1-30-06: Review of (and final proofs for) term equivalence

$$\frac{}{\Gamma \vdash e_1 \approx e_2 : \mathbf{unit}} \qquad \frac{\Gamma \vdash e_1 \Leftrightarrow e_2 : \mathbf{T}}{\Gamma \vdash e_1 \approx e_2 : \mathbf{T}}$$

$$\frac{\forall \Delta \supseteq \Gamma . \Delta \vdash e_1' \approx e_2' : \tau' \implies \Delta \vdash e_1 e_1' \approx e_2 e_2' : \tau''}{\Gamma \vdash e_1 \approx e_2 : \tau' \to \tau''}$$

We proved last time:

Head expansion If $\Gamma \vdash e_1 \approx e_2 : \tau$ and $e_1' \to^*_{\text{wh}} e_1$ and $e_2' \to^*_{\text{wh}} e_2$ then $\Gamma \vdash e_1' \approx e_2' : \tau$.

Monotonicity If $\Gamma \vdash e_1 \approx e_2 : \tau$ and $\Delta \supseteq \Gamma$ then $\Delta \vdash e_1 \approx e_2 : \tau$.

Main lemma (1): If $\Gamma \vdash e_1 \approx e_2 : \tau$ then $\Gamma \vdash e_1 \Leftrightarrow e_2 : \tau$.

Main lemma (2): If $\Gamma \vdash p_1 \leftrightarrow p_2 : \tau$, then $\Gamma \vdash p_1 \approx p_2 : \tau$.

Need to deal with substitutions:

$\boxed{\Delta \vdash \gamma_1 \approx \gamma_2 : \Gamma}$

$$\frac{\forall x : \tau \in \Gamma . \Delta \vdash \gamma_1(x) \approx \gamma_2(x) : \tau}{\Delta \vdash \gamma_1 \approx \gamma_2 : \Gamma}$$

Now we want to prove the "fundamental theorem":

Fundamental theorem If $\Gamma \vdash e_1 \equiv e_2 : \tau$, and $\Delta \vdash \gamma_1 \approx \gamma_2 : \Gamma$ then $\Delta \vdash \gamma_1 e_1 \approx \gamma_2 e_2 : \tau$.

Proof: by induction on the definitional equivalence judgment.

Case: (reflexivity) We end up changing the rules so that reflexivity is no longer a case:

$$\frac{\Gamma, x : \tau' \vdash e_1 \equiv e_2 : \tau''}{\Gamma \vdash \lambda x . e_1 \equiv \lambda x . e_2 : \tau' \to \tau''}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \to \tau'' \qquad \Gamma \vdash e_1' \equiv e_2' : \tau'}{\Gamma \vdash e_1 e_1' \equiv e_2 e_2' : \tau''}$$

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x \equiv x : \tau}$$

Case: (symmetry) We have

$$\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

By induction, $\Delta \vdash \gamma_1 e_2 \approx \gamma_2 e_1 : \tau$.

Now, we go and prove symmetry of the logical relation:

Lemma [Symmetry of the logical relation] If $\Gamma \vdash e_1 \approx e_2 : \tau$ then $\Gamma \vdash e_2 \approx e_1 : \tau$.

Proof: by induction on τ.

Case: $\tau = \mathbf{unit}$. Immediate.

Case: $\tau = \mathbf{T}$. We have that $\Gamma \vdash e_1 \Leftrightarrow e_2 : \mathbf{T}$ (so it boils down to symmetry of algorithmic equivalence). Left as an exercise.

Case: $\tau = \tau' \to \tau''$. Let $\Delta \supseteq \Gamma$. Suppose $\Delta \vdash e_1' \approx e_2' : \tau'$. We need to show that $\Gamma \vdash e_2 e_1' \approx e_1 e_2' : \tau''$. By induction, $\Delta \vdash e_2' \approx e_1' : \tau'$. We have by definition that $\Gamma \vdash e_1 e_2' \approx e_2 e_1' : \tau''$. Finally, by induction $\Gamma \vdash e_2 e_1' \approx e_1 e_2' : \tau''$.

Lemma If $\Delta \vdash \gamma_1 \approx \gamma_2 : \Gamma$ then $\Delta \vdash \gamma_2 \approx \gamma_1 : \Gamma$.

Now, back to the case for the fundamental theorem:

Fund. thm. (continued)

Case: (symmetry) Now this case is easy: by symmetry $\Delta \vdash \gamma_2 \approx \gamma_1 : \Gamma$. By induction, $\Delta \vdash \gamma_2 e_2 \approx \gamma_1 e_1 : \tau$. The final result follows from symmetry of the logical relation.

Case: (transitivity)

$$\frac{\Delta \vdash e_1 \equiv e_2 : \tau \qquad \Delta \vdash e_2 \equiv e_3 : \tau}{\Delta \vdash e_1 \equiv e_3 : \tau}$$

By induction, $\Delta \vdash \gamma_1 e_1 \approx \gamma_2 e_2 : \tau$. Ideally, by "induction" $\Delta \vdash \gamma_2 e_2 \approx \gamma_2 e_3 : \tau$.

Lemma If $\Delta \vdash \gamma_1 \approx \gamma_2 : \Gamma$, then $\Delta \vdash \gamma_2 \approx \gamma_2 : \Gamma$.

Proof: By symmetry $\Delta \vdash \gamma_2 \approx \gamma_1 : \Gamma$. By transitivity, $\Delta \vdash \gamma_2 \approx \gamma_2 : \Gamma$.

(This is a PER, a partial equiv relation that supports symmetry and transitivity, but not necessarily reflexivity.)

By that reasoning, we can take away "ideally". By the lemma, $\Delta \vdash \gamma_2 \approx \gamma_2 : \Gamma$. By transitivity, $\Delta \vdash \gamma_1 e_1 \approx \gamma_2 e_3 : \tau$, so we are done.

Transitivity of the logical relation If $\Gamma \vdash e_1 \approx e_2 : \tau$ and $\Gamma \vdash e_2 \approx e_3 : \tau$, then $\Gamma \vdash e_1 \approx e_3 : \tau$.

Proof: by induction on τ.

Case: $\tau = \mathbf{unit}$. Immediate.

Case: $\tau = \mathbf{T}$. Needs algorithmic transitivity. By induction on the algorithm (do it).

Case: $\tau = \tau' \to \tau''$. Let $\Delta \supseteq \Gamma$. Suppose that $\Delta \vdash e_1' \approx e_3' : \tau'$. By definition of the logical relation, we have $\Delta e_1 e_1' \approx e_2 e_3' : \tau''$. By symmetry, we have $\Delta \vdash e_3' \approx e_1' : \tau'$. By induction, $\Delta \vdash e_3' \approx e_3' : \tau'$. Therefore, by the definition of the logical relation, $\Delta \vdash e_2 e_3' \approx e_3 e_3' : \tau''$. By induction, $\Delta \vdash e_1 e_1' \approx e_3 e_3' : \tau''$.

Now, back to the case for the fundamental theorem (again):

Fund. thm. (continued, again)

Case: (extensionality)

$$\frac{\Gamma, x : \tau' \vdash e_1(x) \equiv e_2(x) : \tau'' \qquad x \notin \mathrm{FV}(e_1) \cup \mathrm{FV}(e_2)}{\Gamma \vdash e_1 \equiv e_2 : \tau' \to \tau''}$$

Let $\Delta' \supseteq \Delta$. Suppose $\Delta' \vdash e_1' \approx e_2' : \tau'$. So $\Delta \vdash \gamma_1, x \mapsto e_1' \approx \gamma_2, x \mapsto e_2' : \Gamma, x : \tau'$. By monotonicity, $\Delta' \vdash \gamma_1, x \mapsto e_1' \approx \gamma_2, x \mapsto e_2' : \Gamma, x : \tau'$. By induction $\Delta' \vdash (\gamma_1, x \mapsto e_1')(e_1 x) \approx (\gamma_2, x \mapsto e_2')(e_2 x) : \tau''$. This means that $\Delta' \vdash \gamma_1 e_1(e_1') \approx \gamma_2 e_2(e_2') : \tau''$.

Case: (β-reduction)

To be proved next time.