2-22-06: More relational parametricity

Church-encodings of lists/Shortcut fusion

Adding lists as a primitive:

$\mathbf{nil} : \mathbf{nlist}$
$\mathbf{cons} : \mathbf{nat} \to \mathbf{nlist} \to \mathbf{nlist}$

$v ::= \dots \mid \mathbf{nil} \mid \mathbf{cons}(n, v)$
$e ::= \dots \mid \mathbf{case}\ e\ \mathbf{of}\ \mathbf{nil} \Rightarrow e_1 \mid \mathbf{cons}(x, L) \Rightarrow e_2$

Operational rules as expected.

$$\frac{v_1 = v_2 = \mathbf{nil}}{\delta \vdash v_1 \approx v_2 : \mathbf{nlist}} \qquad \frac{v_1 = \mathbf{cons}(n, v_1') \quad v_2 = \mathbf{cons}(n, v_2') \quad \delta \vdash v_1' \approx v_2' : \mathbf{nlist}}{\delta \vdash v_1 \approx v_2 : \mathbf{nlist}}$$

(For **nlist**, we could actually use $v_1 = v_2$ instead)

The relational parametricity proof can be easily extended for this extension.

We'll use this instead of **case**:

$\mathbf{fold}[\sigma](v_n)(v_c)(\mathbf{nil}) \mapsto v_n$
$\mathbf{fold}[\sigma](v_n)(v_c)(\mathbf{cons}(n, L)) \mapsto v_c(n)(\mathbf{fold}[\sigma](v_n)(v_c)(L)$

Proposition Suppose $\vdash f : \forall\alpha.\alpha \to (\mathbf{nat} \to \alpha \to \alpha) \to \alpha$. Then there exists $L : \mathbf{nlist}$ such that for all σ and $v_n : \sigma$ and $v_c : \mathbf{nat} \to \sigma \to \sigma$, we have $f[\sigma](v_n)(v_c) \equiv \mathbf{fold}[\sigma](v_n)(v_c)(L)$.

Let $L = f[\mathbf{nlist}](\mathbf{nil})(\mathbf{cons}) = \mathbf{build}\ f$. By parametricity, $\vdash f \approx f : \forall\alpha.\alpha \to (\mathbf{nat} \to \alpha \to \alpha) \to \alpha$. Choose $\sigma_1 = \mathbf{nlist}$, $\sigma_2 = \sigma$. Let $R = \{(L, v) \mid v \equiv \mathbf{fold}[\sigma](v_n)(v_c)(L)\}$. Easy to see that $\mathbf{nil} \approx v_n$ by the definition of **fold**.

To show: $\delta \vdash \mathbf{cons} \approx v_c : \mathbf{nat} \to \alpha \to \alpha$ where $\delta = \alpha \mapsto (\mathbf{nlist}, \sigma, R)$. Let $n : \mathbf{nat}$, let $(L, v) \in R$. Then we want to show $(\mathbf{cons}(n)(L), v_c(n)(v)) \in R$. Suffices to show that $v_c(n)(v) \equiv \mathbf{fold}[\sigma](v_n)(v_c)(\mathbf{cons}(n)(L))$. We have

$$\mathbf{fold}[\sigma](v_n)(v_c)(\mathbf{cons}(n)(L)) \mapsto v_c(n)(\mathbf{fold}[\sigma](v_n)(v_c)(L))$$

Now, because $(L, v) \in R$, $\mathbf{fold}[\sigma](v_n)(v_c)(L) \equiv v$.

We now have $\delta \vdash f[\mathbf{nlist}](\mathbf{nil})(\mathbf{cons}) \approx f[\sigma](v_n)(v_c) : \alpha$. So $(f[\mathbf{nlist}](\mathbf{nil})(\mathbf{cons}), f[\sigma](v_n)(v_c)) \in R$. (That is, $(L, f[\sigma](v_n)(v_c)) \in R$). Therefore, $f[\sigma](v_n)(v_c) \equiv \mathbf{fold}[\sigma](v_n)(v_c)(L)$.

Shortcut fusion (aka the foldr-build rule) If $\vdash f : \forall\alpha.\alpha \to (\mathbf{nat} \to \alpha \to \alpha) \to \alpha$ then for all σ, for all $v_n : \sigma$, for all $v_c : \mathbf{nat} \to \sigma \to \sigma$, we have $\mathbf{fold}[\sigma](v_n)(v_c)(\mathbf{build}(f)) \equiv f[\sigma](v_n)(v_c)$.

Rearrangements

$\mathbf{nil} : \mathbf{list}(\tau)$
$\mathbf{cons} : \tau \to \mathbf{list}(\tau) \to \mathbf{list}(\tau)$

$$\frac{}{\delta \vdash \mathbf{nil} \approx \mathbf{nil} : \mathbf{list}(\tau)} \qquad \frac{\delta \vdash vh_1 \approx vh_2 : \tau \quad \delta \vdash vt_1 \approx vt_2 : \mathbf{list}(\tau)}{\delta \vdash \mathbf{cons}(vh_1, vt_1) \approx \mathbf{cons}(vh_2, vt_2) : \mathbf{list}(\tau)}$$

So $\delta \vdash [v_1, \ldots, v_n] \approx [v_1', \ldots, v_n'] : \mathbf{list}(\tau)$ iff for all $i \in 1, \ldots n$, $\delta \vdash v_i \approx v_i' : \tau$.

Proposition Suppose $\vdash m : \forall\alpha.\mathbf{list}(\alpha) \to \mathbf{list}(\alpha)$. For all $f : \sigma \to \tau$ and all $L : \mathbf{list}(\sigma)$,

$$\mathbf{map}(f)(m[\sigma](L)) \equiv m[\tau](\mathbf{map}(f)(L))$$

Proof: By parametricity, $\vdash m \approx m : \forall\alpha.\mathbf{list}(\alpha) \to \mathbf{list}(\alpha)$. Pick σ, τ. Let $R = \{(v_1, v_2) \mid v_2 \equiv f(v_1)\} \in \mathrm{VRel}(\sigma, \tau)$. Let $\delta = \alpha \mapsto (\sigma, \tau, R)$. So, $\delta \vdash m[\sigma] \approx m[\tau].\mathbf{list}(\alpha) \to \mathbf{list}(\alpha)$. We want to show $\delta \vdash L \approx \mathbf{map}(f)(L) : \mathbf{list}(\alpha)$. We have $L = [v_1, \ldots, v_n]$ and $\mathbf{map}(f)(L) \equiv [f(v_1), \ldots, f(v_n)]$. Suffices to show that $\delta \vdash v_i \approx f(v_i) : \alpha$. (True)

Hence $\delta \vdash m[\sigma](L) \approx m[\tau](\mathbf{map}(f)(L)) : \mathbf{list}(\alpha)$. So, $\mathbf{map}(f)(m[\sigma](L)) \equiv m[\tau](\mathbf{map}(f)(L))$.