1-4-06: Strengthening induction

A general pattern when using induction is that, to get the proof to go through, one must often prove a stronger statement than the one originally desired. This is often called “generalizing the induction hypothesis.” Logical relations are a tool to do this.

Review of $\lambda_-$

Syntax

$\tau ::= T \mid \tau_1 \rightarrow \tau_2$

$e ::= x \mid c \mid \lambda x : \tau.e \mid e_1(e_2)$

$v ::= x \mid c \mid \lambda x : \tau.e$

$\Gamma ::= \emptyset \mid \Gamma, x : \tau$

Typing judgments:

$\Gamma \vdash e : \tau$

\[
\begin{array}{c}
\Gamma \vdash c : T \\
\Gamma \vdash e : \tau \\
\Gamma \vdash \lambda x : \tau.e : \tau \rightarrow \tau' \\
\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \\
\Gamma \vdash e_2 : \tau_1 \\
\end{array}
\]

Evaluation rules:

$e \mapsto e'$

\[
\begin{array}{c}
(\lambda x : \tau.e)v \mapsto e[v/x] \\
e_1 \mapsto e'_1 \\
e_1(e_2) \mapsto e'_1(e_2) \\
e \mapsto e' \\
v(e) \mapsto v(e')
\end{array}
\]

Progress If $\vdash e : \tau$ then either $e$ is a value or $e \mapsto e'$.

Preservation If $\vdash e : \tau$ and $e \mapsto e'$ then $\vdash e' : \tau$.

Proving preservation

By induction on the evaluation derivation.

Case: $e = e_1(e_2)$. We have $\vdash e_1(e_2) : \tau$ and $e_1 \mapsto e'_1$ and $e_1(e_2) \mapsto e'_1(e_2)$. By inversion, $\vdash e_1 : \tau_2 \rightarrow \tau$ and $\vdash e_2 : \tau_2$. We’re done.

Case: $e = (\lambda x : \tau'',e'')v \mapsto e''[v/x]$.

[My handwritten notes stop here. I think we went into why you need to introduce an arbitrary $\Gamma$ (i.e. $\Gamma \vdash e : \tau$ rather than $\vdash e : \tau$) in order to make the induction go through]