e[\sigma] \in \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\sigma, R)\}$
$\mathcal{C}[\![\alpha]\!]\delta = \delta_S(\alpha)$
$\text{Cand} = \{R \mid R \text{ is a set of terms}\}$

Extension to paths:

$p ::= x \mid p(e) \mid p[\sigma]$

Lemma [Main lemma]

1. If $e \in \mathcal{C}[\![\tau]\!]\delta$ then $e \in \mathrm{SN}$.

2. If $p \in \mathrm{SN}$ then $p \in \mathcal{C}[\![\tau]\!]\delta$.

Part 1:

Proof: by induction on τ.

Check $\mathbf{T}$ and $\tau' \to \tau''$ cases.

Case: $\tau = \alpha$. We know that $e \in \mathcal{C}[\![\alpha]\!]\delta$ which means that $e \in \delta_S(\alpha)$. No good – we know nothing about $\delta_S(\alpha)$!

We need to know that $\delta_S(\alpha) \in \mathrm{SN}$. We extend Cand:

$\mathrm{Cand} = \{R \mid R \text{ is a set of terms}, R \subseteq \mathrm{SN}\}$

Case: $\tau = \forall\alpha.\tau'$. Let $\sigma = \mathbf{T}$ and $R = \mathrm{SN} = \mathcal{C}[\![\mathbf{T}]\!]\delta$. (Note $R \in \mathrm{Cand}$). By the definition of the logical relation, $e[\sigma] \in \mathcal{C}[\![\tau']\!]\delta, \alpha \mapsto (\sigma, R)$. By induction, $e[\sigma] \in \mathrm{SN}$. By the subterm property of SN, $e \in \mathrm{SN}$.

Note, we could have chosen $R = \emptyset$, which removes any future proof obligations.

Part 2:

Proof: by induction on τ.

($\mathbf{T}$ and arrow are the same as before, but with δ's)

Case: $\tau = \alpha$. We know $p \in \mathrm{SN}$. Need to show $p \in \delta_S(\alpha)$. We know that $\delta_S(\alpha) \subset \mathrm{SN}$, but this does not imply that $p \in \delta_S(\alpha)$.

Extend Cand:

$\mathrm{Cand} = \{R \mid R \text{ is a set of terms}, R \subseteq \mathrm{SN}, p \in \mathrm{SN} \implies p \in R\}$

Case: $\tau = \forall\alpha.\tau'$. Suppose σ is a type and $R \in \mathrm{Cand}$. To show: $p[\sigma] \in \mathcal{C}[\![\tau']\!]\delta, \alpha \mapsto (\sigma, R)$. Because $p \in \mathrm{SN}$, we have $p[\sigma]$ is also strongly normalizing. By induction, $p[\sigma] \in \mathcal{C}[\![\tau']\!]\delta, \alpha \mapsto (\sigma, R)$.

New definition: $S ::= \cdot \mid S(e) \mid S[\sigma]$. Write $S[e]$ for e plugged in for the hole in S.

$$\frac{e \in \mathrm{SN}}{S[(\lambda x.e')(e)] \to_{\mathrm{wh}} S[e'[e/x]]} \qquad \frac{}{S[(\Lambda\alpha.e)[\sigma]] \to_{\mathrm{wh}} S[e[\sigma/\alpha]]}$$

Lemma [Head expansion] If $e \to_{\mathrm{wh}} e'$ and $e' \in \mathcal{C}[\![\tau]\!]\delta$ then $e \in \mathcal{C}[\![\tau]\!]\delta$.

Proof: by induction on τ.

Case: $\tau = \mathbf{T}$. We need that SN is closed under head expansion. Suppose there were an infinite reduction sequence $(\Lambda\alpha.e)[\sigma] \to_{\mathrm{wh}} \cdots \to_{\mathrm{wh}} (\Lambda\alpha.f)[\sigma] \to_{\mathrm{wh}} f[\sigma/\alpha]$. This means that $e \to_{\mathrm{wh}} \cdots \to_{\mathrm{wh}} f$. To show: $e[\sigma/\alpha] \to^*_{\mathrm{wh}} f[\sigma/\alpha]$ (a lemma).

Case: $\tau = \tau' \to \tau''$. Suppose $f \in \mathcal{C}[\![\tau']\!]\delta$. To show: $e(f) \in \mathcal{C}[\![\tau'']\!]\delta$. By the logical relation, $e'(f) \in \mathcal{C}[\![\tau'']\!]\delta$. By induction, $e(f) \in \mathcal{C}[\![\tau'']\!]\delta$. Mr Pedantic says: $e \to_{\text{wh}} e'$ means $e = S[\dots]$ and $e' = S[\dots]$. So let $S' = S(f)$. Therefore $e(f) = S'[\dots] \to_{\text{wh}} e'(f) = S'[\dots]$.

Case: $\tau = \alpha$. To show: $e \in \delta_S(\alpha)$. We know that $e \to_{\text{wh}} e'$ and $e' \in \delta_S(\alpha)$. So, once more, we need to extend Cand.

Cand $= \{R \mid R \text{ is a set of terms}, R \subseteq \text{SN}, p \in \text{SN} \implies p \in R, e \to_{\text{wh}} e' \wedge e' \in R \implies e \in R\}$