2-6-06: Termination for System F

Review of System F

Types $\sigma, \tau ::= \ldots \mid \forall\alpha.\tau \mid \alpha$
Terms $e ::= \ldots \mid \Lambda\alpha.e \mid e[\tau]$
Values $v ::= \ldots \mid \Lambda\alpha.e$
$\Delta ::= \emptyset \mid \Delta, \alpha$

New reduction rules: $\boxed{e \mapsto e'}$

$$\frac{e_1 \mapsto e_2}{e_1[\tau] \mapsto e_2[\tau]} \qquad \frac{}{(\Lambda\alpha.e)[\tau] \mapsto e[\tau/\alpha]}$$

New typing rules: $\boxed{\Delta; \Gamma \vdash e : \tau}$

$$\frac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.e : \forall\alpha : \tau}$$

$$\frac{\Delta; \Gamma \vdash e : \forall\alpha.\tau \qquad \Delta \vdash \sigma \text{ type}}{\Delta; \Gamma \vdash e[\sigma] : \tau[\sigma/\alpha]}$$

Nice try:

$\mathcal{C}[\![\tau]\!] = \{e \mid e \mapsto^* v \wedge v \in \mathcal{V}[\![\tau]\!]\}$
$\mathcal{V}[\![\mathbf{T}]\!] = \{c\}$
$\mathcal{V}[\![\tau_1 \to \tau_2]\!] = \{v \mid \forall v' \in \mathcal{V}[\![\tau_1]\!].v(v') \in \mathcal{C}[\![\tau_2]\!]\}$
$\mathcal{V}[\![\forall\alpha.\tau]\!] = \{v \mid \forall\sigma.v[\sigma] \in \mathcal{C}[\![\tau[\sigma/\alpha]]\!]\}$

Do this instead:

$\text{Cand} = \{R \mid \forall e \in R.e \text{ is a closed value}\}$

$\mathcal{C}[\![\tau]\!]\delta = \{e \mid e \mapsto^* v \wedge v \in \mathcal{V}[\![\tau]\!]\delta\}$
$\mathcal{V}[\![\mathbf{T}]\!]\delta = \{c\}$
$\mathcal{V}[\![\tau_1 \to \tau_2]\!]\delta = \{v \mid \forall v' \in \mathcal{V}[\![\tau_1]\!]\delta.v(v') \in \mathcal{C}[\![\tau_2]\!]\delta\}$
$\mathcal{V}[\![\forall\alpha.\tau]\!]\delta = \{v \mid \forall\sigma.\emptyset \vdash \sigma \text{ type} \wedge \forall R \in \text{Cand}.v[\sigma] \in \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\sigma, R)\}$
$\mathcal{V}[\![\alpha]\!]\delta = \delta_S(\alpha)$

To the proof!

$$\frac{\forall\alpha \in \Delta.\emptyset \vdash \delta_T(\alpha) \text{ type} \wedge \delta_S(\alpha) \in \text{Cand}}{\delta \in \mathcal{D}[\![\Delta]\!]}$$

$$\frac{\forall x : \tau \in \Gamma.\gamma(x) \in \mathcal{V}[\![\tau]\!]\delta}{\gamma \in \mathcal{G}[\![\Gamma]\!]\delta}$$

Fundamental theorem If $\Delta; \Gamma \vdash e : \tau$, and $\delta \in \mathcal{D}[\![\Delta]\!]$ and $\gamma \in \mathcal{G}[\![\Gamma]\!]\delta$ then $\delta_T(\gamma(e)) \in \mathcal{C}[\![\tau]\!]\delta$.

Proof: by induction on e.

Case: $e = x$. By inversion, $x : \tau \in \Gamma$, which means that $\gamma(x) \in \mathcal{V}[\![\tau]\!]$. Hence $\gamma(x)$ is closed, so δ has no effect, and we're done.

Case:

$$\frac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau}$$

To show: $\delta_T(\gamma(\Lambda\alpha.e)) \in \mathcal{C}[\![\forall\alpha.\tau]\!]\delta$.

Note $\delta_T(\gamma(\Lambda\alpha.e)) = \Lambda\alpha.\delta_T(\gamma e)$. Let $v = \Lambda\alpha.\delta_T(\gamma e)$.

Suffices to show: $\forall\sigma.\forall R \in \text{Cand}.v[\sigma] \in \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\sigma, R)$. Suppose we have σ and $R \in \text{Cand}$. Let $\delta' = \delta, \alpha \mapsto (\sigma, R)$. Then $\delta' \in \mathcal{D}[\![\Delta, \alpha]\!]$. Need to show that $\gamma \in \mathcal{G}[\![\Gamma]\!]\delta'$. We have a weakening property:

Lemma If $\alpha \notin \text{FV}(\tau)$ then $\mathcal{C}[\![\tau]\!]\delta = \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\sigma, R)$. Similarly for $\mathcal{V}[\![]\!]$ and $\mathcal{G}[\![\Gamma]\!]$.

So now we're in a position to apply induction. We get that $\delta'_T(\gamma(e) \in \mathcal{C}[\![\tau]\!]\delta'$. That is, $\delta_T(\gamma e)[\sigma/\alpha] \in \mathcal{C}[\![\tau]\!]\delta'$.

Lemma (expansion) If $e' \in \mathcal{C}[\![\tau]\!]\delta$ and $e \mapsto e'$ then $e \in \mathcal{C}[\![\tau]\!]\delta$.

With the expansion lemma, we're done.

Case:

$$\frac{\Delta; \Gamma \vdash e : \forall\alpha.\tau \quad \Delta \vdash \sigma \text{ type}}{\Delta; \Gamma \vdash e[\sigma] : \tau[\sigma/\alpha]}$$

By induction, $\delta_T(\gamma e) \in \mathcal{C}[\![\forall\alpha.\tau]\!]\delta$. By definition of the LR, $\delta_T(\gamma e) \mapsto^* v \in \mathcal{V}[\![\forall\alpha\tau]\!]\delta$. So, $\forall\sigma'.\forall R \in \text{Cand}.v \in \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\sigma', R)$.

We need to show that $\delta_T(\gamma e[\sigma]) \in \mathcal{C}[\![\tau[\sigma/\alpha]]\!]\delta$. But $\delta_T(\gamma e[\sigma]) = \delta_T(\gamma e)[\delta_T\sigma] \mapsto^* v[\delta_T\sigma]$. Note $\delta_T\sigma$ is closed.

Let $R = \mathcal{V}[\![\sigma]\!]\delta$. Then $v[\delta_T\sigma] \in \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\delta_T\sigma, R)$. Need a lemma:

Lemma (type substitution lemma) To prove next time.