1-25-06: More on equivalence with unit

Wanting to find an algorithmic notion of equivalence (and we find it by trying to prove it correct). Completeness: definitional equivalence implies algorithmic equivalence.

Theorem If $\Gamma \vdash e_1 \equiv e_2 : \tau$, then $\Gamma \vdash e_1 \Leftrightarrow e_2 : \tau$

(Validity: if two things are equiv, they are both well-formed)

$$\frac{}{\Gamma \vdash e_1 \Leftrightarrow e_2 : \mathbf{unit}}$$

Notation for logical two-place relation: $\boxed{\Gamma \vdash e_1 \approx e_2 : \tau}$

$$\frac{}{\Gamma \vdash e_1 \approx e_2 : \mathbf{unit}} \qquad \frac{\Gamma \vdash e_1 \Leftrightarrow e_2 : \mathbf{T}}{\Gamma \vdash e_1 \approx e_2 : \mathbf{T}}$$

$$\frac{\forall f_1, f_2\ \Gamma \vdash f_1 \approx f_2 : \tau' \Rightarrow \Gamma \vdash e_1 f_1 \approx e_2 f_2 : \tau''}{\Gamma \vdash e_1 \approx e_2 : \tau' \to \tau''}$$

Lemma Main lemma (1) If $\Gamma \vdash e_1 \approx e_2 : \tau$ then $\Gamma \vdash e_1 \Leftrightarrow e_2 : \tau$.

Lemma Main lemma (2) If $\Gamma \vdash p_1 \leftrightarrow p_2 : \tau$, then $\Gamma \vdash p_1 \approx p_2 : \tau$.

Proof: part (1) by induction on τ.

Case: $\tau = \mathbf{unit}$. Trivial

Case: $\tau = \mathbf{T}$. Trivial

Case: $\tau = \tau' \to \tau''$. We have $\Gamma, x : \tau' \vdash x \leftrightarrow x : \tau'$. By induction (using part 2) we have $\Gamma, x : \tau' \vdash x \approx x : \tau'$. By the logical relation, $\Gamma, x : \tau' \vdash e_1 x \approx e_2 x : \tau''$. By induction on part 1, $\Gamma, x : \tau' \vdash e_1 x \Leftrightarrow e_2 x : \tau''$.

Suggests eta rule:

$$\frac{\Gamma, x : \tau' \vdash e_1 x \Leftrightarrow e_2 x : \tau''}{\Gamma \vdash e_1 \Leftrightarrow e_2 : \tau' \to \tau''}$$

Also we have:

$$\frac{}{\Gamma, x : \tau \vdash x \leftrightarrow x : \tau}$$

Monotonicity If $\Gamma \vdash e_1 \approx e_2 : \tau$ and $\Delta \supseteq \Gamma$ then $\Delta \vdash e_1 \approx e_2 : \tau$.

Proof: by induction on τ.

Case: $\tau = \tau' \to \tau''$. For all f_1, f_2 assume $\Delta \vdash f_1 \approx f_2 : \tau'$. To show: $\Delta \vdash e_1 f_2 \approx e_2 f_2 : \tau''$.

To make this work, we need to adjust the logical relation:

$$\frac{\forall f_1, f_2 \text{ if } \Delta \supseteq \Gamma \text{ and } \Delta \vdash f_1 \approx f_2 : \tau' \text{ then } \Delta \vdash e_1 f_1 \approx e_2 f_2 : \tau''}{\Gamma \vdash e_1 \approx e_2 : \tau' \to \tau''}$$

Kripke-style logical relation.

Proof: (Main lemma part 2) by induction on τ.

Case: $\tau = \textbf{unit}$. Trivial.

Case: $\tau = \mathbf{T}$. Suggests rule:

$$\frac{\Gamma \vdash p_1 \leftrightarrow p_2 : \mathbf{T}}{\Gamma \vdash p_1 \Leftrightarrow p_2 : \mathbf{T}}$$

Case: $\tau = \tau' \to \tau''$. Let $f_1, f_2, \Delta \supseteq \Gamma$. Assume $\Delta \vdash f_1 \approx f_2 : \tau'$. By induction on part 1, we have $\Gamma \vdash f_1 \Leftrightarrow f_2 : \tau'$. By ?? (to show below), $\Delta \vdash p_1 f_1 \leftrightarrow p_2 f_2 : \tau''$. By induction on part 2, we have $\Delta \vdash p_1 f_1 \approx p_2 f_2 : \tau''$.

To get the ?? step above, we introduce an algorithmic rule:

$$\frac{\begin{array}{c}\Gamma \vdash p_1 \leftrightarrow p_2 : \tau' \to \tau'' \\ \Gamma \vdash e_1 \Leftrightarrow e_2 : \tau'\end{array}}{\Gamma \vdash p_1 e_1 \leftrightarrow p_2 e_2 : \tau''}$$

Want to separate algorithmic equivalence for paths and for arbitrary terms: the rules for paths shrink terms and grow types; for terms, the opposite.

$$\begin{array}{rcl}\mathcal{E} & ::= & \cdot \\ & | & \mathcal{E}(e)\end{array}$$

Closure under head expansion If $\Gamma \vdash \mathcal{E}_1(e_1[f_1/x]) \approx \mathcal{E}_2(e_2[f2/x]) : \tau$ then $\Gamma \vdash \mathcal{E}_1((\lambda x.e_1) f_1) \approx \mathcal{E}_2((\lambda x.e_2) f_2) : \tau$.

Proof: by induction on τ.

Case: $\tau = \textbf{unit}$. Trivial.

Case: $\tau = \mathbf{T}$. We want to define weak head reduction:

$$\mathcal{E}((\lambda x.e) f) \to_{\text{wh}} \mathcal{E}(e[f/x])$$

Now we want a new rule:

$$\frac{e_1 \to_{\mathrm{wh}} e_1' \qquad e_2 \to_{\mathrm{wh}} e_2' \qquad \Gamma \vdash e_1' \Leftrightarrow e_2' : \mathbf{T}}{\Gamma \vdash e_1 \Leftrightarrow e_2 : \mathbf{T}}$$

Karl groups rules into a more general form:

$$\frac{e_1 \to_{\mathrm{wh}}^* p_1 \qquad e_2 \to_{\mathrm{wh}}^* p_2 \qquad \Gamma \vdash p_1 \leftrightarrow p_2 : \mathbf{T}}{\Gamma \vdash e_1 \Leftrightarrow e_2 : \mathbf{T}}$$

Case: $\tau = \tau' \to \tau''$. To prove this case, we need to generalize:

Head expansion version 2 If $\Gamma \vdash e_1 \approx e_2 : \tau$ and $e_1' \to_{\mathrm{wh}}^* e_1$ and $e_2' \to_{\mathrm{wh}}^* e_2$ then $\Gamma \vdash e_1' \approx e_2' : \tau$.

Proof: by induction on τ.

Case: $\tau = \tau' \to \tau''$. Assume $\Delta \supseteq \Gamma$ and $\Delta \vdash f_1 \approx f_2 : \tau'$. To show: $\Gamma \vdash e_1' f_1 \approx e_2' f_2 : \tau''$. Follows directly from the logical relation. We know that $e_1' f_1 \to_{\mathrm{wh}}^* e_1 f_1$ and $e_2' f_2 \to_{\mathrm{wh}}^* e_2 f_2$. By induction, $\Delta \vdash e_1' f_1 \approx e_2' f_2 : \tau''$. We're done.