1-4-06: Strengthening induction

A general pattern when using induction is that, to get the proof to go through, one must often prove a *stronger* statement than the one originally desired. This is often called "generalizing the induction hypothesis." Logical relations are a tool to do this.

Review of $\lambda_{\rightarrow}$

Syntax

$$\begin{array}{rcl} \tau & ::= & \mathbf{T} \mid \tau_1 \rightarrow \tau_2 \\ e & ::= & x \mid c \mid \lambda x : \tau . e \mid e_1(e_2) \\ v & ::= & x \mid c \mid \lambda x : \tau . e \\ \Gamma & ::= & \emptyset \mid \Gamma, x : \tau \end{array}$$

Typing judgments: $\boxed{\Gamma \vdash e : \tau}$

$$\frac{}{\Gamma \vdash c : \mathbf{T}} \qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash e : \tau} \qquad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : \tau \rightarrow \tau'} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1(e_2) : \tau_2}$$

Evaluation rules: $\boxed{e \mapsto e'}$

$$\frac{}{(\lambda x : \tau . e) v \mapsto e[v/x]} \qquad \frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)} \qquad \frac{e \mapsto e'}{v(e) \mapsto v(e')}$$

Progress If $\vdash e : \tau$ then either e is a value or $e \mapsto e'$.

Preservation If $\vdash e : \tau$ and $e \mapsto e'$ then $\vdash e' : \tau$.

Proving preservation

By induction on the evaluation derivation.

Case: $e = e_1(e_2)$. We have $\vdash e_1(e_2) : \tau$ and $e_1 \mapsto e_1'$ and $e_1(e_2) \mapsto e_1'(e_2)$. By inversion, $\vdash e_1 : \tau_2 \rightarrow \tau$ and $\vdash e_2 : \tau_2$. We're done.

Case: $e = (\lambda x : \tau'' . e'')v \mapsto e''[v/x]$.

[My handwritten notes stop here. I think we went into why you need to introduce an arbitrary Γ (i.e. $\Gamma \vdash e : \tau$ rather than $\vdash e : \tau$) in order to make the induction go through)