2-8-06: Strong normalization for System F

Review of the LR

\[
\mathcal{C}[\tau]\delta = \{e \mid e \mapsto^* \tau \land \tau \in \mathcal{V}[\tau]\delta\}
\]

\[
\mathcal{V}[T]\delta = \{c\}
\]

\[
\mathcal{V}[\tau_1 \rightarrow \tau_2]\delta = \{v \mid \forall v' \in \mathcal{V}[\tau_1]\delta. v(v') \in \mathcal{C}[\tau_2]\delta\}
\]

\[
\mathcal{V}[\forall \alpha.\tau]\delta = \{v \mid \forall \sigma. 0 \vdash \sigma \text{ type } \land \forall \tau \in \text{Cand}. v[\sigma] \in \mathcal{C}[\tau]\delta, \alpha \mapsto (\sigma, R)\}
\]

\[
\mathcal{V}[\alpha]\delta = \delta_S(\alpha)
\]

**Lemma [Type substitution]** \[C[\tau]\delta, \alpha \mapsto (\delta_T(\sigma), \mathcal{V}[\sigma]\delta) = C[\tau[\sigma/\alpha]]\delta.\]

Equivalent to show \[\mathcal{V}[\tau]\delta, \alpha \mapsto (\delta_T(\sigma), \mathcal{V}[\sigma]\delta) = \mathcal{V}[\tau[\sigma/\alpha]]\delta.\]

**Proof:** cases and induction on \(\tau\). Let \(\tau' = \delta, \alpha \mapsto (\delta_T(\sigma), \mathcal{V}[\sigma]\delta).\)

**Case:** \(\tau = T\). Trivial because \(T[\sigma/\alpha] = T\).

**Case:** \(\tau = \tau' \rightarrow \tau''\). (\(\subseteq\)) Suppose \(v \in \mathcal{V}[\tau' \rightarrow \tau'']\delta'.\) To show: \(v \in \mathcal{V}[\tau[\sigma/\alpha]]\delta\).

Note \(\tau[\sigma/\alpha] = \tau'[\sigma/\alpha] \rightarrow \tau''[\sigma/\alpha]\). Suppose \(v' \in \mathcal{V}[\tau'[\sigma/\alpha]]\delta\). Suffices to show \(v(v') \in \mathcal{C}[\tau''[\sigma/\alpha]]\delta\). By induction, \(\mathcal{V}[\tau'[\sigma/\alpha]]\delta = \mathcal{V}[\tau']\delta',\) so \(v' \in \mathcal{V}[\tau']\delta'.\) We can apply the definition of the logical relation to deduce: \(v(v') \in \mathcal{C}[\tau''[\sigma/\alpha]]\delta'.\) By induction, \(v(v') \in \mathcal{C}[\tau''[\sigma/\alpha]]\delta\).

Other direction is analogous.

**Case:** \(\tau = \alpha\). Need to show \(\mathcal{V}[\alpha]\delta' = \mathcal{V}[\alpha[\sigma/\alpha]]\delta\). By definition, \(\mathcal{V}[\alpha]\delta' = \mathcal{V}[\sigma]\delta\).

Likewise \(\mathcal{V}[\alpha[\sigma/\alpha]]\delta = \mathcal{V}[\alpha]\delta\).

**Case:** \(\tau = \beta \neq \alpha\). \(\mathcal{V}[\beta]\delta' = \mathcal{V}[\beta[\sigma/\alpha]]\delta\). But \(\mathcal{V}[\beta[\sigma/\alpha]]\delta = \mathcal{V}[\beta]\delta = \delta_S(\beta) = \mathcal{V}[\beta]\delta'.\)

**DO:** last case.

**Strong normalization**

New reduction rules.

\[
e \mapsto e'
\]

\[
\mathcal{V}[\sigma] = \mathcal{V}[\sigma[\alpha]]
\]

\[
(\Lambda \alpha.e)[\sigma] \mapsto e[\sigma/\alpha]
\]

\[
\Lambda \alpha.e \mapsto \Lambda \alpha.e'
\]

Logical relation:

\[
\mathcal{C}[T]\delta = \text{SN}
\]

\[
\mathcal{C}[\tau' \rightarrow \tau'']\delta = \{e \mid \forall e' \in \mathcal{C}[\tau']\delta. e(e') \in \mathcal{C}[\tau'']\delta\}
\]

\[
\mathcal{C}[\forall \alpha.\tau]\delta = \{e \mid \forall \sigma. 0 \vdash \sigma \text{ type } \land \forall \tau \in \text{Cand}. e[\sigma] \in \mathcal{C}[\tau]\delta, \alpha \mapsto (\sigma, R)\}
\]

\[
\mathcal{C}[\alpha]\delta = \delta_S(\alpha)
\]

\(\text{Cand} = \{R \mid R \text{ is a set of terms}\}\)

Extension to paths:

\(p ::= x \mid p(e) \mid p[\sigma]\)
Lemma [Main lemma]

1. If \( e \in C[\tau] \delta \) then \( e \in SN \).
2. If \( p \in SN \) then \( p \in C[\tau] \delta \).

Part 1:

Proof: by induction on \( \tau \).

Check \( T \) and \( \tau' \rightarrow \tau'' \) cases.

Case: \( \tau = \alpha \). We know that \( e \in C[\alpha] \delta \) which means that \( e \in \delta_S(\alpha) \). No good – we know nothing about \( \delta_S(\alpha) \)!

We need to know that \( \delta_S(\alpha) \subseteq SN \). We extend \( \text{Cand} \):

\[ \text{Cand} = \{ R \mid R \text{ is a set of terms, } R \subseteq SN \} \]

Case: \( \tau = \forall \alpha.\tau' \). Let \( \sigma = T \) and \( R = SN = C[T] \delta \). (Note \( R \in \text{Cand} \)). By the definition of the logical relation, \( e[\sigma] \in C[\tau'] \delta, \alpha \mapsto (\sigma, R) \). By induction, \( e[\sigma] \in SN \). By the subterm property of \( SN \), \( e \in SN \).

Note, we could have chosen \( R = \emptyset \), which removes any future proof obligations.

Part 2:

Proof: by induction on \( \tau \).

(\( T \) and arrow are the same as before, but with \( \delta \)'s)

Case: \( \tau = \alpha \). We know \( p \in SN \). Need to show \( p \in \delta_S(\alpha) \). We know that \( \delta_S(\alpha) \subseteq SN \), but this does not imply that \( p \in \delta_S(\alpha) \).

Extend \( \text{Cand} \):

\[ \text{Cand} = \{ R \mid R \text{ is a set of terms, } R \subseteq SN, p \in SN \implies p \in R \} \]

Case: \( \tau = \forall \alpha.\tau' \). Suppose \( \sigma \) is a type and \( R \in \text{Cand} \). To show: \( p[\sigma] \in C[\tau'] \delta, \alpha \mapsto (\sigma, R) \). Because \( p \in SN \), we have \( p[\sigma] \) is also strongly normalizing. By induction, \( p[\sigma] \in C[\tau'] \delta, \alpha \mapsto (\sigma, R) \).

New definition: \( S ::= \cdot \mid S(e) \mid S[\sigma] \). Write \( S[e] \) for \( e \) plugged in for the hole in \( S \).

\[
\begin{align*}
e & \in SN \\
S[(\lambda x.e')(e)] & \rightarrow_{wh} S[e'[e/x]] \quad S[(\Lambda \alpha.e)[\sigma]] & \rightarrow_{wh} S[e[\sigma/\alpha]]
\end{align*}
\]

Lemma [Head expansion] If \( e \rightarrow_{wh} e' \) and \( e' \in C[\tau] \delta \) then \( e \in C[\tau] \delta \).

Proof: by induction on \( \tau \).

Case: \( \tau = T \). We need that \( SN \) is closed under head expansion. Suppose there were an infinite reduction sequence \( (\Lambda \alpha.e)[\sigma] \rightarrow_{wh} \cdots \rightarrow_{wh} (\Lambda \alpha.f)[\sigma] \rightarrow_{wh} f[\sigma/\alpha] \). This means that \( e \rightarrow_{wh} \cdots \rightarrow_{wh} f \). To show: \( e[\sigma/\alpha] \rightarrow_{wh}^* f[\sigma/\alpha] \) (a lemma).
Case: $\tau = \tau' \rightarrow \tau''$. Suppose $f \in C[\tau']\delta$. To show: $e(f) \in C[\tau'']\delta$. By the logical relation, $e'(f) \in C[\tau'']\delta$. By induction, $e(f) \in C[\tau'']\delta$. Mr Pedantic says: $e \rightarrow_{wh} e'$ means $e = S[\ldots]$ and $e' = S[\ldots]$. So let $S' = S(f)$. Therefore $e(f) = S'[\ldots] \rightarrow_{wh} e'(f) = S'[\ldots]$.

Case: $\tau = \alpha$. To show: $e \in \delta_S(\alpha)$. We know that $e \rightarrow_{wh} e'$ and $e' \in \delta_S(\alpha)$. So, once more, we need to extend Cand.

Cand = \{ $R \mid R$ is a set of terms, $R \subseteq SN$, $p \in SN \implies p \in R$, $e \rightarrow_{wh} e' \land e' \in R \implies e \in R$ \}