2-1-06: Wrapping up equivalence; Girard's method

We were proving the fund. thm: if $\Gamma \vdash e_1 \equiv e_2 : \tau$ and $\Delta \vdash \gamma_1 \approx \gamma_2 : \Gamma$, then $\Delta \vdash \gamma_1 e_1 \approx \gamma_2 e_2 : \tau$.

(See pg. 241 in book for a different way to do this proof; we can't strictly use induction here, without an extra lemma)

Extra lemma If $\Gamma \vdash e : \tau$ and $\Delta \vdash \gamma_1 \approx \gamma_2 : \Gamma$, then $\Delta \vdash \gamma_1 e \approx \gamma_2 e : \tau$.

The final case: β-reduction.

$$\frac{\Gamma, x : \tau' \vdash e'' : \tau'' \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (\lambda x : \tau'.e'')e' \equiv e''[e'/x] : \tau''}$$

Using the "extra lemma" above, we have $\Delta \vdash \gamma_1 e' \approx \gamma_2 e' : \tau'$. Let $\gamma_i' = \gamma_i, x \mapsto \gamma_i e'$. We have $\Delta \vdash \gamma_1' \approx \gamma_2' : \Gamma, x : \tau'$ so by the "extra lemma" we have that $\Delta \vdash \gamma_1' \approx \gamma_2' : \tau''$. In other words, $\Delta \vdash \gamma_1 e''[\gamma_1 e'/x] \approx \gamma_2 e''[\gamma_2 e'/x] : \tau''$. We've got the right term on the right-hand side of the LR, but we need head-expansion to get the other. By head-expansion, $\Delta \vdash \gamma_1(\lambda x.e'')(e') \approx \gamma_2(e''[e'/x]) : \tau''$.

Polymorphism/Girard's method

Types $\sigma, \tau ::= \ldots \mid \forall \alpha.\tau \mid \alpha$
Terms $e ::= \ldots \mid \Lambda\alpha.e \mid e[\tau]$
Values $v ::= \ldots \mid \Lambda\alpha.e$
$\Delta ::= \emptyset \mid \Delta, \alpha$

New reduction rules: $\boxed{e \mapsto e'}$

$$\frac{e_1 \mapsto e_2}{e_1[\tau] \mapsto e_2[\tau]} \qquad \frac{}{(\Lambda\alpha.e)[\tau] \mapsto e[\tau/\alpha]}$$

Typing rules: $\boxed{\Delta; \Gamma \vdash e : \tau}$

$$\frac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.e : \forall\alpha : \tau}$$

$$\frac{\Delta; \Gamma \vdash e : \forall\alpha.\tau \qquad \Delta \vdash \sigma \text{ type}}{\Delta; \Gamma \vdash e[\sigma] : \tau[\sigma/\alpha]}$$

Proposed LR:

$$\mathcal{V}[\![\forall\alpha.\tau]\!] = \{v \mid \forall\sigma.v[\sigma] \in \mathcal{C}[\![\tau[\sigma/\alpha]]\!]\}$$

Note: ML is predicative.

Do for next time: read in TAPL about Church encodings.

Here's where we're going:

$$\mathcal{V}[\![\forall\alpha.\tau]\!] = \{v \mid \forall\sigma.\forall R.v[\sigma] \in \mathcal{C}[\![\tau]\!]_{\eta,\alpha\mapsto R}\}$$