2-13-06: Final SN proof; Parametricity

Finishing SN proof for System F

$\mathcal{C}[\![\forall\alpha.\tau]\!]\delta = \{e \mid \forall\sigma.\emptyset \vdash \sigma \text{ type}.\forall R \in \text{Cand}.e[\sigma] \in \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\sigma, R)\}$

Cand $= \{R \mid R$ is a set of terms, $R \subseteq \text{SN}, p \in \text{SN} \implies p \in R, e \to_{\text{wh}} e' \wedge e' \in R \implies e \in R\}$

Lemma [Head expansion] If $e \to_{\text{wh}} e'$ and $e' \in \mathcal{C}[\![\tau]\!]\delta$ then $e \in \mathcal{C}[\![\tau]\!]\delta$.

Proof: by induction on τ.

Case: $\tau = \forall\alpha.\tau'$. Let σ. Let $R \in$ Cand. By definition of the LR we have $e'[\sigma] \in \mathcal{C}[\![\tau']\!]\delta, \alpha \to (\sigma, R)$. We have $e[\sigma] \to_{\text{wh}} e'[\sigma]$ so we can apply induction to get our result.

Recall:

$$\frac{\forall\alpha \in \Delta.\delta_S(\alpha) \in \text{Cand}}{\delta \in \mathcal{D}[\![\Delta]\!]} \qquad \frac{\forall x : \tau \in \Gamma.\gamma(x) \in \mathcal{C}[\![\tau]\!]\delta}{\gamma \in \mathcal{G}[\![\Gamma]\!]\delta}$$

Fundamental theorem If $\Delta; \Gamma \vdash e : \tau$ and $\delta \in \mathcal{D}[\![\Delta]\!]$ and $\gamma \in \mathcal{G}[\![\Gamma]\!]\delta$ then $\delta_T(\gamma(e)) \in \mathcal{C}[\![\tau]\!]\delta$.

Proof: by induction on typing judgments.

Case:

$$\frac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau}$$

Let σ. Let $R \in$ Cand. Let $\delta' = \delta, \alpha \mapsto (\sigma, R)$. Note $\delta' \in \mathcal{D}[\![\Delta, \alpha]\!]$ because $R \in$ Cand. (We have weakening to give that $\gamma \in \mathcal{G}[\![\Gamma]\!]\delta'$) By induction, $\delta'_T(\gamma(e)) \in \mathcal{C}[\![\tau]\!]\delta'$. We know that $\delta'_T(\gamma(e)) = \delta_T(\gamma(e))[\sigma/\alpha]$. By head expansion, we have our result.

DO: type application case.

Parametricity

$\mathcal{C}[\![\tau]\!]\delta = \{e \mid e \mapsto^* v \wedge v \in \mathcal{V}[\![\tau]\!]\delta\}$
$\mathcal{V}[\![\forall\alpha.\tau]\!]\delta = \{v \mid \forall\sigma.\forall R \in \text{Cand}.v[\sigma] \in \mathcal{C}[\![\tau]\!]\delta, \alpha \mapsto (\sigma, R)\}$
Cand $= \{R \mid R$ is a set of closed values$\}$

Theorem $\forall\alpha.\alpha$ is uninhabited.

Proof: Suppose $\vdash v : \forall\alpha.\alpha$. Then by FTLR, $v \in \mathcal{V}[\![\forall\alpha.\alpha]\!]\emptyset$. Let $\sigma = \mathbf{T}$ and $R = \emptyset$. Then by definition of the LR, $v[\mathbf{T}] \in \mathcal{C}[\![\alpha]\!]\alpha \mapsto (\mathbf{T}, \emptyset)$. So $v[\mathbf{T}] \mapsto^* v' \in \mathcal{V}[\![\alpha]\!]\alpha \mapsto (\mathbf{T}, \emptyset) = \emptyset$. Contradiction.

Proposition Suppose $\vdash f : \forall\alpha : \alpha \to \alpha$. Then for any closed value v of type τ, we have $f[\tau](v) \downarrow v$.

Proof: By FTLR, $\forall\sigma, \forall R \in \text{Cand}$, $f[\sigma] \in \mathcal{C}[\![\alpha \to \alpha]\!]\alpha \mapsto (\sigma, R)$. Let $\sigma = \tau$, $R = \{v\}$. So $f[\tau] \downarrow v' \in \mathcal{V}[\![\alpha \to \alpha]\!]\alpha \mapsto (\tau, R)$. We need that $v \in \mathcal{V}[\![\alpha]\!]\alpha \mapsto (\tau, R) = R = \{v\}$, which is obviously true. By the LR, $v'(v) \in \mathcal{C}[\![\alpha]\!]\alpha \mapsto (\tau, R)$ so

$$v'(v) \downarrow v'' \in \mathcal{V}[\![\alpha]\!]\alpha \mapsto (\tau, R) = R = \{e\}$$

Hence $v'' = v$. Therefore $f[\tau](v) \mapsto^* v'(v) \mapsto^* v$.