A Theory of Substructural Types and Control*

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Abstract

Exceptions are invaluable for structured error handling in high-level languages, but they are at odds with linear types. More generally, control effects may delete or duplicate portions of the stack, which, if we are not careful, can invalidate all substructural usage guarantees for values on the stack. We have developed a type-and-effect system that tracks control effects and ensures that values on the stack are never wrongly duplicated or dropped. We present the system first with abstract control effects and prove its soundness. We then give examples of three instantiations with particular control effects, including exceptions and delimited continuations, and show that they meet the soundness criteria for specific control effects.

1 Substructural Types and Control

Consider, for example, a language like Scala (Odersky and Zenger 2005) with mutable references and arithmetic. Here is a method that takes two integers and divides each by the other, returning a pair of references to their quotients:

$$\mathsf{def}\ \mathit{divRef}(z_1\colon \mathrm{Int},\ z_2\colon \mathrm{Int}) = (\mathsf{new}\ \mathrm{Ref}(z_1\ /\ z_2),\ \mathsf{new}\ \mathrm{Ref}(z_2\ /\ z_1))$$

Suppose that references in this language are *linear*, meaning that they cannot be duplicated, and must be explicitly deallocated rather than implicitly dropped. In such a language, *divRef* has a memory leak. Most uses of *divRef* are harmless, but consider the expression $\mathit{divRef}(0, 5)$. The method will raise a division-by-zero exception, but (assuming one reasonable evaluation order) only *after* it has allocated a reference to hold the result of the first division. Because the method raises an exception but does

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not return the successfully allocated reference, there is no way for recovery code that catches the exception to free the reference.

In short, exceptions and linear types refuse to get along, because linear types make promises that exceptions do not let them keep.

With *affine* rather than linear types, however, *divRef* is not a problem, because such a type system does not require that references be freed explicitly. In a language with affine types, implicitly dropping a value is just fine—presumably there is a garbage collector—and only duplication is forbidden. Consider, however, adding delimited continuation operators such as shift and reset to a language with affine types. Assuming a method *unref* that dereferences and deallocates a reference, we might attempt to define a method *squareRef* that takes a reference to an integer, frees it, and returns its contents, squared:

```
def twiceTo(x: Int) =
  shift { (k: Int ⇒ Int) ⇒ k(k(x)) }
def squareRef(r: Ref[Int]) =
  reset { twiceTo(1) × r.unref() }
```

Method *twiceTo* uses shift to capture its continuation up to the nearest enclosing reset, and it then applies the captured continuation k twice to the parameter x. Method *squareRef* provides the context for *twiceTo* to capture, which is to free r and multiply by its contents:

$$[\,] \times r.unref() \;.$$

Since *twiceTo* uses its continuation twice, the second use of the continuation will access a dangling pointer that the first use freed.

Typically, an affine type system works by imposing two syntactic requirements: a variable of affine type, such as r, cannot appear twice in its scope (up to branching), and a function that closes over an affine variable must itself have an affine type. The *squareRef* example violates neither dictum. In the presence of delimited continuations, we need to add a third rule: that *a captured continuation that contains an affine value must not duplicated.* A simple approximation of this rule is to give *all* captured continuations an affine (or in a linear system, linear) type. Such a rule would permit some limited uses of delimited continuations, such as coroutines, but we will show that this simple rule is overly restrictive.

Our solution. The memory leak and dangling pointer in the above examples can be fixed by small changes to the code. For *divRef*, it suffices to ensure that both divisions happen before both allocations:

```
def divRef(z1: Int, z2: Int) = {
  val z12 = z1 / z2
  val z21 = z2 / z1
```

```
    (new Ref(z12), new Ref(z21))
}
```

For *squareRef*, we need the dereferencing to happen once, outside the reset delimiter:

```
def squareRef(r: Ref[Int]) = {
  val z = r.unref()
  reset { twiceTo(1) × z }
}
```

Unfortunately, the conservative approximation suggested above, that all continuations be treated linearly, would still disallow these repaired examples. We have designed a type-and-effect system (Lucassen and Gifford 1988) that permits these two repaired versions of the methods while forbidding the original, erroneous versions. The key idea is to assign to each expression a control effect that reflects whether it may duplicate or drop its continuation, and to prohibit using an expression in a context that cannot be treated as the control effect allows. In this paper, we

- exhibit a *generic* type system for substructural types and control defined in terms of an unspecified, abstract control effect (§4);
- give soundness criteria for the abstract control effect and prove type safety for the generic system, relying on the soundness of the abstract control effect (§5); and
- demonstrate three concrete instantiations of control effects and prove that they meet the soundness criteria (§6).

The generic type-and-effect system in §4 is defined as an extension to λ^{URAL} (Ahmed et al. 2005), a substructural λ calculus, which we review in §3, after discussing related work in §2.

2 Related Work and Comparison

This work is not the first to relate substructural types to control operators and control effects. Thielecke (2003) shows how to use a type-and-effect system to reason about how expressions treat their continuations. In particular, he gives a continuation-passing style transform where continuations that will be used linearly are given a linear type. Thielecke notes that many useful applications of continuations treat them linearly. However, his goals are different than ours. He uses substructural types in his object language to reason about how continuations will be used in a non-substructural source language, whereas we want to reason about continuations in order to safely use substructural types. Thielecke has linear types only in the object language of his

translation, whereas we are interested in linear (and other substructural) types in the source language.

Other recent work relates substructural logics and control. Kiselyov and Shan (2007) use a substructural logic to allow the "dynamic" control operator *shift0* to modify answer types in a typed setting. Unlike this work, their *terms* are structures in substructural logic, not their types. Mazurak and Zdancewic's Lolliproc (2010) relates double negation elimination in classical linear logic to delimited control.

We draw significantly on other work on control operators, effect systems, and substructural types as well.

Control operators. The literature contains a large vocabulary of control operators, extending back to ISWIM's **J** operator (Landin 1965), Reynolds's *escape* (1972), and Scheme's *call/cc* (Clinger 1985). However, for integration in a language with substructural types, control operators with delimited extent, originating with Felleisen's $\mathcal{F}$ (1988), are most appropriate, because without some way to mask out control effects, any use of control pollutes the entire program and severely limits the utility of substructural types.

As examples of control features to add to our calculus, we consider the delimited continuation operators *shift* and *reset* (Danvy and Filinski 1989) and structured exception handling (Goodenough 1975). Both *shift*/*reset* and structured exceptions have been combined with type-and-effect systems to make them more amenable to static reasoning.

Type-and-effect systems for control. Java (Gosling et al. 1996) has checked exceptions, an effect system for tracking the exceptions that a method may raise. Our version of exception effects is similar to Java's, except that we offer effect polymorphism, which makes higher-order programming with checked exceptions more convenient. Our type system for exceptions appears in §6.3.

Because Danvy and Filinski's *shift* (1989) captures a delimited continuation up to the nearest *reset* delimiter, typing *shift* and *reset* requires some nonlocal means of communicating types between delimiters and control operators. They realize this communication with a type-and-effect system, which allows *shift* to capture and compose continuations of varying types. Asai and Kameyama (2007) extend Danvy and Filinski's (monomorphic) type system with polymorphism, which includes polymorphism of answer types. We give two substructural type systems with *shift* and *reset*. Section 6.1 presents a simpler version that severely limits the answer types of continuations that may be captured. Then, in §6.2, we combine the simpler system with a polymorphic version of Danvy and Filinski's, similar to Asai and Kameyama's, to allow answer-type modification and polymorphism in a substructural setting.

Substructural type systems. Researchers have proposed a plethora of substructural type systems. These range from minimalistic models (Wadler 1992; Bierman

1993; Barber 1996; Morrisett et al. 2005) based on Girard's linear logic (1987), to real programming languages, which are often oriented toward specific problems such as safety in low-level languages (Grossman et al. 2002), typestate and protocol checking (Aldrich et al. 2009), or security (Swamy et al. 2010).

We translate our substructural type-and-effect system into Ahmed et al.'s λ^{URAL} (2005), which is a polymorphic λ calculus that supports a variety of substructural typing disciplines. We provide a primer on λ^{URAL} in §3.

Motivation. The software engineering case for structured exception handling is widely acknowledged and understood, but *shift* and *reset* (Danvy and Filinski 1989), the other control operators discussed in this paper, are more obscure. The essential idea is simple: whereas raising an exception discards the context up to some delimiter—the exception handler—*shift* captures and reifies the context up to its delimiter, *reset*, which allows reinstating the context later. These control operators may be used to implement exceptions, by capturing continuations but never reinvoking them, but they may also express other control structures, such as coroutines and cooperative multithreading, and they may be used to abstract non-determinism and search in an elegant way.

Our goal is to safely integrate control operators with substructural types. A substructural type system regulates the order and number of uses of data by statically ensuring that some values be used at most once, at least once, or exactly once (Walker 2005). Like *shift* and *reset*, substructural types are a general facility that can express a variety of specific language features, mostly for the purpose of managing stateful resources, such as typestate, region-based memory management, and session types.

The direct impetus for this work is the design of the programming language Alms (Tov and Pucella 2011), which provides both exceptions and affine types, a variety of substructural type that can prohibit reusing particular values. As demonstrated in §1, the combination of affine types and exceptions is not a problem. However, as we observe in that previous work, "we anticipate that safely combining linearity with exceptions requires a type-and-effect system to track when raising an exception would implicitly discard linear values." Our desire to add linear types to Alms motivates this development of a general theory of substructural types and control effects.

3 Syntax and Semantics of λ^{URAL}

In this paper, we add control effects to Ahmed et al.'s λ^{URAL} (2005), a substructural λ calculus. Our presentation of λ^{URAL} is heavily based on theirs, with a few small changes.

The syntax of λ^{URAL} appears in Figure 1. Those non-terminals that appear in blue are different from their counterparts in the calculus with control effects (§4), which

v	::=		values
	|	x	variable
	|	$\lambda x.e$	abstraction
	|	$\Lambda.e$	type abstraction
	|	$\mathsf{inl}\ v$	sum construction, left
	|	$\mathsf{inr}\ v$	sum construction, right
	|	$[v_1, v_2]$	sum elimination
	|	$\langle v_1, v_2 \rangle$	pair construction
	|	$\mathsf{uncurry}\ v$	pair elimination
	|	$\langle\rangle$	the nil value
	|	$\mathsf{ignore}\ v$	nil elimination
	|	ℓ	location (run-time only)
e	::=		expressions
	|	v	values
	|	$e_1\, e_2$	application
	|	$e\ \_$	type application
	|	$\mathsf{new}^{\mathsf{q}}\ e$	reference allocation
	|	$\mathsf{free}\ e$	reference deallocation
	|	$\mathsf{read}\ e$	reference read
	|	$\mathsf{swap}\ e_1\, e_2$	reference read and write

Figure 1: λ^{URAL} syntax (i): expression level

τ	::=		types
	|	α	type variable
	|	$^{\xi}\overline{\tau}$	qualified pretype
$\mathfrak{q}$	::=		constant qualifiers
	|	U	unlimited
	|	R	relevant
	|	A	affine
	|	L	linear
ξ	::=		qualifiers
	|	α	qualifier variable
	|	$\mathfrak{q}$	qualifier constant
$\overline{\tau}$	::=		pretypes
	|	α	pretype variable
	|	$\mathsf{1}$	multiplicative unit
	|	$\tau_1 \otimes \tau_2$	multiplicative conjunction
	|	$\tau_1 \oplus \tau_2$	additive disjunction
	|	$\tau_1 \multimap \tau_2$	function
	|	$\mathsf{ref}\ \tau$	reference
	|	$\forall\alpha{:}\kappa.\tau$	universal quantification
ι	::=		type-level terms
	|	ξ	qualifier
	|	$\overline{\tau}$	pretype
	|	τ	type
κ	::=		kinds
	|	QUAL	qualifiers
	|	$\overline{\star}$	pretypes
	|	$\star$	types

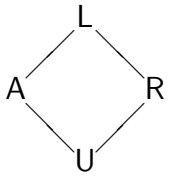
Figure 2: λ^{URAL} syntax (ii): type and kind level

will appear in red.[1]

The expression level. Values include abstractions, type abstractions, and introduction and elimination forms for sums, products, and the unit value. At run time, values also include location names. (This differs from Ahmed et al.'s presentation of λ^{URAL} by including sums—additive disjunctions, to be precise.) Expressions include values, application, type application, and operations on mutable references. Following Ahmed et al., we elide the formal parameter in type abstractions and the actual parameter in type applications.

The type level. Expressions in λ^{URAL} are classified by types (τ), but the language at the type level is much richer. Four constant qualifiers ($\mathfrak{q}$) distinguish four substructural properties that may be enforced for values:

- L as in *linear*, for values that may be neither duplicated nor implicitly dropped;
- A as in *affine*, for values that may be dropped (weakening) but not duplicated;
- R as in *relevant*, for values that may be duplicated (contraction) but not dropped; and
- U as in *unlimited*, for ordinary values that allow both dropping and duplication.



The four constant qualifiers form a lattice, whereby it is always safe to treat a value as if it has a higher qualifier than its own.

Qualifiers (ξ) include both qualifier constants and type variables, allowing for qualifier polymorphism. Pretypes ($\overline{\tau}$) specify the representation of a value, and its introduction and elimination rules. Pretypes include type variables, function types, universal quantification, the unit type, and additive disjunction. Types (τ) classify expressions. A type is either a pretype decorated with its qualifier ($^{\xi}\overline{\tau}$) or a type variable. We use non-terminal ι to refer to the three kinds of type-level terms as a group.

The kind level. Types in λ^{URAL} are classified by three kinds (κ): QUAL for qualifiers, $\overline{\star}$ for pretypes, and $\star$ for types. Type variables may have any of these three kinds, which is why universal quantification ($\forall\alpha{:}\kappa.\,\tau$) specifies the kind of α.

[1] This is the color version of this paper; a black-and-white version, which is more suitable for printing, is available online at www.ccs.neu.edu/~tov/pubs/substructural-control.

3.1 Operational Semantics

The operational semantics of λ^{URAL} is completely standard and appears in Figure 3. Reduction is call-by-value and evaluates operators before operands, which is important when we consider the sequencing of effects in §4.

3.2 Static Semantics

Type judgments for λ^{URAL} use two kinds of contexts:

Δ	::=		kind contexts
	$\mid$	$\bullet$	empty
	$\mid$	$\Delta, \alpha{:}\kappa$	kind of type variable
Γ	::=		type contexts
	$\mid$	$\bullet$	empty
	$\mid$	$\Gamma, x{:}\tau$	type of variable

Figure 4 contains the kinding judgment ($\Delta \vdash \iota : \kappa$), which assigns kinds to type-level terms. This judgment enforces the type/pretype structure, whereby type constructors such as $\oplus$ form a pretype from types (rule K-Sum), and decorating a pretype with a qualifier forms a type (rule K-Type).

In Figure 5, three judgments relate qualifiers to each other, to types, and to type contexts. Qualifier subsumption ($\Delta \vdash \xi_1 \preceq \xi_2$) defines the qualifier order, with top L and bottom U. The next judgment bounds a type by a qualifier; judgment $\Delta \vdash \tau \preceq \xi$ means that values of type τ may safely be used according to the structural rules implied by ξ. Finally, bounding a type context by a qualifier ($\Delta \vdash \Gamma \preceq \xi$) means that every type in context Γ is bounded by qualifier ξ.

Figure 6 gives rules for splitting a type context into two ($\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2$), which is necessary for distributing typing assumptions to multiple subterms of a term. Any variable may be distributed to one side or the other. Rule S-Contract implements the contraction structural rule, whereby variables whose type is unlimited or relevant may be duplicated to both contexts.

Finally, Figure 7 gives the judgment for assigning types to expressions ($\Delta; \Gamma \vdash e : \tau$). Several points are worthy of note:

- The weakening rule, T-Weak, allows discarding portions of the context that are upper-bounded by A, which means that all the values dropped are either affine or unlimited.
- The rules for application and reference swapping, T-App, T-SwapStrong, and T-SwapWeak, split the context to distribute assumptions to subterms.
- Rule T-Abs selects a qualifier ξ for a function type based on bounding the context, Γ. This means that the qualifier of a function type must upper bound

$$s ::= \{\} \mid \{\ell \overset{\mathsf{q}}{\mapsto} v\} \mid s_1 \uplus s_2 \qquad \textit{(stores)}$$

$$E ::= [] \mid E\, e_2 \mid v_1\, E \mid E\, \_ \mid \mathsf{new}^{\mathsf{q}}\, E \mid \mathsf{free}\ E \mid \mathsf{read}\ E \mid \mathsf{swap}\ E\, e \mid \mathsf{swap}\ v\, E \qquad \textit{(evaluation contexts)}$$

$\boxed{(s, e) \longmapsto (s, e')}$ *(reduction)*

$$\begin{aligned}
(s,)(\lambda x.e_1)\, v_2 &\longmapsto (s,)\{v_2/x\}e_2 \\
(s,)(\Lambda.e)\, _ &\longmapsto (s,)e \\
(s,)\mathsf{ignore}\, \langle\rangle\, v &\longmapsto (s,)v \\
(s,)\mathsf{uncurry}\ v\, \langle v_1, v_2\rangle &\longmapsto (s,)v\, v_1\, v_2 \\
(s,)[v_1, v_2]\, (\mathsf{inl}\ v) &\longmapsto (s,)v\, v_1 \\
(s,)[v_1, v_2]\, (\mathsf{inr}\ v) &\longmapsto (s,)v\, v_2 \\
(s,)\mathsf{new}^{\mathsf{q}}\, v &\longmapsto (s \uplus \{\ell \overset{\mathsf{q}}{\mapsto} v\},)\ell \\
(s \uplus \{\ell \overset{\mathsf{q}}{\mapsto} v\},)\mathsf{free}\ \ell &\longmapsto (s,)v \\
(s \uplus \{\ell \overset{\mathsf{q}}{\mapsto} v\},)\mathsf{read}\ \ell &\longmapsto (s \uplus \{\ell \overset{\mathsf{q}}{\mapsto} v\},)v \\
(s \uplus \{\ell \overset{\mathsf{q}}{\mapsto} v_1\},)\mathsf{swap}\ \ell\, v_2 &\longmapsto (s \uplus \{\ell \overset{\mathsf{q}}{\mapsto} v_2\},)\langle \ell, v_1\rangle \\
\frac{(s,)e \longmapsto (s',)e'}{(s,)E[e] \longmapsto (s',)E[e']}&
\end{aligned}$$

Figure 3: λ^{URAL} operational semantics

$\boxed{\Delta \vdash \iota : \kappa}$ *(kinding type-level terms)*

$$\textsc{K-Var}\ \frac{\alpha{:}\kappa \in \Delta}{\Delta \vdash \alpha : \kappa} \qquad \textsc{K-Qual}\ \frac{}{\Delta \vdash \mathsf{q} : \mathsf{QUAL}} \qquad \textsc{K-Arr}\ \frac{\Delta \vdash \tau_1 : \star \qquad \Delta \vdash \tau_2 : \star}{\Delta \vdash \tau_1 \multimap \tau_2 : \overline{\star}} \qquad \textsc{K-All}\ \frac{\Delta, \alpha{:}\kappa \vdash \tau : \star}{\Delta \vdash \forall \alpha{:}\kappa.\tau : \overline{\star}}$$

$$\textsc{K-Unit}\ \frac{}{\Delta \vdash 1 : \overline{\star}} \qquad \textsc{K-Sum}\ \frac{\Delta \vdash \tau_1 : \star \qquad \Delta \vdash \tau_2 : \star}{\Delta \vdash \tau_1 \oplus \tau_2 : \overline{\star}} \qquad \textsc{K-Prod}\ \frac{\Delta \vdash \tau_1 : \star \qquad \Delta \vdash \tau_2 : \star}{\Delta \vdash \tau_1 \otimes \tau_2 : \overline{\star}} \qquad \textsc{K-Ref}\ \frac{\Delta \vdash \tau : \star}{\Delta \vdash \mathsf{ref}\ \tau : \overline{\star}}$$

$$\textsc{K-Type}\ \frac{\Delta \vdash \overline{\tau} : \overline{\star} \qquad \Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash {}^{\xi}\overline{\tau} : \star}$$

Figure 4: λ^{URAL} statics (i): kinding

$\boxed{\Delta \vdash \xi_1 \preceq \xi_2}$ *(qualifier subsumption)*

$$\textsc{QSub-Bot}\ \frac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \mathsf{U} \preceq \xi} \qquad \textsc{QSub-Top}\ \frac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \xi \preceq \mathsf{L}} \qquad \textsc{QSub-Refl}\ \frac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \xi \preceq \xi}$$

$\boxed{\Delta \vdash \tau \preceq \xi}$ *(qualifier bound for types)*

$$\textsc{B-Var}\ \frac{\Delta \vdash \alpha : \star}{\Delta \vdash \alpha \preceq \mathsf{L}} \qquad \textsc{B-Type}\ \frac{\Delta \vdash \overline{\tau} : \overline{\star} \qquad \Delta \vdash \xi' \preceq \xi}{\Delta \vdash {}^{\xi'}\overline{\tau} \preceq \xi}$$

$\boxed{\Delta \vdash \Gamma \preceq \xi}$ *(qualifier bound for type contexts)*

$$\textsc{B-Nil}\ \frac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \bullet \preceq \xi} \qquad \textsc{B-Cons}\ \frac{\Delta \vdash \Gamma \preceq \xi \qquad \Delta \vdash \tau \preceq \xi}{\Delta \vdash \Gamma, x{:}\tau \preceq \xi}$$

Figure 5: λ^{URAL} statics (ii): qualifiers

$\boxed{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2}$ *(type context splitting)*

$$\textsc{S-Nil}\ \frac{}{\Delta \vdash \bullet \rightsquigarrow \bullet \boxplus \bullet} \qquad \textsc{S-ConsL}\ \frac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \qquad \Delta \vdash \tau : \star}{\Delta \vdash \Gamma, x{:}\tau \rightsquigarrow (\Gamma_1, x{:}\tau) \boxplus \Gamma_2}$$

$$\textsc{S-ConsR}\ \frac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \qquad \Delta \vdash \tau : \star}{\Delta \vdash \Gamma, x{:}\tau \rightsquigarrow \Gamma_1 \boxplus (\Gamma_2, x{:}\tau)} \qquad \textsc{S-Contract}\ \frac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \qquad \Delta \vdash \tau \preceq \mathsf{R}}{\Delta \vdash \Gamma, x{:}\tau \rightsquigarrow (\Gamma_1, x{:}\tau) \boxplus (\Gamma_2, x{:}\tau)}$$

Figure 6: λ^{URAL} statics (iii): context splitting

$\boxed{\Delta;\Gamma \vdash e : \tau}$ *(typing expressions)*

T-Weak
$$\frac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \qquad \Delta \vdash \Gamma_2 \preceq \mathsf{A} \qquad \Delta;\Gamma_1 \vdash e : \tau}{\Delta;\Gamma \vdash e : \tau}$$

T-Var
$$\frac{\Delta \vdash \tau : \star}{\Delta;\bullet, x{:}\tau \vdash x : \tau}$$

T-Abs
$$\frac{\Delta \vdash \Gamma \preceq \xi \qquad \Delta;\Gamma, x{:}\tau_1 \vdash e : \tau_2}{\Delta;\Gamma \vdash \lambda x.e : {}^{\xi}(\tau_1 \multimap \tau_2)}$$

T-TAbs
$$\frac{\Delta \vdash \Gamma \preceq \xi \qquad \Delta, \alpha{:}\kappa;\Gamma \vdash e : \tau}{\Delta;\Gamma \vdash \Lambda.e : {}^{\xi}\forall\alpha{:}\kappa.\tau}$$

T-Unit
$$\frac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta;\bullet \vdash \langle\rangle : {}^{\xi}1}$$

T-Inl
$$\frac{\Delta \vdash \tau_1 \preceq \xi \qquad \Delta \vdash \tau_2 : \star \qquad \Delta;\Gamma \vdash v_1 : \tau_1}{\Delta;\Gamma \vdash \mathsf{inl}\ v_1 : {}^{\xi}(\tau_1 \oplus \tau_2)}$$

T-Inr
$$\frac{\Delta \vdash \tau_2 \preceq \xi \qquad \Delta \vdash \tau_1 : \star \qquad \Delta;\Gamma \vdash v_2 : \tau_2}{\Delta;\Gamma \vdash \mathsf{inr}\ v_2 : {}^{\xi}(\tau_1 \oplus \tau_2)}$$

T-App
$$\frac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \qquad \Delta;\Gamma_1 \vdash e_1 : {}^{\xi}(\tau_1 \multimap \tau_2) \qquad \Delta;\Gamma_2 \vdash e_2 : \tau_1}{\Delta;\Gamma \vdash e_1\, e_2 : \tau_2}$$

T-TApp
$$\frac{\Delta;\Gamma \vdash e : {}^{\xi}\forall\alpha{:}\kappa.\tau \qquad \Delta \vdash \iota : \kappa}{\Delta;\Gamma \vdash e\_ : \{\iota/\alpha\}\tau}$$

T-Prod
$$\frac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \qquad \Delta;\Gamma_1 \vdash v_1 : \tau_1 \qquad \Delta \vdash \tau_1 \preceq \xi \qquad \Delta;\Gamma_2 \vdash v_2 : \tau_2 \qquad \Delta \vdash \tau_2 \preceq \xi}{\Delta;\Gamma \vdash \langle v_1, v_2\rangle : {}^{\xi}(\tau_1 \otimes \tau_2)}$$

T-SumE
$$\frac{\Delta \vdash \xi' : \mathsf{QUAL} \qquad \Delta;\Gamma \vdash v_1 : {}^{\xi_1}(\tau_1 \multimap \tau) \qquad \Delta \vdash \xi_1 \preceq \xi \qquad \Delta;\Gamma \vdash v_2 : {}^{\xi_2}(\tau_2 \multimap \tau) \qquad \Delta \vdash \xi_2 \preceq \xi}{\Delta;\Gamma \vdash [v_1, v_2] : {}^{\xi}({}^{\xi'}(\tau_1 \oplus \tau_2) \multimap \tau)}$$

T-ProdE
$$\frac{\Delta \vdash \xi : \mathsf{QUAL} \qquad \Delta;\Gamma \vdash v : {}^{\xi'}(\tau_1 \multimap {}^{\xi'}(\tau_2 \multimap \tau))}{\Delta;\Gamma \vdash \mathsf{uncurry}\ v : {}^{\xi'}({}^{\xi}(\tau_1 \otimes \tau_2) \multimap \tau)}$$

T-UnitE
$$\frac{\Delta \vdash \xi : \mathsf{QUAL} \qquad \Delta \vdash \tau : \star \qquad \Delta;\Gamma \vdash v : {}^{\xi'}1}{\Delta;\Gamma \vdash \mathsf{ignore}\ v : {}^{\xi}(\tau \multimap \tau)}$$

Figure 7: λ^{URAL} statics (iv): typing

(continued in Figure 8)

(continued from Figure 7)

$\boxed{\Delta; \Gamma \vdash e : \tau}$ *(typing expressions)*

$$\textsc{T-NewUA}\ \frac{\mathsf{q} \preceq \mathsf{A} \qquad \Delta; \Gamma \vdash e : \tau \qquad \Delta \vdash \tau \preceq \mathsf{A}}{\Delta; \Gamma \vdash \mathsf{new}^{\mathsf{q}}\, e : {}^{\mathsf{q}}\mathsf{ref}\ \tau}$$

$$\textsc{T-NewRL}\ \frac{\mathsf{R} \preceq \mathsf{q} \qquad \Delta; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \mathsf{new}^{\mathsf{q}}\, e : {}^{\mathsf{q}}\mathsf{ref}\ \tau}$$

$$\textsc{T-Delete}\ \frac{\Delta; \Gamma \vdash e : {}^{\xi}\mathsf{ref}\ \tau \qquad \Delta \vdash \mathsf{A} \preceq \xi}{\Delta; \Gamma \vdash \mathsf{free}\ e : \tau}$$

$$\textsc{T-Read}\ \frac{\Delta; \Gamma \vdash e : {}^{\xi}\mathsf{ref}\ \tau \qquad \Delta \vdash \tau \preceq \mathsf{R}}{\Delta; \Gamma \vdash \mathsf{read}\ e : \tau}$$

$$\textsc{T-SwapStrong}\ \frac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \qquad \Delta; \Gamma_1 \vdash e_1 : {}^{\xi}\mathsf{ref}\ \tau_1 \qquad \Delta \vdash \mathsf{A} \preceq \xi \qquad \Delta; \Gamma_2 \vdash e_2 : \tau_2 \qquad \Delta \vdash \tau_2 \preceq \xi}{\Delta; \Gamma \vdash \mathsf{swap}\ e_1\, e_2 : {}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ \tau_2 \otimes \tau_1)}$$

$$\textsc{T-SwapWeak}\ \frac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \qquad \Delta; \Gamma_1 \vdash e_1 : {}^{\xi}\mathsf{ref}\ \tau \qquad \Delta; \Gamma_2 \vdash e_2 : \tau}{\Delta; \Gamma \vdash \mathsf{swap}\ e_1\, e_2 : {}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ \tau \otimes \tau)}$$

Figure 8: λ^{URAL} statics (v): typing

the qualifiers of the types of the function's free variables. As we will see in §5, this property is key to our soundness theorem.

4 Generic Control Effects

Rather than add a specific control effect, such as exceptions or delimited continuations, to λ^{URAL}, we aim to design a substructural type system with a general notion of control effect. Thus, in this section, we define a new calculus, $\lambda^{\mathsf{URAL}}(\mathcal{C})$, parameterized by an unspecified control effect.

4.1 The Control Effect Parameter

In this subsection, we give the form of the parameter that stands for a particular control effect. Our definition of $\lambda^{\mathsf{URAL}}(\mathcal{C})$ relies only on this abstract specification of the formal parameter. In §5, we specify several properties of the parameter that are sufficient for a generic soundness theorem to hold, and in §6 we give three examples of actual control effect parameters.

Definition 4.1 (Control effect)**.**

A control effect instance is a triple $(\mathcal{C}, \bot_e, \fatsemi)$ *where* $\mathcal{C}$ *is a set of control effects (*c*),* $\bot_e \in \mathcal{C}$ *is a distinguished* pure *effect that denotes no actual control, and* $\fatsemi \colon \mathcal{C} \times \mathcal{C} \rightharpoonup \mathcal{C}$ *is an associative, partial, binary operation denoting effect sequencing.*

For example, in §6.3 we add exception handling to $\lambda^{\mathsf{URAL}}(\mathcal{C})$. An exception effect is the set of exceptions that may be raised by an expression, the distinguished pure effect $\bot_e$ is the empty set, and sequencing is set union. A non-empty effect indicates that an expression may discard part of its continuation, whereas the empty effect guarantees that an expression treats its continuation linearly.

In simple cases, as with exceptions, effects form a join semilattice where sequencing is the join, but this is not necessarily true in general (§6.2).

4.2 Updated Syntax

In $\lambda^{\mathsf{URAL}}(\mathcal{C})$, control effects constitute a fourth kind of type-level term, in addition to qualifiers, pretypes, and types. We add a new kind, CTL, and include abstract control effects ($c \in \mathcal{C}$) among the type-level terms:

k	::=		kinds
	|	CTL	control effects
	|	QUAL	qualifiers
	|	$\overline{\star}$	pretypes
	|	$\star$	types
i	::=		type-level terms
	|	c	control effect
	|	ξ	qualifier
	|	$\overline{t}$	pretype
	|	t	type

Function and universal pretypes now have latent effects, which record the effect that will happen when an abstraction is applied. We update the definition of pretypes to include these latent effects:

$\overline{t}$	::=		updated pretypes
	|	$\cdots$	other productions as before
	|	$t_1 \overset{c}{\multimap} t_2$	function with latent effect
	|	$\forall^c \alpha{:}k.t$	universal with latent effect
t	::=		updated types
	|	α	type variable
	|	$^{\xi}\overline{t}$	qualified pretype

$\boxed{\mathrm{D} \vDash_{\mathcal{C}} i : k}$ *(kinding type-level terms)*

$$\textsc{C-K-Bot}\ \frac{}{\mathrm{D} \vDash_{\mathcal{C}} \bot_{\mathcal{C}} : \mathsf{CTL}} \qquad \textsc{C-K-Arr}\ \frac{\mathrm{D} \vDash_{\mathcal{C}} t_1 : \star \qquad \mathrm{D} \vDash_{\mathcal{C}} t_2 : \star \qquad \mathrm{D} \vDash_{\mathcal{C}} c : \mathsf{CTL}}{\mathrm{D} \vDash_{\mathcal{C}} t_1 \overset{c}{\multimap} t_2 : \overline{\star}}$$

$$\textsc{C-K-All}\ \frac{\mathrm{D}, \alpha{:}k \vDash_{\mathcal{C}} t : \star \qquad \mathrm{D} \vDash_{\mathcal{C}} c : \mathsf{CTL}}{\mathrm{D} \vDash_{\mathcal{C}} \forall^{c}\alpha{:}k.t : \overline{\star}}$$

Figure 9: $\lambda^{\mathsf{URAL}}(\mathcal{C})$ statics (i): updated kinding rules

$\boxed{\mathrm{D} \vDash_{\mathcal{C}} c \succeq \xi}$ *(qualifier bound for control effects)*

$$\textsc{C-B-Pure}\ \frac{\mathrm{D} \vDash_{\mathcal{C}} \xi : \mathsf{QUAL}}{\mathrm{D} \vDash_{\mathcal{C}} \bot_{\mathcal{C}} \succeq \xi} \qquad \textsc{C-B-Unl}\ \frac{\mathrm{D} \vDash_{\mathcal{C}} c : \mathsf{CTL}}{\mathrm{D} \vDash_{\mathcal{C}} c \succeq \mathsf{U}}$$

$\boxed{\mathrm{D} \vDash_{\mathcal{C}} c_1 \preceq c_2}$ *(control effect subsumption)*

$$\textsc{CSub-Refl}\ \frac{\mathrm{D} \vDash_{\mathcal{C}} c : \mathsf{CTL}}{\mathrm{D} \vDash_{\mathcal{C}} c \preceq c} \qquad \textsc{CSub-Trans}\ \frac{\mathrm{D} \vDash_{\mathcal{C}} c_1 \preceq c' \qquad \mathrm{D} \vDash_{\mathcal{C}} c' \preceq c_2}{\mathrm{D} \vDash_{\mathcal{C}} c_1 \preceq c_2}$$

Figure 10: $\lambda^{\mathsf{URAL}}(\mathcal{C})$ statics (ii): control effect judgments

The other pretype ($\bar{t}$) productions remain unchanged.

For non-terminal symbols that differ between λ^{URAL} and $\lambda^{\mathsf{URAL}}(\mathcal{C})$, we use red Roman letters (t, k, G, ...) for $\lambda^{\mathsf{URAL}}(\mathcal{C})$ to distinguish them from λ^{URAL}, where they appeared in blue Greek (τ, κ, Γ, ...).

4.3 Static Semantics of $\lambda^{\mathsf{URAL}}(\mathcal{C})$

All type system judgments from λ^{URAL} are updated for $\lambda^{\mathsf{URAL}}(\mathcal{C})$, and $\lambda^{\mathsf{URAL}}(\mathcal{C})$ adds two new judgments as well. The kinding and expression typing judgments are the only two to change significantly. The judgments for bounding types ($\mathrm{D} \vDash_{\mathcal{C}} t \preceq \xi$), bounding type contexts $\mathrm{D} \vDash_{\mathcal{C}} \mathrm{G} \preceq \xi$), and splitting type contexts ($\mathrm{D} \vDash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2$) are isomorphic to the λ^{URAL} versions of those judgments from Figures 5 and 6. They

$\boxed{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c}$ *(typing expressions)*

C-T-Subsume
$$\frac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c' \qquad \mathrm{D} \vdash_{\mathcal{C}} c' \preceq c}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c}$$

C-T-Weak
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \qquad \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e : t \,;\, c \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \mathsf{A}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c}$$

C-T-Var
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} t : \star}{\mathrm{D}; \bullet, x{:}t \vdash_{\mathcal{C}} x : t \,;\, \bot_{\mathcal{C}}}$$

C-T-Abs
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \preceq \xi \qquad \mathrm{D}; \mathrm{G}, x{:}t_1 \vdash_{\mathcal{C}} e : t_2 \,;\, c}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \lambda x.e : {}^{\xi}(t_1 \xrightarrow{c} t_2) \,;\, \bot_{\mathcal{C}}}$$

C-T-TAbs
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \preceq \xi \qquad \mathrm{D}, \alpha{:}k; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \Lambda.e : {}^{\xi}\forall^{c}\alpha{:}k.t \,;\, \bot_{\mathcal{C}}}$$

C-T-Unit
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL}}{\mathrm{D}; \bullet \vdash_{\mathcal{C}} \langle\rangle : {}^{\xi}1 \,;\, \bot_{\mathcal{C}}}$$

C-T-Inl
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} t_1 \preceq \xi \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 : \star \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} v_1 : t_1 \,;\, \bot_{\mathcal{C}}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{inl}\ v_1 : {}^{\xi}(t_1 \oplus t_2) \,;\, \bot_{\mathcal{C}}}$$

C-T-Inr
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 \preceq \xi \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} v_2 : t_2 \,;\, \bot_{\mathcal{C}}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{inr}\ v_2 : {}^{\xi}(t_1 \oplus t_2) \,;\, \bot_{\mathcal{C}}}$$

C-T-App
$$\frac{\begin{array}{c}\mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e_1 : {}^{\xi_1}(t_1 \xrightarrow{c} t_2) \,;\, c_1 \qquad \mathrm{D}; \mathrm{G}_2 \vdash_{\mathcal{C}} e_2 : t_1 \,;\, c_2 \\ \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \xi_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1 \succeq \xi_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_2 \succeq \xi_1 \\ \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1 \otimes c_2 \otimes c : \mathsf{CTL}\end{array}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e_1\, e_2 : t_2 \,;\, c_1 \otimes c_2 \otimes c}$$

C-T-TApp
$$\frac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : {}^{\xi}\forall^{c'}\alpha{:}k.t \,;\, c \qquad \mathrm{D} \vdash_{\mathcal{C}} i : k \qquad \mathrm{D} \vdash_{\mathcal{C}} c \otimes c' : \mathsf{CTL}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e\ \_ : \{i/\alpha\}t \,;\, c \otimes c'}$$

C-T-Prod
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \qquad \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} v_1 : t_1 \,;\, \bot_{\mathcal{C}} \qquad \mathrm{D} \vdash_{\mathcal{C}} t_1 \preceq \xi \qquad \mathrm{D}; \mathrm{G}_2 \vdash_{\mathcal{C}} v_2 : t_2 \,;\, \bot_{\mathcal{C}} \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 \preceq \xi}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \langle v_1, v_2 \rangle : {}^{\xi}(t_1 \otimes t_2) \,;\, \bot_{\mathcal{C}}}$$

C-T-SumE
$$\frac{\begin{array}{c}\mathrm{D} \vdash_{\mathcal{C}} \xi' : \mathsf{QUAL} \\ \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} v_1 : {}^{\xi_1}(t_1 \xrightarrow{c} t) \,;\, \bot_{\mathcal{C}} \qquad \mathrm{D} \vdash_{\mathcal{C}} \xi_1 \preceq \xi \\ \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} v_2 : {}^{\xi_2}(t_2 \xrightarrow{c} t) \,;\, \bot_{\mathcal{C}} \qquad \mathrm{D} \vdash_{\mathcal{C}} \xi_2 \preceq \xi\end{array}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} [v_1, v_2] : {}^{\xi}({}^{\xi'}(t_1 \oplus t_2) \xrightarrow{c} t) \,;\, \bot_{\mathcal{C}}}$$

C-T-ProdE
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL} \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1 \otimes c_2 : \mathsf{CTL} \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} v : {}^{\xi'}(t_1 \xrightarrow{c_1} {}^{\xi'}(t_2 \xrightarrow{c_2} t)) \,;\, \bot_{\mathcal{C}}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{uncurry}\ v : {}^{\xi'}({}^{\xi}(t_1 \otimes t_2) \xrightarrow{c_1 \otimes c_2} t) \,;\, \bot_{\mathcal{C}}}$$

C-T-UnitE
$$\frac{\mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL} \qquad \mathrm{D} \vdash_{\mathcal{C}} t : \star \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} v : {}^{\xi'}1 \,;\, \bot_{\mathcal{C}}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{ignore}\ v : {}^{\xi}(t \xrightarrow{\bot_{\mathcal{C}}} t) \,;\, \bot_{\mathcal{C}}}$$

Figure 11: $\lambda^{\mathsf{URAL}}(\mathcal{C})$ statics (iii): typing

(continued in Figure 12)

(continued from Figure 11)

$\boxed{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c}$ *(typing expressions)*

C-T-NewUA
$$\frac{\mathsf{q} \preceq \mathsf{A} \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c \qquad \mathrm{D} \vdash_{\mathcal{C}} t \preceq \mathsf{A}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{new}^{\mathsf{q}}\, e : {}^{\mathsf{q}}\mathsf{ref}\ t \,;\, c}$$

C-T-NewRL
$$\frac{\mathsf{R} \preceq \mathsf{q} \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{new}^{\mathsf{q}}\, e : {}^{\mathsf{q}}\mathsf{ref}\ t \,;\, c}$$

C-T-Delete
$$\frac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : {}^{\xi}\mathsf{ref}\ t \,;\, c \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathsf{A} \preceq \xi}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{free}\ e : t \,;\, c}$$

C-T-Read
$$\frac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : {}^{\xi}\mathsf{ref}\ t \,;\, c \qquad \mathrm{D} \vdash_{\mathcal{C}} t \preceq \mathsf{R}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{read}\ e : t \,;\, c}$$

C-T-SwapStrong
$$\frac{\begin{array}{c}\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \\ \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e_1 : {}^{\xi_1}\mathsf{ref}\ t_1 \,;\, c_1 \\ \mathrm{D}; \mathrm{G}_2 \vdash_{\mathcal{C}} e_2 : t_2 \,;\, c_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \xi_2 \\ \mathrm{D} \vdash_{\mathcal{C}} c_1 \succeq \xi_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_2 \succeq \xi_1 \\ \mathrm{D} \vdash_{\mathcal{C}} \mathsf{A} \preceq \xi_1 \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 \preceq \xi_1 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1 \oslash c_2 : \mathsf{CTL}\end{array}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{swap}\ e_1\, e_2 : {}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ t_2 \otimes t_1) \,;\, c_1 \oslash c_2}$$

C-T-SwapWeak
$$\frac{\begin{array}{c}\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \\ \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e_1 : {}^{\xi_1}\mathsf{ref}\ t \,;\, c_1 \\ \mathrm{D}; \mathrm{G}_2 \vdash_{\mathcal{C}} e_2 : t \,;\, c_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \xi_2 \\ \mathrm{D} \vdash_{\mathcal{C}} c_1 \succeq \xi_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_2 \succeq \xi_1 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1 \oslash c_2 : \mathsf{CTL}\end{array}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{swap}\ e_1\, e_2 : {}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ t \otimes t) \,;\, c_1 \oslash c_2}$$

Figure 12: $\lambda^{\mathsf{URAL}}(\mathcal{C})$ statics (iv): typing

are merely updated with new non-terminals as appropriate (*i.e.*, κ to k, $\overline{\tau}$ to $\overline{t}$, and τ to t).

Kinding. We identify control effects as the type-level terms (i) that are assigned kind CTL by the kinding judgment. Figure 9 shows one new kinding rule, C-K-Bot, which assigns kind CTL to the pure effect $\bot_{\mathcal{C}}$. We update rules C-K-Arr and C-K-All to account for latent effects in function and universal pretypes. The remaining kinding rules are the same as for λ^{URAL}, with non-terminals *mutatis mutandis*. Specific control effect instances (§6) must define additional kinding rules for their particular effects.

Control effect judgments. The first new judgment for control effects ($\mathrm{D} \vdash_{\mathcal{C}} c \succeq \xi$, Figure 10) relates control effects to qualifiers. This gives the meaning of a control effect in terms of a lower bound for how an expression with that effect may treat its own continuation. For example, if an expression e has some effect c such that $\mathrm{D} \vdash_{\mathcal{C}} c \succeq \mathsf{A}$, this indicates that e may drop but not duplicate its continuation. We give two rules here:

- Rule C-B-Pure says that the pure effect is bounded by any qualifier, which means that a pure expression satisfies any requirement for how it treats its continuation.

- Rule C-B-Unl says that all control effects are bounded by U, which means that we may assume, conservatively, that any expression might freely duplicate or drop its continuation.

Specific instances of the control effect parameter will extend this judgment to take into account the properties of a particular control effect.

The second judgment for control effects ($\mathrm{D} \vdash_{\mathcal{C}} c_1 \preceq c_2$) defines a subsumption order for control effects. This means that an expression whose effect is c_1 may be safely considered to have effect c_2. Only two rules for the judgment appear in Figure 10, which together ensure that control effect subsumption is a preorder. As with control effect bounding, specific control effect instances will extend this judgment.

Expression typing. The expression typing judgment for $\lambda^{\mathsf{URAL}}(\mathcal{C})$ (Figure 11) assigns not only a type t but an effect c to expressions: $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$. Having seven premises, the rule for applications (C-T-App) is unwieldy, but it likely gives

the most insight into how $\lambda^{\mathsf{URAL}}(\mathcal{C})$ works:

$$\frac{\begin{array}{ll}(1) & \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \\ (2) & \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e_1 : {}^{\xi_1}(t_1 \xrightarrow{c} t_2) \,;\, c_1 \\ (3) & \mathrm{D}; \mathrm{G}_2 \vdash_{\mathcal{C}} e_2 : t_1 \,;\, c_2 \\ (4) & \mathrm{D} \vdash_{\mathcal{C}} c_2 \succeq \xi_1 \\ (5) & \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \xi_2 \\ (6) & \mathrm{D} \vdash_{\mathcal{C}} c_1 \succeq \xi_2 \\ (7) & \mathrm{D} \vdash_{\mathcal{C}} c_1 \mathbin{⊜} c_2 \mathbin{⊜} c : \mathsf{CTL}\end{array}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e_1\, e_2 : t_2 \,;\, c_1 \mathbin{⊜} c_2 \mathbin{⊜} c}$$

We consider the premises in order:

(1) The first premise, as in λ^{URAL}, splits the type context G into G_1 for typing e_1 and G_2 for typing e_2.

(2–3) As in λ^{URAL}, these premises assign types to expressions e_1 and e_2, but they assign control effects c_1 and c_2 as well.

(4) This premise relates the type of e_1 to the effect of e_2 to ensure that e_2's effect does not violate e_1's invariants. Because we fix a left-to-right evaluation order, by the time e_2 gets to run, e_1 has reduced to a value of type ${}^{\xi_1}(t_1 \xrightarrow{c} t_2)$, which thus may be treated according to qualifier ξ_1. Because that value is part of e_2's continuation, we require that e_2's effect, c_2, be lower-bounded by ξ_1. In other words, e_2 will treat its continuation no more liberally than ξ_1 allows.

(5–6) These premises relate the free variables of e_2 to the effect of e_1. Due to the evaluation order, e_2 appears unevaluated in e_1's continuation, which means that if e_1 drops or duplicates its continuation then e_2 may be evaluated never or more than once. Premise (5) says that the type context for typing e_2, and thus e_2's free variables, are bounded above by some qualifier ξ_2, and this qualifier thus indicates how many times it is safe to evaluate e_2. Premise (6) lower bounds e_1's effect, c_1, by ξ_2, ensuring that e_1's effect treats e_2 properly.

(7) The net effect of the application expression is a sequence of the effect of e_1 (c_1), then the effect of e_2 (c_2), and finally the latent effect of the function to which e_1 must evaluate (c): $c_1 \mathbin{⊜} c_2 \mathbin{⊜} c$. This premise checks that those three effects may be sequenced in that order according to a particular control effect's definition of sequencing and the kinding judgment.

Rules C-T-SwapStrong and C-T-SwapStrong (reference swap) are similar, since they need to safely sequence two subexpressions. Both rules follow rule C-T-App in relating the effect of the first subexpression to the type context of the second and effect of the second to the qualifier of the first. Rule C-T-TApp (type application),

while dealing with only one effectful subexpression, needs to sequence the effect of evaluating the expression in a type application with the latent effect of the resulting type abstraction value.

The subsumption rule C-T-SUBSUME implements control effect subsumption, whereby an expression of effect c may also be considered to have effect c' if c is less than c' in the control effect subsumption order. C-T-WEAK, which handles weakening, is unchanged from λ^{URAL}.

The remaining rules are for typing values, which always have the pure effect $\bot_{\mathcal{C}}$. Rules C-T-UNIT, C-T-INL, and C-T-INR, for unit and sum introduction, are unchanged from λ^{URAL}, except that each of them assigns the pure effect. Rules C-T-ABS and C-T-TABS also assign the pure effect to their values, but each records the effect of the abstraction body as the latent effect in the resulting type.

5 The Generic Theory

To prove type safety for $\lambda^{\mathsf{URAL}}(\mathcal{C})$, we define a type-preserving translation to λ^{URAL}. Rather than provide a reduction semantics for $\lambda^{\mathsf{URAL}}(\mathcal{C})$, we define its operational semantics in terms of the translation and the reduction semantics of λ^{URAL} (§3.1). Thus, if we can show that all well-typed $\lambda^{\mathsf{URAL}}(\mathcal{C})$ programs translate to well-typed λ^{URAL} programs, then λ^{URAL}'s type safety theorem applies to $\lambda^{\mathsf{URAL}}(\mathcal{C})$ as well.

The translation is into what Danvy and Filinski (1989) call *continuation-composing style* (henceforth "CCoS"). It is similar to continuation-passing style, but unlike continuation-passing style it still relies on the object language's order of evaluation.

In order to specify the translation and prove the propositions specified later in this section, we impose several more requirements on the abstract control effect parameter. As the semantics of $\lambda^{\mathsf{URAL}}(\mathcal{C})$ was parameterized by an abstract control effect, so is the theory of $\lambda^{\mathsf{URAL}}(\mathcal{C})$ parameterized by several definitions and properties that a control effect must satisfy.

The development of this section is constrained by several dependencies, so we provide an outline:

The Translation Parameter (§5.1). A control effect instance must supply a few definitions to fully specify its particular CCoS translation.

The Translation (§5.2). The definition of the CCoS translation relies on the definitions supplied by the control effect parameter.

Parameter Properties (§5.3). A control effect instance must satisfy several properties on which the generic type safety theorem relies.

Generic Type Safety (§5.4). The section culminates in a generic proof of type safety for $\lambda^{\mathsf{URAL}}(\mathcal{C})$.

5.1 The Translation Parameter

Definition 5.1 (Translation parameter)**.**
The definition of the generic CCoS translation relies on the following effect-specific definitions:

- *a metafunction* $(\cdot)^*$ *from effects to qualifiers, such that* $\bot_{\mathcal{E}}{}^* = \mathsf{L}$ *and* $\alpha^* = \alpha$*;*
- *a value* $\mathsf{done}_{\mathcal{E}}$*, to use as the initial continuation for a CCoSed program; and*
- *a pair of* answer-type *metafunctions* $\langle\!\langle \cdot, \cdot \rangle\!\rangle^-_c$ *and* $\langle\!\langle \cdot, \cdot \rangle\!\rangle^+_c$*, each of which maps a* λ^{URAL} *type and a* $\lambda^{\mathsf{URAL}}(\mathcal{C})$ *effect to a* λ^{URAL} *type.*

Intuitively, we can understand metafunctions $(\cdot)^*$, $\langle\!\langle \cdot, \cdot \rangle\!\rangle^-_c$, and $\langle\!\langle \cdot, \cdot \rangle\!\rangle^+_c$ as relating the effect of a $\lambda^{\mathsf{URAL}}(\mathcal{C})$ expression to the type of its translation into λ^{URAL}. Typically, the CPS translation of an expression of some type τ yields a type like

$$(\tau \to \mathsf{Answer}) \to \mathsf{Answer}.$$

Given a $\lambda^{\mathsf{URAL}}(\mathcal{C})$ expression whose translated type is τ and whose effect is c, our translation yields type

$$^{c^*}(\tau \multimap \langle\!\langle \tau_0, c \rangle\!\rangle^-_{\mathcal{E}}) \multimap \langle\!\langle \tau_0, c \rangle\!\rangle^+_{\mathcal{E}}$$

for some answer type τ_0. That is, $(\cdot)^*$ gives the qualifier of the continuation, and the other two metafunctions give the answer types, which may depend on the nature of the control effect. Because they give the answer types in negative and positive positions, respectively, we call $\langle\!\langle \tau, c \rangle\!\rangle^-_{\mathcal{E}}$ the *negative answer type* and $\langle\!\langle \tau, c \rangle\!\rangle^+_{\mathcal{E}}$ the *positive answer type*.

5.2 The Translation

In this subsection, we specify the CCoS translation from $\lambda^{\mathsf{URAL}}(\mathcal{C})$ to λ^{URAL}. In several places, we rely on the definitions of c^*, $\mathsf{done}_{\mathcal{E}}$, $\langle\!\langle \tau, c \rangle\!\rangle^-_{\mathcal{E}}$, and $\langle\!\langle \tau, c \rangle\!\rangle^+_{\mathcal{E}}$ supplied by the control effect parameter.

The translation for kinds and kind contexts appears in Figure 13. The control effect kind CTL translates to QUAL, and the other three kinds translate to themselves. The translation of a kind context merely translates each kind in its range.

Figure 14 presents the translation for pretypes, types, and type contexts. Most of this translation is straightforward: type variables and the unit pretype translate to themselves, sum, product, and reference types translate homomorphically, types composed of a qualifier and a pretype translate the pretype, and type contexts translate all the types in their range. The two interesting cases are for function and universal pretypes. These follow the usual CPS translation for function and universal types, with several refinements:

$$
\begin{aligned}
\mathsf{QUAL}^* &= \mathsf{QUAL} && \textit{(kinds)}\\
\overline{\star}^* &= \overline{\star}\\
\star^* &= \star\\
\mathsf{CTL}^* &= \mathsf{QUAL}\\
\bullet^* &= \bullet && \textit{(kind contexts)}\\
(\mathrm{D}, \alpha{:}k)^* &= \mathrm{D}^*, \alpha{:}k^*
\end{aligned}
$$

Figure 13: CCoS translation (i): kinds and kind contexts

$$
\begin{aligned}
\alpha^* &= \alpha && \textit{(pretypes)}\\
1^* &= 1\\
(t_1 \oplus t_2)^* &= {t_1}^* \oplus {t_2}^*\\
(t_1 \otimes t_2)^* &= {t_1}^* \otimes {t_2}^*\\
(\mathsf{ref}\ t)^* &= \mathsf{ref}\ t^*\\
(t_1 \xrightarrow{c} t_2)^* &= \forall\alpha{:}\star.{}^{\mathsf{L}}({t_1}^* \multimap {}^{\mathsf{L}}({c^*}({t_2}^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle_{\mathrm{e}}^{-}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle_{\mathrm{e}}^{+}))\\
(\forall^{c}\beta{:}k.t)^* &= \forall\alpha{:}\star.{}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({c^*}(t^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle_{\mathrm{e}}^{-}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle_{\mathrm{e}}^{+})\\
\alpha^* &= \alpha && \textit{(types)}\\
({}^{\xi}\overline{t})^* &= {}^{\xi}\overline{t}^*\\
\bullet^* &= \bullet && \textit{(type contexts)}\\
(\mathrm{G}, x{:}t)^* &= \mathrm{G}^*, x{:}t^*
\end{aligned}
$$

Figure 14: CCoS translation (ii): type-level terms and contexts

- Each adds an extra universal quantifier in front of its result, which is used to make (type) abstractions polymorphic in their answer types.
- Because the effect of an expression limits how it may use its continuation, the translation c^* of latent effect c becomes the qualifier of the continuation.
- All other qualifiers of the translated pretype are L. (This is because the translation never needs to duplicate partially-applied continuations, so L is a sufficiently permissive qualifier for those continuations. Furthermore, because the type rules for abstractions always allow a qualifier of L, using L wherever possible simplifies the proof.)

Translation of values and expressions is defined by mutual induction in Figure 15. Value translation (v^*) is mostly straightforward. Both value and type abstraction have an additional type abstraction added to the front, which matches the addition of the universal quantifier in the type translation, and both translate the body according to the expression translation $[\![e]\!]_{\mathcal{C}}$. The expression translation is standard except for two unusual aspects of the translation of applications and type applications:

- The result of evaluating e_1, bound to x_1, is in each case instantiated by a type application, which compensates for the new type abstraction in the translation of abstractions. For the type application case, $x_1\,\_$ is instantiated then again, corresponding to the instantiation from the source expression.
- Curiously, the continuation y is η-expanded to $\lambda x.\, y\, x$. While η-expanding a variable may seem useless, it is actually necessary to obtain a type-preserving translation.

In particular, the reason for this η expansion is to handle effect subsumption. Effects in $\lambda^{\mathsf{URAL}}(\mathcal{C})$ are translated to qualifiers in λ^{URAL}, and while $\lambda^{\mathsf{URAL}}(\mathcal{C})$ supports effect subsumption directly, there is no analogous qualifier subsumption in λ^{URAL}. However, qualifier subsumption for function types can be done explicitly using η expansion:

Lemma 5.2 (Dereliction)**.**
If $\Delta; \Gamma \vdash v : {}^{\xi}(\tau_1 \multimap \tau_2)$ *and* $\Delta \vdash \xi \preceq \xi'$ *then* $\Delta; \Gamma \vdash \lambda x.\, v\, x : {}^{\xi'}(\tau_1 \multimap \tau_2)$.

The proof of Lemma 5.2 relies on another lemma:

Lemma 5.3 (Value strengthening)**.**
Any qualifier that upper bounds the type of a value also bounds the portion of the type context necessary for typing that value. That is, if $\Delta; \Gamma \vdash v : \tau$ *and* $\Delta \vdash \tau \preceq \xi$ *then there exist some* Γ_1 *and* Γ_2 *such that*

- $\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2$,
- $\Delta; \Gamma_1 \vdash v : \tau$,

- $\Delta \vdash \Gamma_1 \preceq \xi$, *and*
- $\Delta \vdash \Gamma_2 \preceq \mathsf{A}$.

Proof. See p. 48. ▷

Proof of Lemma 5.2. Choose type contexts Γ_1 and Γ_2 according to Lemma 5.3. Then $\Delta; \Gamma_1, x{:}\tau_1 \vdash v\, x : \tau_2$ by rule T-App. By induction on the length of Γ_1 and transitivity of qualifier subsumption, we know that $\Delta \vdash \Gamma_1 \preceq \xi'$. Then by rule T-Abs, $\Delta; \Gamma_1 \vdash \lambda x.\, v\, x : {}^{\xi'}(\tau_1 \multimap \tau_2)$, and we change Γ_1 to Γ by rule T-Weak.

See p. 51 for additional details. ▷

Operational semantics of $\lambda^{\mathsf{URAL}}(\mathcal{C})$. Having defined the translation, we run a program e by applying the CCoS translation and passing it the initial continuation $\mathsf{done}_{\mathcal{C}}$. We define the operational semantics of $\lambda^{\mathsf{URAL}}(\mathcal{C})$ as a partial function *eval* : *Expressions* $\rightharpoonup$ *Values* $\cup$ {Wrong}:

$$eval(e) = \begin{cases} v & \text{if } [\![e]\!]_{\mathcal{C}}\ \mathsf{done}_{\mathcal{C}} \longmapsto^{*} v; \\ \textsc{Wrong} & \text{if } [\![e]\!]_{\mathcal{C}}\ \mathsf{done}_{\mathcal{C}} \longmapsto^{*} e' \\ & \quad \text{such that } e' \text{ is not a value} \\ & \quad \text{and } \neg\exists e''.e' \longmapsto e''. \end{cases}$$

5.3 Parameter Properties

Having defined the CCoS translation, we are now ready to state the additional properties that the abstract control effect parameter must satisfy for the generic type safety theorem (§5.4) to hold:

Parameter Property 1 (Answer types)**.**

1. *For all* τ, $\langle\!\langle \tau, \bot_{\mathcal{C}} \rangle\!\rangle^{-}_{\mathcal{C}} = \langle\!\langle \tau, \bot_{\mathcal{C}} \rangle\!\rangle^{+}_{\mathcal{C}}$.

   Rationale. For pure expressions, the negative and positive answer types agree, because a pure expression finishes by calling its continuation. Henceforth, we are justified defining the *pure answer type* $\langle\!\langle \tau \rangle\!\rangle_{\mathcal{C}} \triangleq \langle\!\langle \tau, \bot_{\mathcal{C}} \rangle\!\rangle^{+}_{\mathcal{C}}$.

2. *If* $\mathrm{D}^* \vdash \tau : \star$ *and* $\mathrm{D} \vdash_{\mathcal{C}} c : \mathsf{CTL}$ *then* $\mathrm{D}^* \vdash \langle\!\langle \tau, c \rangle\!\rangle^{-}_{\mathcal{C}} : \star$ *and* $\mathrm{D}^* \vdash \langle\!\langle \tau, c \rangle\!\rangle^{+}_{\mathcal{C}} : \star$.

   Rationale. For the translation to be well typed, well-kinded types and effects must become well-kinded answer types.

3. *For all* D, τ, $c_1 \neq \bot_{\mathcal{C}}$, *and* $c_2 \neq \bot_{\mathcal{C}}$ *such that* $\mathrm{D} \vdash_{\mathcal{C}} c_1 \oslash c_2 : \mathsf{CTL}$,

   (a) $\langle\!\langle \tau, c_1 \oslash c_2 \rangle\!\rangle^{-}_{\mathcal{C}} = \langle\!\langle \tau, c_2 \rangle\!\rangle^{-}_{\mathcal{C}}$,

   (b) $\langle\!\langle \tau, c_1 \oslash c_2 \rangle\!\rangle^{+}_{\mathcal{C}} = \langle\!\langle \tau, c_1 \rangle\!\rangle^{+}_{\mathcal{C}}$, *and*

$$
\begin{aligned}
x^* &= x && \textit{(values)}\\
(\lambda x.e)^* &= \Lambda.\lambda x.\ [\![e]\!]_{\mathfrak{e}}\\
(\Lambda.e)^* &= \Lambda.\Lambda.\ [\![e]\!]_{\mathfrak{e}}\\
(\mathsf{inl}\ v)^* &= \mathsf{inl}\ v^*\\
(\mathsf{inr}\ v)^* &= \mathsf{inr}\ v^*\\
[v_1, v_2]^* &= \Lambda.[\lambda x.{v_1}^*\ \_\ x, \lambda x.{v_2}^*\ \_\ x]\\
\langle v_1, v_2\rangle^* &= \langle {v_1}^*, {v_2}^*\rangle\\
(\mathsf{uncurry}\ v)^* &= \Lambda.\ \mathsf{uncurry}\,(\lambda x_1.\lambda x_2.\ [\![v\,x_1\,x_2]\!]_{\mathfrak{e}})\\
\langle\rangle^* &= \langle\rangle\\
(\mathsf{ignore}\ v)^* &= \Lambda.\lambda x.\ \mathsf{ignore}\ v^*\ [\![x]\!]_{\mathfrak{e}}
\end{aligned}
$$

$$
\begin{aligned}
[\![v]\!]_{\mathfrak{e}} &= \lambda y.\, y\, v^* && \textit{(expressions)}\\
[\![e_1\, e_2]\!]_{\mathfrak{e}} &= \lambda y.\ [\![e_1]\!]_{\mathfrak{e}}\ (\lambda x_1.\ [\![e_2]\!]_{\mathfrak{e}}\ (\lambda x_2.x_1\ \_\ x_2\,(\lambda x.y\,x)))\\
[\![e\ \_]\!]_{\mathfrak{e}} &= \lambda y.\ [\![e]\!]_{\mathfrak{e}}\ (\lambda x_1.x_1\ \_\ \_\,(\lambda x.y\,x))\\
[\![\mathsf{new}^{\mathsf{q}}\, e]\!]_{\mathfrak{e}} &= \lambda y.\ [\![e]\!]_{\mathfrak{e}}\ (\lambda x.y\,(\mathsf{new}^{\mathsf{q}}\,x))\\
[\![\mathsf{read}\ e]\!]_{\mathfrak{e}} &= \lambda y.\ [\![e]\!]_{\mathfrak{e}}\ (\lambda x.y\,(\mathsf{read}\ x))\\
[\![\mathsf{free}\ e]\!]_{\mathfrak{e}} &= \lambda y.\ [\![e]\!]_{\mathfrak{e}}\ (\lambda x.y\,(\mathsf{free}\ x))\\
[\![\mathsf{swap}\ e_1\, e_2]\!]_{\mathfrak{e}} &= \lambda y.\ [\![e_1]\!]_{\mathfrak{e}}\ (\lambda x_1.\ [\![e_2]\!]_{\mathfrak{e}}\ (\lambda x_2.y\,(\mathsf{swap}\ x_1\,x_2)))
\end{aligned}
$$

Figure 15: CCoS translation (iii): values and expressions

(c) $\langle\!\langle \tau, c_1 \rangle\!\rangle^-_{\mathfrak{e}} = \langle\!\langle \tau, c_2 \rangle\!\rangle^+_{\mathfrak{e}}$.

RATIONALE. Effect sequencing must maintain answer types in order for the continuations of sequenced expressions to compose.

4. *If* $\mathrm{D} \vdash_{\mathfrak{e}} c_1 \preceq c_2$, *then for every type* τ *there exists some type* τ' *such that* $\langle\!\langle \tau', c_1 \rangle\!\rangle^-_{\mathfrak{e}} = \langle\!\langle \tau, c_2 \rangle\!\rangle^-_{\mathfrak{e}}$ *and* $\langle\!\langle \tau', c_1 \rangle\!\rangle^+_{\mathfrak{e}} = \langle\!\langle \tau, c_2 \rangle\!\rangle^+_{\mathfrak{e}}$.

RATIONALE. For control effect subsumption to be valid, related control effects must generate related answer types.

Parameter Property 2 (Done)**.**
If $\Delta \vdash \tau \preceq \mathsf{A}$ *then* $\Delta; \bullet \vdash \mathsf{done}_{\mathfrak{e}} : {}^{\mathsf{L}}(\tau \multimap \langle\!\langle \tau \rangle\!\rangle_{\mathfrak{e}})$.

RATIONALE. The $\mathsf{done}_{\mathfrak{e}}$ value must be well typed for the translation of a whole program to be well typed.

Parameter Property 3 (Effect sequencing)**.**
If $\mathrm{D} \vdash_{\mathfrak{e}} c_1 \mathbin{⊙} c_2 : \mathsf{CTL}$ *then* $\mathrm{D}^* \vdash (c_1 \mathbin{⊙} c_2)^* \preceq {c_1}^*$ *and* $\mathrm{D}^* \vdash (c_1 \mathbin{⊙} c_2)^* \preceq {c_2}^*$.

RATIONALE. Sequencing lowers the translation of control effects in the qualifier order. This makes sense, because if either of two sequenced expressions may duplicate or discard their continuations, then the compound expression may do the same.

Parameter Property 4 (Bottom and lifting)**.**

1. $c_1 \mathbin{⊙} c_2 = \bot_{\mathfrak{e}}$ *if and only if* $c_1 = c_2 = \bot_{\mathfrak{e}}$.

   RATIONALE. Sequencing impure expressions should not result in a pure expression.

2. *If* $\mathrm{D} \vdash_{\mathfrak{e}} c_1 \mathbin{⊙} c_2 : \mathsf{CTL}$ *and* $c_1 \mathbin{⊙} c_2 \neq \bot_{\mathfrak{e}}$, *then there exist some* $c_1' \neq \bot_{\mathfrak{e}}$ *and* $c_2' \neq \bot_{\mathfrak{e}}$ *such that*

   - $\mathrm{D} \vdash_{\mathfrak{e}} c_1 \preceq c_1'$,
   - $\mathrm{D} \vdash_{\mathfrak{e}} c_2 \preceq c_2'$,
   - $c_1' \mathbin{⊙} c_2' = c_1 \mathbin{⊙} c_2$, *and*
   - $\mathrm{D} \vdash_{\mathfrak{e}} c_1' \mathbin{⊙} c_2' : \mathsf{CTL}$.

   RATIONALE. This assumption is likely not necessary, but it significantly simplifies the proof by allowing the effects in a sequence to be considered either all pure or all impure.

The final property concerns four lemmas that we state and prove for the generic system in the next subsection. An actual control effect instance needs to extend these lemmas to cover any additional rules added to the relevant judgments:

Parameter Property 5 (New rules)**.**

1. *Lemma 5.4 (§5.4) must be extended, by induction on derivations, for any rules added to the kinding judgment* $\mathrm{D} \vdash_{\mathfrak{e}} i : k$.
2. *Lemma 5.5 (§5.4) must be extended, by induction on derivations, for any rules added to the control effect bounding judgment* $\mathrm{D} \vdash_{\mathfrak{e}} c \succeq \xi$.
3. *Lemma 5.6 (§5.4) must be extended, by induction on derivations, for any rules added to the control effect subsumption judgment* $\mathrm{D} \vdash_{\mathfrak{e}} c_1 \preceq c_2$.
4. *Lemma 5.7 (§5.4) must be extended, by induction on derivations, for any rules added to the expression typing judgment* $\mathrm{D}; \mathrm{G} \vdash_{\mathfrak{e}} e : t \,;\, c$.

In §6, we give several example control effects and show that they satisfy the above properties.

5.4 Generic Type Safety

Assuming that the above properties hold of the control effect parameter, we can now prove a type safety theorem for $\lambda^{\mathsf{URAL}}(\mathfrak{C})$ that leaves the control effect abstract. We sketch the proof here, but the full proof is available in the appendices.

We begin with a lemma that ensures that control effects translate to well-formed qualifiers:

Lemma 5.4 (Translation of kinding)**.**
For all D, i, *and* k, *if* $\mathrm{D} \vdash_{\mathfrak{e}} i : k$ *then* $\mathrm{D}^* \vdash i^* : k^*$.

Proof. See p. 52. ▷

We continue with two lemmas concerning how the translation of control effects to qualifiers relates to qualifier subsumption. The former ensures that the control effect bound used by typing rules such as C-T-App matches the qualifier assigned to the type of a continuation by the CCoS translation. The latter shows that a larger control effect, which indicates more liberal treatment of a continuation, maps to a smaller qualifier, which indicates more liberal treatment of any value.

Lemma 5.5 (Translation of effect bounds)**.**
If $\mathrm{D} \vdash_{\mathfrak{e}} c \succeq \xi$ *then* $\mathrm{D}^* \vdash \xi \preceq c^*$.

Proof. See p. 55. ▷

Lemma 5.6 (Translation of effect subsumption)**.**
If $\mathrm{D} \vdash_{\mathfrak{e}} c_1 \preceq c_2$ *then* $\mathrm{D}^* \vdash {c_2}^* \preceq {c_1}^*$.

Proof. See p. 55. ▷

The most difficult lemma, and the heart of the proof, is about typing translated expressions. Given a $\lambda^{\mathsf{URAL}}(\mathcal{C})$ expression whose control effect is c, the translation of the control effect, c^*, is the qualifier of the continuation of the translated expression:

Lemma 5.7 (Translation of term typing)**.**
If $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$ *then*

$$\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{c^*}(t^* \multimap \langle\!\langle t^*, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle t^*, c \rangle\!\rangle^+_{\mathcal{C}}).$$

Proof. By induction on the typing derivation, generalizing the induction hypothesis thus:

> If $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$, then for all τ_0 such that $\mathrm{D}^* \vdash \tau_0 : \star$, and for all ξ_0 such that $\mathrm{D}^* \vdash \xi_0 \preceq c^*$, we have $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap \langle\!\langle \tau_0, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau_0, c \rangle\!\rangle^+_{\mathcal{C}})$.

We consider two cases here:

Case $\dfrac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c' \qquad \mathrm{D} \vdash_{\mathcal{C}} c' \preceq c}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c}$.

> By Property 5 (part 3), $\mathrm{D}^* \vdash c^* \preceq c'^*$, and thus by Property 1 (part 4), there exists some type τ_0' such that $\langle\!\langle \tau_0', c' \rangle\!\rangle^-_{\mathcal{C}} = \langle\!\langle \tau_0, c \rangle\!\rangle^-_{\mathcal{C}}$ and $\langle\!\langle \tau_0', c' \rangle\!\rangle^+_{\mathcal{C}} = \langle\!\langle \tau_0, c \rangle\!\rangle^+_{\mathcal{C}}$. By the lemma assumption, $\mathrm{D}^* \vdash \xi_0 \preceq c^*$, and by transitivity of qualifier subsumption, $\mathrm{D}^* \vdash \xi_0 \preceq c'^*$. Thus, we can apply the induction hypothesis at $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c'$, using the same ξ_0 but with τ_0' for τ_0, yielding
>
> $$\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap \langle\!\langle \tau_0', c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau_0', c' \rangle\!\rangle^+_{\mathcal{C}}).$$
>
> Then it suffices to substitute $\langle\!\langle \tau_0, c \rangle\!\rangle^-_{\mathcal{C}}$ for $\langle\!\langle \tau_0', c' \rangle\!\rangle^-_{\mathcal{C}}$ and $\langle\!\langle \tau_0, c \rangle\!\rangle^+_{\mathcal{C}}$ for $\langle\!\langle \tau_0', c' \rangle\!\rangle^+_{\mathcal{C}}$, which we know to be equal by Property 1 (part 4).

Case $\dfrac{\begin{array}{c}\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \xi_2 \\ \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e_1 : {}^{\xi_1}(t_1 \xrightarrow{c} t_2) \,;\, c_1 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1 \succeq \xi_2 \\ \mathrm{D}; \mathrm{G}_2 \vdash_{\mathcal{C}} e_2 : t_1 \,;\, c_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_2 \succeq \xi_1 \\ \mathrm{D} \vdash_{\mathcal{C}} c_1 \oslash c_2 \oslash c : \mathsf{CTL}\end{array}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e_1\, e_2 : t_2 \,;\, c_1 \oslash c_2 \oslash c}$.

> For rule C-T-App, we want to show that $[\![e_1\, e_2]\!]_{\mathcal{C}}$ has type
>
> $${}^{\mathsf{L}}(^{\xi_0}({t_2}^* \multimap \langle\!\langle \tau_0, c_1 \oslash c_2 \oslash c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau_0, c_1 \oslash c_2 \oslash c \rangle\!\rangle^+_{\mathcal{C}}).$$
>
> Consider the translation of $e_1\, e_2$,
>
> $$\lambda y.\, [\![e_1]\!]_{\mathcal{C}}\, (\lambda x_1.\, [\![e_2]\!]_{\mathcal{C}}\, (\lambda x_2.\, x_1 \_ x_2\, (\lambda x.\, y\, x))).$$
>
> The type derivation is too large to show here in detail, but it hinges on giving the right qualifiers to the types of continuations. We will consider the continuation

passed to the whole expression and the continuations constructed for e_1, e_2, and the function application itself, in turn.

First we consider y, the continuation of the whole application expression. Given the type that we need to derive for the whole expression, the qualifier of y's type must be ξ_0. Furthermore, from the assumptions of the lemma, we know that $\mathrm{D}^* \vdash \xi_0 \preceq (c_1 \otimes c_2 \otimes c)^*$. By Property 3, each of ${c_1}^*$, ${c_2}^*$, and c^* is greater than $(c_1 \otimes c_2 \otimes c)^*$, so by transitivity, ξ_0 is less than each of these.

Expression e_1 has effect c_1, so by the induction hypothesis, its continuation may have qualifier ${c_1}^*$. The continuation passed to $[\![e_1]\!]_{\mathrm{e}}$ is

$$\lambda x_1.\, [\![e_2]\!]_{\mathrm{e}}\, (\lambda x_2.\, x_1\, \_\, x_2\, (\lambda x.\, y\, x)),$$

whose free variables are $\{y\} \cup \mathrm{fv}(e_2)$. Thus, the qualifier of this function must upper bound both ξ_0 and the qualifiers of the types in G_2 (the type context for e_2). We have $\mathrm{D}^* \vdash \xi_0 \preceq {c_1}^*$ from the previous paragraph. Furthermore, looking at the premises of rule T-App, we see that ξ_2 upper bounds the types in G_2 and is less than ${c_1}^*$ (by Property 5 (part 2)), so by transitivity, $\mathrm{D}^* \vdash {\mathrm{G}_2}^* \preceq {c_1}^*$, as desired.

Expression e_2 has effect c_2, so similarly, its continuation should have qualifier ${c_2}^*$. The free variables of e_2's continuation are only y and x_1, which is the value of e_1. We handle y as before. The type of x_1 is ${}^{\xi_1}((t_1 \xrightarrow{c}\!\!\circ\ t_2)^*)$, so it remains to show that $\mathrm{D}^* \vdash \xi_1 \preceq {c_2}^*$, by Property 5 (part 2) applied to the premise $\mathrm{D} \vdash_{\mathrm{e}} c_2 \succeq \xi_1$.

Finally, given that x_1 has type ${}^{\xi_1}((t_1 \xrightarrow{c}\!\!\circ\ t_2)^*)$, it expects a continuation whose qualifier is c^*. The type of y has qualifier ξ_0, which is less than c^*. Then by Lemma 5.2 (Dereliction), the type of the η expansion $\lambda x.\, y\, x$ may be given qualifier c^*.

See p. 56 for the remaining cases. ▷

Corollary 5.8 (Translation of program typing)**.**
If $\mathrm{D}; \mathrm{G} \vdash_{\mathrm{e}} e : t \,;\, \bot_{\mathrm{e}}$ *where* $\mathrm{D} \vdash_{\mathrm{e}} t \preceq \mathsf{A}$*, then*

$$\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathrm{e}}\, \mathsf{done}_{\mathrm{e}} : \langle\!\langle t^* \rangle\!\rangle_{\mathrm{e}}.$$

Proof. By Lemma 5.4, Lemma 5.7, Property 2, and rules QSub-Refl and T-App. See p. 74 for details. ▷

Lemma 5.9 (λ^{URAL} safety)**.**
If $\bullet; \bullet \vdash e_1 : \tau$ *and* $e_1 \overset{*}{\longmapsto} e_2$*, then either* $\exists v_2.e_2 \equiv v_2$ *or* $\exists e_3.e_2 \longmapsto e_3$.

Proof. See the proof in Ahmed et al. (2005). □

Theorem 5.10 ($\lambda^{\mathsf{URAL}}(\mathcal{C})$ safety)**.**
If $\bullet;\bullet \vdash_{\mathcal{C}} e : t \,;\, \bot_{\mathcal{C}}$*, and* $\bullet \vdash_{\mathcal{C}} t \preceq \mathsf{A}$ *then* $\mathit{eval}(e) \neq \textsc{Wrong}$.

Proof. By Corollary 5.8, $\bullet;\bullet \vdash [\![e]\!]_{\mathcal{C}}\ \mathsf{done}_{\mathcal{C}} : \langle\!\langle t^* \rangle\!\rangle_{\mathcal{C}}$. Then by Lemma 5.9, either $[\![e]\!]_{\mathcal{C}}\ \mathsf{done}_{\mathcal{C}}$ reduces to a value v, in which case $\mathit{eval}(e) = v$, or $[\![e]\!]_{\mathcal{C}}\ \mathsf{done}_{\mathcal{C}}$ diverges, in which case $\mathit{eval}(e)$ is undefined. □

6 Example Control Effects

In the previous section, we proved type safety for $\lambda^{\mathsf{URAL}}(\mathcal{C})$, a substructural λ calculus parameterized by abstract control effects. In this section, we give three instances of control effects as described by Definition 4.1 and show that they satisfy the properties on which the generic type safety theorem depends.

It will be useful, when stating several later definitions, to have a definition for meets and joins of qualifiers.

Definition 6.1 (Qualifier meets and joins)**.**
We define meets and joins of qualifiers as follows:

$$\mathsf{L} \sqcap \xi = \xi \sqcap \mathsf{L} = \xi \sqcap \xi = \xi \qquad \mathsf{U} \sqcup \xi = \xi \sqcup \mathsf{U} = \xi \sqcup \xi = \xi$$
$$\mathsf{U} \sqcap \xi = \xi \sqcap \mathsf{U} = \mathsf{A} \sqcap \mathsf{R} = \mathsf{R} \sqcap \mathsf{A} = \mathsf{U} \qquad \mathsf{L} \sqcup \xi = \xi \sqcup \mathsf{L} = \mathsf{A} \sqcup \mathsf{R} = \mathsf{R} \sqcup \mathsf{A} = \mathsf{L}$$
otherwise, $\xi \sqcap \xi'$ *is undefined* *otherwise,* $\xi \sqcup \xi'$ *is undefined*

6.1 Shift and Reset

We define here a control effect instance for delimited continuations. In this example, we restrict answer types to the unit type $^{\mathsf{U}}1$ in order to keep the effects simple. In §6.2, we show how to define a more general control effect instance that allows answer-type modification.

We add shift and reset to $\lambda^{\mathsf{URAL}}(\mathcal{C})$ as follows. First, we extend the syntax:

e	::=		new expressions
	|	$\cdots$	*extending syntax from Figure 1*
	|	$\mathsf{reset}\ e$	delimiter
	|	$\mathsf{shift}\ x\ \mathsf{in}\ e$	control operator

We give the dynamics of the new expressions by defining their CCoS translations, which are standard:

$$[\![\mathsf{reset}\ e]\!]_{\mathcal{D}} = \lambda y.\, y\, ([\![e]\!]_{\mathcal{D}}\ (\lambda x.\, x))$$
$$[\![\mathsf{shift}\ x\ \mathsf{in}\ e]\!]_{\mathcal{D}} = \lambda y.\, (\lambda x.\ [\![e]\!]_{\mathcal{D}}\ (\lambda x'.\, x'))\, (\Lambda.\, \lambda x.\, \lambda y'.\, y'\, (y\, x))$$

To type shift and reset, we define delimited continuation effects d as the dual lattice of the qualifier lattice ξ with a new point $\bot_{\mathcal{D}}$:

$\boxed{\mathrm{D} \vdash_{\mathcal{D}} i : k}$ *(kinding delimited control effects)*

$$\textsc{D-K-Qual}\ \frac{\mathrm{D} \vdash_{\mathcal{D}} \xi : \mathsf{QUAL}}{\mathrm{D} \vdash_{\mathcal{D}} \overline{\xi} : \mathsf{CTL}} \qquad \textsc{D-K-Join}\ \frac{\mathrm{D} \vdash_{\mathcal{D}} d_1 : \mathsf{CTL} \qquad \mathrm{D} \vdash_{\mathcal{D}} d_2 : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 : \mathsf{CTL}}$$

$\boxed{\mathrm{D} \vdash_{\mathcal{D}} d \succeq \xi}$ *(qualifier bound for delimited control effects)*

$$\textsc{D-B-Qual}\ \frac{\mathrm{D} \vdash_{\mathcal{D}} \xi \preceq \xi'}{\mathrm{D} \vdash_{\mathcal{D}} \overline{\xi'} \succeq \xi} \qquad \textsc{D-B-Join}\ \frac{\mathrm{D} \vdash_{\mathcal{D}} d_1 \succeq \xi \qquad \mathrm{D} \vdash_{\mathcal{D}} d_2 \succeq \xi}{\mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 \succeq \xi}$$

$\boxed{\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_2}$ *(delimited control effect subsumption)*

$$\textsc{DSub-Bot}\ \frac{\mathrm{D} \vdash_{\mathcal{D}} d : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{D}} \bot_{\mathcal{D}} \preceq d} \qquad \textsc{DSub-Lin}\ \frac{\mathrm{D} \vdash_{\mathcal{D}} \xi : \mathsf{QUAL}}{\mathrm{D} \vdash_{\mathcal{D}} \overline{\mathsf{L}} \preceq \overline{\xi}} \qquad \textsc{DSub-Top}\ \frac{\mathrm{D} \vdash_{\mathcal{D}} d : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{D}} d \preceq \overline{\mathsf{U}}}$$

$$\textsc{DSub-Join}\ \frac{\begin{array}{cc}\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_1' & \mathrm{D} \vdash_{\mathcal{D}} d_2 \preceq d_2' \\ \mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 : \mathsf{CTL} & \mathrm{D} \vdash_{\mathcal{D}} d_1' \sqcup d_2' : \mathsf{CTL}\end{array}}{\mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 \preceq d_1' \sqcup d_2'}$$

$\boxed{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{D}} e : t \,;\, d}$ *(delimited control expression typing)*

$$\textsc{D-T-Reset}\ \frac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{D}} e : {}^{\mathsf{U}}1 \,;\, d}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{D}} \mathsf{reset}\ e : {}^{\mathsf{U}}1 \,;\, \bot_{\mathcal{D}}} \qquad \textsc{D-T-Shift}\ \frac{\mathrm{D}; \mathrm{G}, x{:}^{\xi}(t \xrightarrow{\bot_{\mathcal{D}}} {}^{\mathsf{U}}1) \vdash_{\mathcal{D}} e : {}^{\mathsf{U}}1 \,;\, d}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{D}} \mathsf{shift}\, x\, \mathsf{in}\, e : t \,;\, d \sqcup \overline{\xi}}$$

Figure 16: Statics for delimited continuation effects

d	::=		delimited continuation effects
		$\bot_{\mathcal{D}}$	no effect
	|	α	an effect variable
	|	$\overline{\xi}$	treats continuation like ξ
	|	$d_1 \sqcup d_2$	effect join

Let $\mathcal{D}$ be the set of delimited continuation effects (d) quotiented by the following equivalences:

$$\overline{\xi_1} \sqcup \overline{\xi_2} = \overline{(\xi_1 \sqcap \xi_2)} \text{ when } \xi_1 \sqcap \xi_2 \text{ is defined;}$$
$$d \sqcup \bot_{\mathcal{D}} = \bot_{\mathcal{D}} \sqcup d = d \sqcup d = d.$$

(The quotient simplifies defining other functions and relations on delimited continuation effects.) Then we define delimited continuation effects as the triple $(\mathcal{D}, \bot_{\mathcal{D}}, \sqcup)$.

We extend the type system of $\lambda^{\mathsf{URAL}}(\mathcal{C})$ with the new rules in Figure 16. The new kinding rules say that qualifiers-as-effects ($\overline{\xi}$) and joins ($d_1 \sqcup d_2$) are well-kinded if their components are. The new control effect bound rules say that a control effect $\overline{\xi'}$ is bounded by all qualifiers ξ that are less than ξ' and that any bound of both effects in a join bounds the join as well. The rules added for effect subsumption effectively axiomatize the delimited continuation effect lattice. Finally, we add two rules for typing shift and reset. To type an expression $\mathsf{reset}\ e$, subexpression e may have any effect whatsoever, but must return type $^{\mathsf{U}}1$. (We lift this restriction in §6.2.) Then $\mathsf{reset}\ e$ is pure and also has type $^{\mathsf{U}}1$. To type $\mathsf{shift}\, x\, \mathsf{in}\, e$, we give x type $^{\xi}(t \xrightarrow{\bot_{\mathcal{D}}}\circ\ ^{\mathsf{U}}1)$ for checking e, where $\overline{\xi}$ is joined with the effect of e to get the effect of the whole shift expression. That is, because shift captures its continuation and gives the reified continuation qualifier ξ, its effect must be at least $\overline{\xi}$, since that qualifier determines how it might treat its captured continuation.

Type safety. To prove type safety for $\lambda^{\mathsf{URAL}}(\mathcal{C})$ extended with delimited continuation effects, we need to give the translation parameter as described by Definition 5.1. We define the translation parameter as follows:

$$\langle\!\langle \tau, d \rangle\!\rangle^{-}_{\mathcal{D}} = \langle\!\langle \tau, d \rangle\!\rangle^{+}_{\mathcal{D}} = {}^{\mathsf{U}}1$$
$$\mathsf{done}_{\mathcal{D}} = \lambda x.\, \langle\rangle$$
$$d^* = \begin{cases} \mathsf{L} & \text{if } d = \bot_{\mathcal{D}} \\ \alpha & \text{if } d = \alpha \\ \xi & \text{if } d = \overline{\xi} \\ \mathsf{U} & \text{otherwise} \end{cases}$$

Then, we must show that this definition satisfies the properties of §5.3:

Theorem 6.2 (Delimited continuation properties)**.**
Delimited continuation effects $(\mathcal{D}, \bot_{\mathcal{D}}, \sqcup)$ *satisfy Properties 1–5.*

Proof.

Property 1 (Answer types). We must show several equalities on answer types, such as $\langle\!\langle \tau, d_1 \rangle\!\rangle^-_{\mathcal{D}} = \langle\!\langle \tau, d_2 \rangle\!\rangle^+_{\mathcal{D}}$, hold whenever $d_1 \sqcup d_2$ is well formed. All of the equalities are trivial because $\langle\!\langle \tau, d \rangle\!\rangle^-_{\mathcal{D}} = \langle\!\langle \tau, d \rangle\!\rangle^+_{\mathcal{D}} = {}^{\cup}1$.

Property 2 (Done). We need to show that $\Delta; \bullet \vdash \mathsf{done}_{\mathcal{D}} : {}^{\mathsf{L}}(\tau \multimap \langle\!\langle \tau \rangle\!\rangle_{\mathcal{D}})$. Given the definition of $\mathsf{done}_{\mathcal{D}}$, we can show $\Delta; \bullet \vdash \lambda x.\langle\rangle : {}^{\mathsf{L}}(\tau \multimap \langle\!\langle \tau \rangle\!\rangle_{\mathcal{D}})$ by a straightforward type derivation.

Property 3 (Effect sequencing). We need to show that $\mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 : \mathsf{CTL}$ implies that $\mathrm{D}^* \vdash (d_1 \sqcup d_2)^* \preceq {d_1}^*$ and $\mathrm{D}^* \vdash (d_1 \sqcup d_2)^* \preceq {d_2}^*$. By symmetry, it suffices to show the former:

(1) $\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_1$	by CSub-Refl
(2) $\mathrm{D} \vdash_{\mathcal{D}} \bot_{\mathcal{D}} \preceq d_2$	by DSub-Bot
(3) $\mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup \bot_{\mathcal{D}} \preceq d_1 \sqcup d_2$	by (1–2), DSub-Join
(4) $\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_1 \sqcup d_2$	by (3), $d_1 \sqcup \bot_{\mathcal{D}} = d_1$
(5) $\mathrm{D}^* \vdash (d_1 \sqcup d_2)^* \preceq {d_1}^*$	by (4), Lemma 5.6.

Property 4 (Bottom and lifting).

1. To show that $d_1 \sqcup d_2 = \bot_{\mathcal{D}}$ if and only if $d_1 = d_2 = \bot_{\mathcal{D}}$, we consider the quotienting of $\mathcal{D}$.
2. We must also show that if $\mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 : \mathsf{CTL}$ and $d_1 \sqcup d_2 \neq \bot_{\mathcal{D}}$, then there exist some $d'_1 \neq \bot_{\mathcal{D}}$ and $d'_2 \neq \bot_{\mathcal{D}}$ with particular properties. For each d_i $^{(i \in \{1,2\})}$, if $d_i = \bot_{\mathcal{D}}$ then let $d'_i = \overline{\mathsf{L}}$; otherwise, let $d'_i = d_i$. This ensures that 1–2) each $\mathrm{D} \vdash_{\mathcal{D}} d_i \preceq d'_i$, 3) $d_1 \sqcup d_2 = d'_1 \sqcup d'_2$, and 4) $d'_1 \sqcup d'_2$ is well formed.

Property 5 (New rules).

1. We show that $\mathrm{D} \vdash_{\mathcal{D}} d \succeq \xi$ implies that $\mathrm{D}^* \vdash \xi \preceq d^*$, by induction on the derivation. The only new cases to consider are for rules D-B-Qual and D-B-Join. These require a lemma about the translation of qualifier subsumption derivations.
2. We show that $\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_2$ implies that $\mathrm{D}^* \vdash {d_2}^* \preceq {d_1}^*$, again by induction on the derivation. The only nontrivial case is when

$$\frac{\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d'_1 \qquad \mathrm{D} \vdash_{\mathcal{D}} d_2 \preceq d'_2 \qquad \mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 : \mathsf{CTL} \qquad \mathrm{D} \vdash_{\mathcal{D}} d'_1 \sqcup d'_2 : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 \preceq d'_1 \sqcup d'_2}.$$

   We show that $\mathrm{D}^* \vdash (d'_1 \sqcup d'_2)^* \preceq (d_1 \sqcup d_2)^*$ by exhaustively enumerating the possibilities for d_1, d_2, d'_1, and d'_2 such that the premises hold.

3. For translation of kinding, we show that $\mathrm{D} \vdash_{\mathsf{c}} d : \mathsf{CTL}$ implies that $\mathrm{D}^* \vdash d^* : \mathsf{QUAL}$. We proceed, as usual, by a simple induction on the derivation, considering the two new kinding rules for delimited continuation effects.

4. For translation of typing, we use the generalized induction hypothesis as in the proof of Lemma 5.7. There are two cases, for shift and reset, each of which requires a large type derivation.

See p. 75 for additional details. ▷

6.2 Shift and Reset with Answer-Type Modification

The type-and-effect system for shift and reset described in §6.1 requires that all *answer types*—the type of all reset expressions—be $^{\cup}1$. Our second example adds answer-type modification (*à la* Danvy and Filinski 1989), which allows shift to capture and compose continuations of differing types and allows the answer delivered by reset to have any type. Both the syntax and CCoS translation are as in §6.1, but we change the definition of control effects as follows. An answer-type control effect a is either the pure effect $\bot_{\mathcal{A}}$ or a collection of qualifiers $\xi_1, \ldots, \xi_j$ along with old and new answer types t_1 and t_2:

a	::=		answer-type modification effects
	|	$\bot_{\mathcal{A}}$	pure
	|	$^{\Xi}(t_1 \rightarrowtail t_2)$	captures continuation
Ξ	::=	$\xi_1, \ldots, \xi_j$	qualifier collections

A type derivation $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{A}} e : t \; ; \; ^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2)$ may be understood as follows:

- The collection of qualifiers $\xi_1, \ldots, \xi_j$ keeps track of all the ways that expression e may treat its context; expression e may be considered to treat its context according to any qualifier ξ that lower bounds all of $\xi_1, \ldots, \xi_j$. We need a collection of qualifiers because qualifiers do not, in the presence of qualifier variables, have greatest lower bounds.

- Evaluated in a context expecting type t whose original answer type was t_1, expression e changes the answer type to t_2. This means that our type-and-effect judgment, disregarding substructural considerations, is equivalent to the type judgment that Danvy and Filinski write as $\Gamma, t_1 \vdash e : t, t_2$.

$\boxed{\mathrm{D} \vdash_{\mathcal{A}} i : k}$ *(kinding answer-type effects)*

A-K-Effect
$$\frac{\mathrm{D} \vdash_{\mathcal{A}} \xi_1 : \mathsf{QUAL} \quad \cdots \quad \mathrm{D} \vdash_{\mathcal{A}} \xi_k : \mathsf{QUAL} \qquad \mathrm{D} \vdash_{\mathcal{A}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{A}} t_2 : \star}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_k}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}$$

$\boxed{\mathrm{D} \vdash_{\mathcal{A}} a \succeq \xi}$ *(qualifier bound for answer-type effects)*

A-B-Qual
$$\frac{\mathrm{D} \vdash_{\mathcal{A}} \xi \preceq \xi_1 \quad \cdots \quad \mathrm{D} \vdash_{\mathcal{A}} \xi \preceq \xi_j \qquad \mathrm{D} \vdash_{\mathcal{A}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{A}} t_2 : \star}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2) \succeq \xi}$$

$\boxed{\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a_2}$ *(answer-type effect subsumption)*

ASub-Bot
$$\frac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi}(t \rightarrowtail t) : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} \bot_{\mathcal{A}} \preceq {}^{\Xi}(t \rightarrowtail t)}$$

ASub-L
$$\frac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\mathsf{L}}(t_1 \rightarrowtail t_2) \preceq {}^{\Xi}(t_1 \rightarrowtail t_2)}$$

ASub-Top
$$\frac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi}(t_1 \rightarrowtail t_2) \preceq {}^{\mathsf{U}}(t_1 \rightarrowtail t_2)}$$

ASub-Join
$$\frac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_1}(t_1 \rightarrowtail t_2) \preceq {}^{\Xi'_1}(t_1 \rightarrowtail t_2) \qquad \mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_2}(t_1 \rightarrowtail t_2) \preceq {}^{\Xi'_2}(t_1 \rightarrowtail t_2)}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_1,\Xi_2}(t_1 \rightarrowtail t_2) \preceq {}^{\Xi'_1,\Xi'_2}(t_1 \rightarrowtail t_2)}$$

$\boxed{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{A}} e : t \,;\, a}$ *(answer-type effect expression typing)*

A-T-Reset
$$\frac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{A}} e : t_0 \,;\, {}^{\Xi}(t_0 \rightarrowtail t)}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{A}} \mathsf{reset}\ e : t \,;\, \bot_{\mathcal{A}}}$$

A-T-Shift
$$\frac{\mathrm{D}; \mathrm{G}, x{:}^{\xi}(t_1 \xrightarrow{\bot_{\mathcal{A}}} t_2) \vdash_{\mathcal{A}} e : t_0 \,;\, {}^{\Xi}(t_0 \rightarrowtail t)}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{A}} \mathsf{shift}\ x\ \mathsf{in}\ e : t_1 \,;\, {}^{\Xi,\xi}(t_2 \rightarrowtail t)}$$

Figure 17: Statics for answer-type effects

For answer-type modification effects, we define the partial sequencing operation as follows:

$$\bot_{\mathcal{A}} \circ a = a$$
$$a \circ \bot_{\mathcal{A}} = a$$
$$^{\Xi}(t' \rightarrowtail t_2) \circ {}^{\Xi'}(t_1 \rightarrowtail t') = {}^{\Xi,\Xi'}(t_1 \rightarrowtail t_2).$$

Any other cases are undefined.

Collections of qualifiers are quotiented by the following equivalence:

$$\xi_1, \xi_2 = \xi_1 \sqcap \xi_2 \text{ when } \xi_1 \sqcap \xi_2 \text{ is defined.}$$

Then we define answer-type modification effects as the triple $(\mathcal{A}, \bot_{\mathcal{A}}, \circ)$.

The new type rules for answer-type effects appear in Figure 17. For the most part, these rules treat the collection of qualifiers $\xi_1, \ldots, \xi_j$ similarly to the delimited continuation effect $\overline{\xi_1} \sqcup \cdots \sqcup \overline{\xi_j}$ from §6.1. However, there is some subtlety to the definition of answer-type effect subsumption: the only non-bottom effects related by subsumption are those whose before and after answer types match, pairwise, but the pure effect $\bot_{\mathcal{A}}$ is less than any effect whose before and after answer types match *each other* (rule ASub-Bot). This makes sense, as pure expressions do not change the answer type.

The rules for typing shift and reset expressions are a hybrid of the rules from §6.1, which they follow for the qualifier portion, and the rules from Danvy and Filinski (1989), which they follow for maintaining answer types.

Type safety. To prove type safety for $\lambda^{\mathsf{URAL}}(\mathcal{C})$ extended with answer-type modification, we define the translation parameter as follows:

$$\langle\!\langle \tau, \bot_{\mathcal{A}} \rangle\!\rangle^{-}_{\mathcal{A}} = \tau$$
$$\langle\!\langle \tau, {}^{\Xi}(t_1 \rightarrowtail t_2) \rangle\!\rangle^{-}_{\mathcal{A}} = {t_1}^*$$
$$\langle\!\langle \tau, \bot_{\mathcal{A}} \rangle\!\rangle^{+}_{\mathcal{A}} = \tau$$
$$\langle\!\langle \tau, {}^{\Xi}(t_1 \rightarrowtail t_2) \rangle\!\rangle^{+}_{\mathcal{A}} = {t_2}^*$$
$$\mathsf{done}_{\mathcal{A}} = \lambda x.x$$
$$a^* = \begin{cases} \mathsf{L} & \text{if } a = \bot_{\mathcal{A}} \\ \xi & \text{if } a = {}^{\xi}(t_1 \rightarrowtail t_2) \\ \mathsf{U} & \text{otherwise} \end{cases}$$

Theorem 6.3 (Answer-type effect properties)**.**
Answer-type modification effects $(\mathcal{A}, \bot_{\mathcal{A}}, \circ)$ *satisfy Properties 1–5.*

Proof. See p. 83. ▷

$\boxed{\mathrm{D} \vdash_{\mathfrak{X}} i : k}$ *(kinding exception effects)*

$$\frac{}{\mathrm{D} \vdash_{\mathfrak{X}} \{\psi\} : \mathsf{CTL}}\ \textsc{X-K-Sing} \qquad \frac{\mathrm{D} \vdash_{\mathfrak{X}} \Psi_1 : \mathsf{CTL} \qquad \mathrm{D} \vdash_{\mathfrak{X}} \Psi_2 : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathfrak{X}} \Psi_1 \cup \Psi_2 : \mathsf{CTL}}\ \textsc{X-K-Union}$$

$\boxed{\mathrm{D} \vdash_{\mathfrak{X}} \Psi \succeq \xi}$ *(qualifier bound for exception effects)*

$$\frac{\mathrm{D} \vdash_{\mathfrak{X}} \Psi : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathfrak{X}} \Psi \succeq \mathsf{A}}\ \textsc{X-B-Raise}$$

$\boxed{\mathrm{D}; \mathrm{G} \vdash_{\mathfrak{X}} e : t \,;\, \Psi}$ *(exception effect expression typing)*

$$\frac{\mathrm{D} \vdash_{\mathfrak{X}} t : \star}{\mathrm{D}; \bullet \vdash_{\mathfrak{X}} \mathsf{raise}\ \psi : t \,;\, \{\psi\}}\ \textsc{X-T-Raise} \qquad \frac{\mathrm{D} \vdash_{\mathfrak{X}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \quad \mathrm{D}; \mathrm{G}_1 \vdash_{\mathfrak{X}} e_1 : t \,;\, \{\psi\} \cup \Psi \quad \mathrm{D}; \mathrm{G}_2 \vdash_{\mathfrak{X}} e_2 : t \,;\, \Psi \quad \mathrm{D} \vdash_{\mathfrak{X}} \mathrm{G}_2 \preceq \mathsf{A}}{\mathrm{D}; \mathrm{G} \vdash_{\mathfrak{X}} e_1\ \mathsf{handle}\ \psi \to e_2 : t \,;\, \Psi}\ \textsc{X-T-Handle}$$

Figure 18: Statics for exception effects

6.3 Exceptions

We add exceptions to $\lambda^{\mathsf{URAL}}(\mathcal{C})$ as follows. We assume a set *Exn* of exception names ψ and extend the syntax of expressions:

ψ	$\in$	*Exn*	exception names
e	::=		new expressions
	$\mid$	$\cdots$	*extending syntax from Figure 1*
	$\mid$	e_1 handle $\psi \to e_2$	delimiter
	$\mid$	raise ψ	control operator

While these exceptions are simple tags, it would not be difficult to have exceptions carry values. As in the previous example, we define the dynamics by the CCoS translation. However, because the CCoS translation for exceptions is type directed, we show how the type system is extended first.

To type exceptions, we instantiate $\lambda^{\mathsf{URAL}}(\mathcal{C})$ as follows. Exception effects, Ψ, are sets of primitive exception names ψ:

Ψ	::=		exception effect sets
		$\varnothing$	the empty effect
		α	an effect variable
		$\{\psi\}$	singleton effect
		$\Psi_1 \cup \Psi_2$	effect union

Let $\mathcal{X}$ be the set of exception effect sets (Ψ). Then we define exception effects as the triple $(\mathcal{X}, \varnothing, \cup)$. We consider exception effects as true sets, not merely as the free algebra generated by the syntax. Thus, the subsumption order is set containment:

$\boxed{\mathrm{D} \vdash_{\mathcal{X}} \Psi_1 \preceq \Psi_2}$ *(exception effect subsumption)*

$$\textsc{XSub-Subset}\ \frac{\Psi_1 \subseteq \Psi_2 \qquad \mathrm{D} \vdash_{\mathcal{X}} \Psi_1 : \mathsf{CTL} \qquad \mathrm{D} \vdash_{\mathcal{X}} \Psi_2 : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{X}} \Psi_1 \preceq \Psi_2}$$

The other new type rules for exception effects appear in Figure 18. Note that rule X-B-Raise says that all exception effects are bounded below by A; this is because exceptions allow an expression to discard its context but not duplicate it. (Of course, the empty exception set $\varnothing$ is bounded by L by rule C-B-Pure.)

To define the CCoS translation, we assume a run-time representation of exceptions and exception sets as follows:

- There is an exception pretype exn such that $\Delta \vdash \mathsf{exn} : \overline{\star}$.
- Each exception ψ is represented by a λ^{URAL} value ψ^*, such that $\Delta; \bullet \vdash \psi^* : {}^{\mathsf{U}}\mathsf{exn}$.
- For each exception ψ and pair of λ^{URAL} values v_1 and v_2, there is a λ^{URAL} value $[v_1, v_2]_\psi$ such that

$$\frac{}{[v_1, v_2]_\psi\, \psi^* \longmapsto v_1\, \psi^*} \qquad \frac{\psi \neq \psi'}{[v_1, v_2]_\psi\, \psi'^* \longmapsto v_2\, \psi'^*}.$$

$$\frac{\Delta; \Gamma \vdash v_1 : {}^{\xi_1}({}^{\mathsf{U}}\mathsf{exn} \multimap \tau) \qquad \Delta \vdash \xi_1 \preceq \xi \qquad \Delta; \Gamma \vdash v_2 : {}^{\xi_2}({}^{\mathsf{U}}\mathsf{exn} \multimap \tau) \qquad \Delta \vdash \xi_2 \preceq \xi}{\Delta; \Gamma \vdash [v_1, v_2]_\psi : {}^{\xi}({}^{\mathsf{U}}\mathsf{exn} \multimap \tau)}$$

Intuitively, $[v_1, v_2]_\psi$ performs case analysis on exception values: when applied to exception ψ, it passes the exception to v_1, and when applied to any other exception, it passes the exception to v_2.

For exception effects, we use a typed CCoS translation that takes an extra parameter: the exception effect of the expression to be translated. We assume that the generic CCoS has been updated to translate type derivations as well in order

to propagate control effects correctly. Then we can give the CCoS translation for exceptions:

$$\begin{aligned}[\![\mathsf{raise}\ \psi]\!]^{\Psi}_{\mathfrak{X}} &= \lambda\_.\ \mathsf{inl}\ \psi^* \\ [\![e_1\ \mathsf{handle}\ \psi \to e_2]\!]^{\Psi}_{\mathfrak{X}} &= \lambda y.\,[v, y]\,([\![e_1]\!]^{\{\psi\}\cup\Psi}_{\mathfrak{X}}\ (\lambda x.\ \mathsf{inr}\ x))\end{aligned}$$

$$\text{where } v = \begin{cases} \lambda\_.\,[\![e_2]\!]^{\varnothing}_{\mathfrak{X}}\ y & \text{if } \Psi = \varnothing; \\ [\lambda\_.\,[\![e_2]\!]^{\Psi}_{\mathfrak{X}}\ y, \lambda x.\ \mathsf{inl}\ x]_{\psi} & \text{if } \Psi \neq \varnothing. \end{cases}$$

Example. The first Scala example from §1 may be recast in λ^{URAL} (with integer division) as follows:

$$\lambda z_1\, z_2.\,\mathsf{pair}\,(\mathsf{ref}\,(z_1 \,/\, z_2))\,(\mathsf{ref}\,(z_2 \,/\, z_1))$$

Let us assume the following (monomorphic, for brevity) types for the operations:

$$\begin{aligned}\cdot\,/\,\cdot &: {}^{\mathsf{U}}(\mathsf{int} \xrightarrow{\varnothing}_{\circ} {}^{\mathsf{U}}(\mathsf{int} \xrightarrow{\{\mathsf{DivBy0}\}}_{\circ} \mathsf{int})) \\ \mathsf{ref} &: {}^{\mathsf{U}}(\mathsf{int} \xrightarrow{\varnothing}_{\circ} {}^{\mathsf{L}}\mathsf{intref}) \\ \mathsf{pair} &: {}^{\mathsf{U}}({}^{\mathsf{L}}\mathsf{intref} \xrightarrow{\varnothing}_{\circ} {}^{\mathsf{L}}({}^{\mathsf{L}}\mathsf{intref} \xrightarrow{\varnothing}_{\circ} {}^{\mathsf{L}}({}^{\mathsf{L}}\mathsf{intref} \otimes {}^{\mathsf{L}}\mathsf{intref})))\end{aligned}$$

To type the application of term $\mathsf{pair}\,(\mathsf{ref}\,(z_1 \,/\, z_2))$ to term $\mathsf{ref}\,(z_2 \,/\, z_1)$, according to premise (6) of rule C-T-App, the effect of the operator must be bounded by the qualifier of the type of the operand. The effect of the operator, $\mathsf{pair}\,(\mathsf{ref}\,(z_1 \,/\, z_2))$, is $\{\mathsf{DivBy0}\}$, based on the type of /; the type of the operand, $\mathsf{ref}\,(z_2 \,/\, z_1)$, is ${}^{\mathsf{L}}\mathsf{intref}$. But $\bullet \vdash_{\mathfrak{X}} \{\mathsf{DivBy0}\} \succeq \mathsf{L}$ is not derivable—a term that can raise an exception does not necessarily treat it context linearly—so the original code has a type error in $\lambda^{\mathsf{URAL}}(\mathcal{C})$.

We can repair the example, as we did in §1, by explicitly ordering the effects so that both divisions happen before any references are allocated:

$$\lambda z_1\, z_2.\,(\lambda x_1\, x_2.\,\mathsf{pair}\,(\mathsf{ref}\,x_1)\,(\mathsf{ref}\,x_2))\,(z_1 \,/\, z_2)\,(z_2 \,/\, z_1)$$

Term $\lambda x_1\, x_2.\,\mathsf{pair}\,(\mathsf{ref}\,x_1)\,(\mathsf{ref}\,x_2)$ has an unlimited type:

$${}^{\mathsf{U}}({}^{\mathsf{U}}\mathsf{int} \xrightarrow{\varnothing}_{\circ} {}^{\mathsf{U}}({}^{\mathsf{U}}\mathsf{int} \xrightarrow{\varnothing}_{\circ} {}^{\mathsf{L}}({}^{\mathsf{L}}\mathsf{intref} \otimes {}^{\mathsf{L}}\mathsf{intref})))$$

Thus, it does not matter that its argument, $z_1 \,/\, z_2$, has non-trivial effect. Similarly, because the codomain of that type is unlimited, it is permissible that the second argument, $z_2 \,/\, z_1$, has non-trivial effect as well. Thus, the repaired example is typeable in $\lambda^{\mathsf{URAL}}(\mathcal{C})$.

Type safety. To prove type safety for $\lambda^{\mathsf{URAL}}(\mathcal{C})$ extended with exceptions, we define the translation parameter as follows:

$$\langle\!\langle \tau, \Psi \rangle\!\rangle_x^- = \langle\!\langle \tau, \Psi \rangle\!\rangle_x^+ = {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau)$$

$$\mathsf{done}_x = \lambda x.\ \mathsf{inr}\ x$$

$$\Psi^* = \begin{cases} \mathsf{L} & \text{if } \Psi = \varnothing \\ \mathsf{A} & \text{if } \Psi \neq \varnothing \end{cases}$$

Theorem 6.4 (Exception effect properties)**.**
Exception effects $(\mathcal{X}, \varnothing, \cup)$ *satisfy Properties 1–5.*

Proof. See p. 88. ▷

7 Conclusion

We began this study with the desire to add linear types to Alms, a general-purpose programming language with affine types and exceptions. The treatment of exceptions in §6.3 points the way toward that goal. One question that remains, however, concerns the pragmatics of checked exceptions in a higher-order language such as Alms, where latent exception effects are likely to appear on many function arrows. We believe that with appropriate defaults most function arrows will not require annotation, but more research is required in that direction.

Another potential direction for future research is to consider how other control effects fit into our general framework. We suspect that some control operators common to imperative languages, such as *return*, *break*, and *goto*, absent first-class labels, would be straightforward. More exotic forms of control may be harder. Some control operators, such as *shift0*, are very difficult to type even in a simpler setting (Kiselyov and Shan 2007), which is why have not considered them. Others, such as Felleisen's *prompt* and *control* (1988) are probably tractable with a more expressive version of our generic type system, because effects need to reflect not only how an expression treats its continuation, but how a continuation, if captured and reinvoked, treats *its* new continuation.

For the cases we consider, however, $\lambda^{\mathsf{URAL}}(\mathcal{C})$ provides a simple and generic framework for integrating substructural types and control effects. We have shown that our type system for $\lambda^{\mathsf{URAL}}(\mathcal{C})$ is sound provided that the particular instantiation of control effects meets several criteria, and we have exhibited three instances of control effects that meet these criteria. We contend that this provides a solid grounding for the extension of realistic substructural programming languages with control effects.

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A Properties of λ^{URAL}

In this section, we state and prove several propositions about λ^{URAL}, including two lemmas from §5.2.

Lemma A.1 (Qualifier subsumption transitivity)**.**
If $\Delta \vdash \xi_1 \preceq \xi'$ *and* $\Delta \vdash \xi' \preceq \xi_2$ *then* $\Delta \vdash \xi_1 \preceq \xi_2$.

Proof. By cases on the derivation of $\Delta \vdash \xi_1 \preceq \xi'$:

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \mathsf{U} \preceq \xi}$**.**

That is, $\xi_1 = \mathsf{U}$, so by rule QSub-Bot.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \xi \preceq \mathsf{L}}$**.**

That is, $\xi' = \mathsf{L}$. By cases on the derivation of $\Delta \vdash \xi' \preceq \xi_2$:

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \mathsf{U} \preceq \xi}$**.**

That is, $\xi' = \mathsf{U}$, but since $\xi' = \mathsf{L}$, this case is vacuous.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \xi \preceq \mathsf{L}}$**.**

That is, $\xi_2 = \mathsf{L}$, so by rule QSub-Top.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \xi \preceq \xi}$**.**

That is, $\xi_2 = \xi' = \mathsf{L}$, so by rule QSub-Top.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \xi \preceq \xi}$**.**

That is, $\xi_1 = \xi'$. Then by a simple substitution. □

Lemma A.2 (Meet and join properties)**.**

Commutativity

- $\xi_1 \sqcap \xi_2 = \xi_2 \sqcap \xi_1$
- $\xi_1 \sqcup \xi_2 = \xi_2 \sqcup \xi_1$

Associativity

- $\xi_1 \sqcap (\xi_2 \sqcap \xi_3) = (\xi_1 \sqcap \xi_2) \sqcap \xi_3$
- $\xi_1 \sqcup (\xi_2 \sqcup \xi_3) = (\xi_1 \sqcup \xi_2) \sqcup \xi_3$

Completeness

If $\Delta \vdash \xi_1 \preceq \xi_2$ *then*

- $\xi_1 \sqcap \xi_2 = \xi_1$
- $\xi_1 \sqcup \xi_2 = \xi_2$

Soundness

If $\Delta \vdash \xi_1 : \mathsf{QUAL}$ *and* $\Delta \vdash \xi_2 : \mathsf{QUAL}$ *then*

- $\Delta \vdash \xi_1 \sqcap \xi_2 \preceq \xi_2$ *when* $\xi_1 \sqcap \xi_2$ *is defined*
- $\Delta \vdash \xi_1 \preceq \xi_1 \sqcup \xi_2$ *when* $\xi_1 \sqcup \xi_2$ *is defined*

Optimality

If $\Delta \vdash \xi_1 : \mathsf{QUAL}$ *and* $\Delta \vdash \xi_2 : \mathsf{QUAL}$ *then*

- *if* $\Delta \vdash \xi \preceq \xi_1$ *and* $\Delta \vdash \xi \preceq \xi_2$ *then* $\Delta \vdash \xi \preceq \xi_1 \sqcap \xi_2$ *whenever* $\xi_1 \sqcap \xi_2$ *is defined*
- *if* $\Delta \vdash \xi_1 \preceq \xi$ *and* $\Delta \vdash \xi_2 \preceq \xi$ *then* $\Delta \vdash \xi_1 \sqcup \xi_2 \preceq \xi$ *whenever* $\xi_1 \sqcup \xi_2$ *is defined*

Domain

The meet $\xi_1 \sqcap \xi_2$ *is not defined if and only if one of:*

- ξ_1 *is a variable and* ξ_2 *is* A *or* R*;*
- ξ_2 *is a variable and* ξ_1 *is* A *or* R*; or*
- ξ_1 *and* ξ_2 *are two distinct variables.*

Likewise for joins.

Substitution

Meet and join respect substitution:

- $\{\iota/\alpha\}(\xi_1 \sqcap \xi_2) = \{\iota/\alpha\}\xi_1 \sqcap \{\iota/\alpha\}\xi_2$ *when* $\xi_1 \sqcap \xi_2$ *is defined*
- $\{\iota/\alpha\}(\xi_1 \sqcup \xi_2) = \{\iota/\alpha\}\xi_1 \sqcup \{\iota/\alpha\}\xi_2$ *when* $\xi_1 \sqcup \xi_2$ *is defined*

Proof.

Commutativity By inspection.

Associativity By inspection. Any multi-meet containing only Ls and one qualifier ξ (possibly repeated) is ξ; if it contains U at all or both A and R then it is U; otherwise it is undefined. Any multi-join containing only Us and one qualifier ξ (possibly repeated) is ξ; if it contains L at all or both A and R then it is L; otherwise it is undefined.

Completeness By induction on the qualifier subsumption derivation:

Case $\dfrac{\Delta \vdash \alpha : \mathsf{QUAL}}{\Delta \vdash \mathsf{U} \preceq \alpha}$**.**
Then $\mathsf{U} \sqcap \alpha = \mathsf{U}$ and $\mathsf{U} \sqcup \alpha = \alpha$.

Case $\dfrac{\mathfrak{q}_1 \preceq \mathfrak{q}_2}{\Delta \vdash \mathfrak{q}_1 \preceq \mathfrak{q}_2}$**.**
By cases on the derivation of $\mathfrak{q}_1 \preceq \mathfrak{q}_2$:

Case $\mathfrak{q} \preceq \mathfrak{q}$**.**
Then $\mathfrak{q} \sqcap \mathfrak{q} = \mathfrak{q} = \mathfrak{q} \sqcup \mathfrak{q}$.

Case $\mathsf{U} \preceq \mathfrak{q}$**.**
Then $\mathsf{U} \sqcap \mathfrak{q} = \mathsf{U}$ and $\mathsf{U} \sqcup \mathfrak{q} = \mathfrak{q}$.

Case $\mathfrak{q} \preceq \mathsf{L}$**.**
Then $\mathfrak{q} \sqcap \mathsf{L} = \mathfrak{q}$ and $\mathfrak{q} \sqcup \mathsf{L} = \mathsf{L}$.

Case $\dfrac{\Delta \vdash \alpha : \mathsf{QUAL}}{\Delta \vdash \alpha \preceq \mathsf{L}}$**.**
Then $\alpha \sqcap \mathsf{L} = \alpha$ and $\alpha \sqcup \mathsf{L} = \mathsf{L}$.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta \vdash \xi \preceq \xi}$**.**
Then $\xi \sqcap \xi = \xi = \xi \sqcup \xi$.

Soundness

- We consider whether $\Delta \vdash \xi_1 \sqcap \xi_2 \preceq \xi_2$ by cases on ξ_1 and ξ_2:

$\Delta \vdash \xi_1 \sqcap \xi_2 \preceq \xi_2$		ξ_2 = U	R	A	L	α
ξ_1	U	$\mathsf{U} \preceq \mathsf{U}$	$\mathsf{U} \preceq \mathsf{R}$	$\mathsf{U} \preceq \mathsf{A}$	$\mathsf{U} \preceq \mathsf{L}$	$\mathsf{U} \preceq \alpha$
	R	$\mathsf{U} \preceq \mathsf{U}$	$\mathsf{R} \preceq \mathsf{R}$	$\mathsf{U} \preceq \mathsf{A}$	$\mathsf{R} \preceq \mathsf{L}$	$\times$
	A	$\mathsf{U} \preceq \mathsf{U}$	$\mathsf{U} \preceq \mathsf{R}$	$\mathsf{A} \preceq \mathsf{A}$	$\mathsf{A} \preceq \mathsf{L}$	$\times$
	L	$\mathsf{U} \preceq \mathsf{U}$	$\mathsf{R} \preceq \mathsf{R}$	$\mathsf{A} \preceq \mathsf{A}$	$\mathsf{L} \preceq \mathsf{L}$	$\alpha \preceq \alpha$
	α	$\mathsf{U} \preceq \mathsf{U}$	$\times$	$\times$	$\alpha \preceq \mathsf{L}$	$\alpha \preceq \alpha$
	β	$\mathsf{U} \preceq \mathsf{U}$	$\times$	$\times$	$\beta \preceq \mathsf{L}$	$\times$

($\times$ indicates that $\xi_1 \sqcap \xi_2$ is undefined)

- We consider whether $\Delta \vdash \xi_1 \preceq \xi_1 \sqcup \xi_2$ by cases on ξ_1 and ξ_2:

$\Delta \vdash \xi_1 \preceq \xi_1 \sqcup \xi_2$		ξ_2 = U	R	A	L	α
ξ_1	U	$\mathsf{U} \preceq \mathsf{U}$	$\mathsf{U} \preceq \mathsf{R}$	$\mathsf{U} \preceq \mathsf{A}$	$\mathsf{U} \preceq \mathsf{L}$	$\mathsf{U} \preceq \alpha$
	R	$\mathsf{R} \preceq \mathsf{R}$	$\mathsf{R} \preceq \mathsf{R}$	$\mathsf{R} \preceq \mathsf{L}$	$\mathsf{R} \preceq \mathsf{L}$	$\times$
	A	$\mathsf{A} \preceq \mathsf{A}$	$\mathsf{A} \preceq \mathsf{L}$	$\mathsf{A} \preceq \mathsf{A}$	$\mathsf{A} \preceq \mathsf{L}$	$\times$
	L	$\mathsf{L} \preceq \mathsf{L}$	$\mathsf{L} \preceq \mathsf{L}$	$\mathsf{L} \preceq \mathsf{L}$	$\mathsf{L} \preceq \mathsf{L}$	$\mathsf{L} \preceq \mathsf{L}$
	α	$\alpha \preceq \alpha$	$\times$	$\times$	$\alpha \preceq \mathsf{L}$	$\alpha \preceq \alpha$
	β	$\beta \preceq \beta$	$\times$	$\times$	$\beta \preceq \mathsf{L}$	$\times$

($\times$ indicates that $\xi_1 \sqcup \xi_2$ is undefined)

Optimality

Let $\Delta \vdash \xi_1 : \mathsf{QUAL}$ and $\Delta \vdash \xi_2 : \mathsf{QUAL}$. Then:

- Suppose that $\Delta \vdash \xi \preceq \xi_1$ and $\Delta \vdash \xi \preceq \xi_2$ and consider the possibilities by which $\xi_1 \sqcap \xi_2$ may be defined:

  Case $\mathsf{L} \sqcap \xi_2 = \xi_2$.
  By the assumption that $\Delta \vdash \xi \preceq \xi_2$.

  Case $\xi_1 \sqcap \mathsf{L} = \xi$.
  By symmetry.

  Case $\xi_1 \sqcap \xi_1 = \xi_1$.
  That is, $\xi_1 = \xi_2$. Then by the assumption that $\Delta \vdash \xi \preceq \xi_1$.

  Case $\mathsf{U} \sqcap \xi_2 = \mathsf{U}$.
  That is, $\xi_1 = \mathsf{U}$. Then $\Delta \vdash \xi \preceq \mathsf{U}$. By inspection of the rules for qualifier subsumption, this implies that $\xi = \mathsf{U}$, so by rule QSub-Bot.

  Case $\xi_1 \sqcap \mathsf{U} = \mathsf{U}$.
  By symmetry.

  Case $\mathsf{A} \sqcap \mathsf{R} = \mathsf{U}$.
  By inspection of the rules for qualifier subsumption, $\Delta \vdash \xi \preceq \mathsf{A}$ only if ξ is U or A, and $\Delta \vdash \xi \preceq \mathsf{A}$ only if ξ is U or R. Then $\xi = \mathsf{U}$, so by rule QSub-Bot.

  Case $\mathsf{R} \sqcap \mathsf{A} = \mathsf{U}$.
  By symmetry.

- By duality.

Domain

By inspection of the tables in the soundness case.

Substitution

By cases on the definition of meet:

Case $\mathsf{L} \sqcap \xi = \xi$**.**
Then $\{\iota/\alpha\}(\mathsf{L} \sqcap \xi) = \{\iota/\alpha\}\xi = \mathsf{L} \sqcap \{\iota/\alpha\}\xi = \{\iota/\alpha\}\mathsf{L} \sqcap \{\iota/\alpha\}\xi$.

Case $\xi \sqcap \mathsf{L} = \xi$**.**
By symmetry.

Case $\xi \sqcap \xi = \xi$**.**
Then $\{\iota/\alpha\}(\xi \sqcap \xi) = \{\iota/\alpha\}\xi = \{\iota/\alpha\}\xi \sqcap \{\iota/\alpha\}\xi$.

Case $\mathsf{U} \sqcap \xi = \mathsf{U}$**.**
Then $\{\iota/\alpha\}(\mathsf{U} \sqcap \xi) = \{\iota/\alpha\}\mathsf{U} = \mathsf{U} = \mathsf{U} \sqcap \{\iota/\alpha\}\xi = \{\iota/\alpha\}\mathsf{U} \sqcap \{\iota/\alpha\}\xi$.

Case $\xi \sqcap \mathsf{U} = \mathsf{U}$**.**
By symmetry.

Case $\mathsf{A} \sqcap \mathsf{R} = \mathsf{U}$**.**
Then $\{\iota/\alpha\}(\mathsf{A} \sqcap \mathsf{R}) = \{\iota/\alpha\}\mathsf{U} = \mathsf{U} = \mathsf{A} \sqcap \mathsf{R} = \{\iota/\alpha\}\mathsf{A} \sqcap \{\iota/\alpha\}\mathsf{R}$.

Case $\mathsf{R} \sqcap \mathsf{A} = \mathsf{U}$**.**
By symmetry.

Dually for join. □

Lemma A.3 (Lower bound of undefined meets)**.**
If $\xi_1 \sqcap \xi_2$ *is undefined,* $\Delta \vdash \xi \preceq \xi_1$*, and* $\Delta \vdash \xi \preceq \xi_2$*, then* $\xi = \mathsf{U}$.

Proof. If either ξ_1 or ξ_2 is U or L, or if $\xi_1 = \xi_2$, or if one is A and the other R, then the meet is defined. That leaves only two possibilities:

- One is A or R and the other is a variable α. The only possibilities for ξ to be less than α are if ξ is α or U. But since α is not less than A nor R, we know that $\xi = \mathsf{U}$.
- They are different variables α and β. As before, the only ways that $\Delta \vdash \xi \preceq \alpha$ is if ξ is either α or U. Similarly, for $\Delta \vdash \xi \preceq \beta$, ξ must be β or U. Since $\alpha \neq \beta$, we know that $\xi = \mathsf{U}$ □

Note that we do not simply define the meet in such cases to be U, because then meets would not be preserved by substitution.

Lemma A.4 (Properties of bounds)**.**

1. *If* $\Delta \vdash \xi_1 \preceq \xi_2$ *then* $\Delta \vdash \xi_1 : \mathsf{QUAL}$ *and* $\Delta \vdash \xi_2 : \mathsf{QUAL}$.
2. *If* $\Delta \vdash \tau \preceq \xi$ *then* $\Delta \vdash \tau : \star$.
3. *If* $\Delta \vdash \tau : \star$ *then* $\Delta \vdash \tau \preceq \mathsf{L}$.
4. *If* $\Delta \vdash \Gamma \preceq \xi$ *and* $x{:}\tau \in \Gamma$ *then* $\Delta \vdash \tau \preceq \xi$.
5. *If* $\Delta \vdash \tau : \star$ *for all* τ *such that* $x{:}\tau \in \Gamma$*, then* $\Delta \vdash \Gamma \preceq \mathsf{L}$.

Proof.

1. By induction on the derivation.
2. By cases on the derivation and the previous part.
3. If τ is a type variable, then by rule B-Var. Otherwise, by inversion of rules K-Type, QSub-Top, and B-Type.
4. By induction on the derivation.
5. By induction on Γ and the rules for context bounding. □

Lemma 5.3 (Value strengthening, restated from p. 23)**.**

Any qualifier that upper bounds the type of a value also bounds the portion of the type context necessary for typing that value. That is, if $\Delta; \Gamma \vdash v : \tau$ *and* $\Delta \vdash \tau \preceq \xi$ *then there exist some* Γ_1 *and* Γ_2 *such that*

- $\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2$,
- $\Delta; \Gamma_1 \vdash v : \tau$,
- $\Delta \vdash \Gamma_1 \preceq \xi$, *and*
- $\Delta \vdash \Gamma_2 \preceq \mathsf{A}$.

Proof. By induction on the derivation of $\Delta; \Gamma \vdash v : \tau$:

Case $\dfrac{\Delta \vdash \tau : \star}{\Delta; \bullet, x{:}\tau \vdash x : \tau}$.

Let $\Gamma_1 = \bullet, x{:}\tau$ and $\Gamma_2 = \bullet$.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL} \qquad \Delta \vdash \Gamma \preceq \xi \qquad \Delta; \Gamma, x{:}\tau_1 \vdash e : \tau_2}{\Delta; \Gamma \vdash \lambda x.e : {}^{\xi}(\tau_1 \multimap \tau_2)}$.

Let $\Gamma_1 = \Gamma$ and $\Gamma_2 = \bullet$.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL} \qquad \Delta \vdash \Gamma \preceq \xi \qquad \Delta, \alpha{:}\kappa; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda.e : {}^{\xi}\forall \alpha{:}\kappa.\tau}$.

Let $\Gamma_1 = \Gamma$ and $\Gamma_2 = \bullet$.

Case $\dfrac{\Delta \vdash \tau_1 \preceq \xi \qquad \Delta \vdash \tau_2 : \star \qquad \Delta; \Gamma \vdash v_1 : \tau_1}{\Delta; \Gamma \vdash \mathsf{inl}\ v_1 : {}^{\xi}(\tau_1 \oplus \tau_2)}$.

By the induction hypothesis, there exist some Γ_1 and Γ_2 such that

(1) $\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2$,
(2) $\Delta; \Gamma_1 \vdash v_1 : \tau_1$,
(3) $\Delta \vdash \Gamma_2 \preceq \mathsf{A}$, and
(4) $\Delta \vdash \Gamma_1 \preceq \xi$.

Then,

(5) $\Delta; \Gamma_1 \vdash \mathsf{inl}\ v_1 : {}^{\xi}(\tau_1 \oplus \tau_2)$ by rule T-Inl .

Case $\dfrac{\Delta \vdash \tau_2 \preceq \xi \qquad \Delta \vdash \tau_1 : \star \qquad \Delta; \Gamma \vdash v_2 : \tau_2}{\Delta; \Gamma \vdash \mathsf{inr}\ v_2 : {}^{\xi}(\tau_1 \oplus \tau_2)}$.

By symmetry.

Case $\dfrac{\Delta \vdash \xi' : \mathsf{QUAL} \qquad \Delta; \Gamma \vdash v_1 : {}^{\xi_1}(\tau_1 \multimap \tau) \quad \Delta \vdash \xi_1 \preceq \xi \qquad \Delta; \Gamma \vdash v_2 : {}^{\xi_2}(\tau_2 \multimap \tau) \qquad \Delta \vdash \xi_2 \preceq \xi}{\Delta; \Gamma \vdash [v_1, v_2] : {}^{\xi}({}^{\xi'}(\tau_1 \oplus \tau_2) \multimap \tau)}$.

By the induction hypothesis, there exist some Γ_{11} and Γ_{21} such that

(1) $\Delta \vdash \Gamma \rightsquigarrow \Gamma_{11} \boxplus \Gamma_{21}$,
(2) $\Delta; \Gamma_{12} \vdash v_1 : {}^{\xi_1}(\tau_1 \multimap \tau)$,
(3) $\Delta \vdash \Gamma_{21} \preceq \mathsf{A}$, and
(4) $\Delta \vdash \Gamma_{11} \preceq \xi_1$.

Likewise, by the induction hypothesis, there exist some Γ_{12} and Γ_{22} such that

(5) $\Delta \vdash \Gamma \rightsquigarrow \Gamma_{12} \boxplus \Gamma_{22}$,
(6) $\Delta; \Gamma_{13} \vdash v_2 : {}^{\xi_2}(\tau_2 \multimap \tau)$,
(7) $\Delta \vdash \Gamma_{22} \preceq \mathsf{A}$, and
(8) $\Delta \vdash \Gamma_{12} \preceq \xi_2$.

Then let $\Gamma_1 = \Gamma_{11} \cup \Gamma_{12}$ and let $\Gamma_2 = \Gamma_{21} \cap \Gamma_{22}$. Note that because each pair is split from the same Γ, they agree everywhere that they are defined. Furthermore, note that if ξ_1 upper bounds the qualifiers of the codomain of Γ_{11} and ξ_2 upper bounds the qualifiers of the codomain of Γ_{12}, then ξ upper bounds the qualifiers of the codomain of Γ_1:

(9) $\Delta \vdash \Gamma_1 \preceq \xi$.

Furthermore,

(10) $\Delta; \Gamma_1 \vdash v_1 : {}^{\xi_1}(\tau_1 \multimap \tau)$ by weak.
(11) $\Delta; \Gamma_1 \vdash v_2 : {}^{\xi_2}(\tau_2 \multimap \tau)$ by weak.
(12) $\Delta; \Gamma_1 \vdash [v_1, v_2] : {}^{\xi}({}^{\xi'}(\tau_1 \oplus \tau_2) \multimap \tau)$ by rule T-SumE .

Case $\dfrac{\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2 \quad \Delta; \Gamma_1 \vdash v_1 : \tau_1 \quad \Delta \vdash \tau_1 \preceq \xi \quad \Delta; \Gamma_2 \vdash v_2 : \tau_2 \quad \Delta \vdash \tau_2 \preceq \xi}{\Delta; \Gamma \vdash \langle v_1, v_2 \rangle : {}^{\xi}(\tau_1 \otimes \tau_2)}$.

By the induction hypothesis, there exist some Γ_{11} and Γ_{21} such that

(1) $\Delta \vdash \Gamma_1 \rightsquigarrow \Gamma_{11} \boxplus \Gamma_{12}$,
(2) $\Delta; \Gamma_{11} \vdash v_1 : \tau_1$,
(3) $\Delta \vdash \Gamma_{12} \preceq \mathsf{A}$, and
(4) $\Delta \vdash \Gamma_{11} \preceq \xi$.

Likewise, by the induction hypothesis, there exist some Γ_{21} and Γ_{21} such that

(5) $\Delta \vdash \Gamma_2 \rightsquigarrow \Gamma_{21} \boxplus \Gamma_{22}$,
(6) $\Delta; \Gamma_{21} \vdash v_2 : \tau_2$,
(7) $\Delta \vdash \Gamma_{22} \preceq \mathsf{A}$, and
(8) $\Delta \vdash \Gamma_{21} \preceq \xi$.

Then let $\Gamma_1 = \Gamma_{11} \cup \Gamma_{21}$ and let $\Gamma_2 = \Gamma_{12} \cup \Gamma_{22}$. Note that because each pair is split from the same Γ, they agree everywhere that they are defined. Note also that Γ can be split as

(9) $\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2$.

Furthermore, note that because ξ upper bounds the qualifiers of the codomains of both Γ_{11} and Γ_{21}, it also upper bounds the qualifiers of the codomain of Γ_1:

(10) $\Delta \vdash \Gamma_1 \preceq \xi$.

Furthermore,

(11) $\Delta; \Gamma_1 \vdash v_1 : \tau_1$ by weak.
(12) $\Delta; \Gamma_1 \vdash v_2 : \tau_2$ by weak.
(13) $\Delta; \Gamma_1 \vdash \langle v_1, v_2 \rangle : {}^{\xi}(\tau_1 \otimes \tau_2)$ by rule T-Prod .

Case $\dfrac{\Delta \vdash \xi' : \mathsf{QUAL} \qquad \Delta; \Gamma \vdash v : {}^{\xi}(\tau_1 \multimap {}^{\xi}(\tau_2 \multimap \tau))}{\Delta; \Gamma \vdash \mathsf{uncurry}\ v : {}^{\xi}({}^{\xi'}(\tau_1 \otimes \tau_2) \multimap \tau)}$.

As in the $\mathsf{inl}\ v$ case.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL}}{\Delta; \bullet \vdash \langle\rangle : {}^{\xi}1}$.

Let $\Gamma_1 = \Gamma_2 = \bullet$.

Case $\dfrac{\Delta \vdash \xi : \mathsf{QUAL} \qquad \Delta \vdash \tau : \star \qquad \Delta; \Gamma \vdash v : {}^{\xi'}1}{\Delta; \Gamma \vdash \mathsf{ignore}\ v : {}^{\xi}(\tau \multimap \tau)}$.

As in the $\mathsf{inl}\ v$ case.

Case $\dfrac{\Delta \vdash \Gamma \rightsquigarrow \Gamma'_1 \boxplus \Gamma'_2 \qquad \Delta; \Gamma'_1 \vdash v : \tau \qquad \Delta \vdash \Gamma'_2 \preceq \mathsf{A}}{\Delta; \Gamma \vdash v : \tau}$.

By the induction hypothesis, there exist some Γ_{11} and Γ_{12} such that

(1) $\Delta \vdash \Gamma'_1 \rightsquigarrow \Gamma_{11} \boxplus \Gamma_{12}$,
(2) $\Delta; \Gamma_{11} \vdash v : \tau$,
(3) $\Delta \vdash \Gamma_{12} \preceq \mathsf{A}$, and
(4) $\Delta \vdash \Gamma_{11} \preceq \xi$.

Then let $\Gamma_1 = \Gamma_{11}$ and let $\Gamma_2 = \Gamma'_2 \cup \Gamma_{12}$. Note that because $\Delta \vdash \Gamma'_2 \preceq \mathsf{A}$ and $\Delta \vdash \Gamma_{12} \preceq \mathsf{A}$, we know that $\Delta \vdash \Gamma_2 \preceq \mathsf{A}$ as well. □

Lemma 5.2 (Dereliction, restated from p. 23)**.**
If $\Delta; \Gamma \vdash v : {}^{\xi}(\tau_1 \multimap \tau_2)$ *and* $\Delta \vdash \xi \preceq \xi'$ *then* $\Delta; \Gamma \vdash \lambda x.\, v\, x : {}^{\xi'}(\tau_1 \multimap \tau_2)$.

Proof. By Lemma 5.3, there exist some Γ_1 and Γ_2 such that:

(1) $\Delta \vdash \Gamma \rightsquigarrow \Gamma_1 \boxplus \Gamma_2$,
(2) $\Delta; \Gamma_1 \vdash v : {}^{\xi}(\tau_1 \multimap \tau_2)$,
(3) $\Delta \vdash \Gamma_2 \preceq \mathsf{A}$, and
(4) $\Delta \vdash \Gamma_1 \preceq \xi$.

Then:

(5) $\Delta; \bullet, x{:}\tau_1 \vdash x : \tau_1$ — by rule T-Var
(6) $\Delta; \Gamma_1, x{:}\tau_1 \vdash v\, x : \tau_2$ — by rule T-App
(7) $\Delta \vdash \Gamma_1 \preceq \xi'$ — by ind. Γ_1, trans.
(8) $\Delta; \Gamma_1 \vdash \lambda x.\, v\, x : {}^{\xi'}(\tau_1 \multimap \tau_2)$ — by rule T-Abs. □

B Properties of $\lambda^{\mathsf{URAL}}(\mathcal{C})$ and the Translation

In this section, we prove several propositions about $\lambda^{\mathsf{URAL}}(\mathcal{C})$ and the CCoS translation, including those from §5.

Lemma B.1 ($\lambda^{\mathsf{URAL}}(\mathcal{C})$ Regularity)**.**

1. *If* $\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_1$ *and* $x{:}t' \in \mathrm{G}$, G_1, *or* G_2, *then* $\mathrm{D} \vdash_{\mathcal{C}} t' : \star$.
2. *If* $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$ *and* $x{:}t' \in \mathrm{G}$ *then* $\mathrm{D} \vdash_{\mathcal{C}} t' : \star$
3. *If* $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$ *then* $\mathrm{D} \vdash_{\mathcal{C}} t : \star$
4. *If* $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$ *then* $\mathrm{D} \vdash_{\mathcal{C}} c : \mathsf{CTL}$

Proof.

1. By induction on the derivation.
2. By induction on the derivation, using the previous part.
3. By induction on the derivation, using the previous part.
4. By induction on the derivation, considering that "derivations" with malformed effect sequences are not valid derivations. □

Lemma 5.4 (Translation of kinding, restated from p. 27)**.**
For all D, i, *and* k, *if* $\mathrm{D} \vdash_{\mathcal{C}} i : k$ *then* $\mathrm{D}^* \vdash i^* : k^*$.

Proof. By simple induction on the kinding derivation:

Case $\dfrac{\alpha{:}k \in \mathrm{D}}{\mathrm{D} \vdash_{\mathcal{C}} \alpha : k}$**.**

Then

(1) $\alpha{:}k^* \in \mathrm{D}^*$ by the definition of D^* and

(2) $\alpha^* = \alpha$

and thus $\mathrm{D}^* \vdash \alpha^* : k^*$ by rule K-Var.

Case $\mathrm{D} \vdash_{\mathcal{C}} \mathfrak{q} : \mathsf{QUAL}$**.**

By rule K-Qual.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} \overline{t} : \overline{\star} \qquad \mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL}}{\mathrm{D} \vdash_{\mathcal{C}} {}^{\xi}\overline{t} : \star}$**.**

(1) $\mathrm{D}^* \vdash \overline{t}^* : \overline{\star}$ by IH,
(2) $\mathrm{D}^* \vdash \xi^* : \mathsf{QUAL}$ by IH,
(3) $(\xi \overline{t})^* = {}^{\xi}(\overline{t}^*)$ by def. t^*, and
(4) $\xi^* = \xi$ by def. ξ^*.

Thus, by rule K-Type.

Case $\mathrm{D} \vdash_{\mathcal{C}} 1 : \overline{\star}$.

By rule K-Unit.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 : \star}{\mathrm{D} \vdash_{\mathcal{C}} t_1 \otimes t_2 : \overline{\star}}$.

(1) $\mathrm{D}^* \vdash {t_1}^* : \star$ by IH and
(2) $\mathrm{D}^* \vdash {t_2}^* : \star$ by IH.

Thus by rule K-Prod.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 : \star}{\mathrm{D} \vdash_{\mathcal{C}} t_1 \oplus t_2 : \overline{\star}}$.

Likewise.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 : \star \qquad \mathrm{D} \vdash_{\mathcal{C}} c : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{C}} t_1 \overset{c}{\multimap} t_2 : \overline{\star}}$.

(1) $\mathrm{D}^*, \alpha{:}\star \vdash {t_1}^* : \star$ by IH and weak.
(2) $\mathrm{D}^*, \alpha{:}\star \vdash {t_2}^* : \star$ by IH and weak.
(3) $\mathrm{D}^*, \alpha{:}\star \vdash c^* : \mathsf{QUAL}$ by IH and weak.
(4) $\mathrm{D}^*, \alpha{:}\star \vdash \mathsf{L} : \mathsf{QUAL}$ by rule K-Qual
(5) $\mathrm{D}^*, \alpha{:}\star \vdash \alpha : \star$ by rule K-Var
(6) $\mathrm{D}^*, \alpha{:}\star \vdash \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}} : \star$ by (5), Property 1.2
(7) $\mathrm{D}^*, \alpha{:}\star \vdash {t_2}^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}} : \star$ by (2, 6), rule K-Arr
(8) $\mathrm{D}^*, \alpha{:}\star \vdash {}^{c^*}({t_2}^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) : \star$ by (3, 7), rule K-Type
(9) $\mathrm{D}^*, \alpha{:}\star \vdash \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}} : \star$ by (5), Property 1.2
(10) $\mathrm{D}^*, \alpha{:}\star \vdash {}^{c^*}({t_2}^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}} : \overline{\star}$ by (8–9), rule K-Arr
(11) $\mathrm{D}^*, \alpha{:}\star \vdash {}^{\mathsf{L}}({}^{c^*}({t_2}^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}}) : \star$ by (4, 10), rule K-Type

(12) $\mathrm{D}^*, \alpha{:}\star \vdash {t_1}^* \multimap {}^{\mathsf{L}}({}^{c^*}({t_2}^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}}) : \overline{\star}$
by (1, 11), rule K-Arr

(13) $\mathrm{D}^*, \alpha{:}\star \vdash {}^{\mathsf{L}}({t_1}^* \multimap {}^{\mathsf{L}}({}^{c^*}({t_2}^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}})) : \star$
by (4, 12), rule K-Type

(14) $\mathrm{D}^* \vdash \forall \alpha{:}\star\,.\,{}^{\mathsf{L}}({t_1}^* \multimap {}^{\mathsf{L}}({}^{c^*}({t_2}^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}})) : \overline{\star}$
by (13), rule K-All

(15) $\mathrm{D}^* \vdash (t_1 \overset{c}{\multimap} t_2)^* : \overline{\star}$ by (14), def. $\overline{t}^*$.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} t : \star}{\mathrm{D} \vdash_{\mathcal{C}} \mathsf{ref}\ t : \overline{\star}}$.

As in product and sum cases.

Case $\dfrac{\mathrm{D}, \beta{:}k \vdash_{\mathcal{C}} t : \star \qquad \mathrm{D} \vdash_{\mathcal{C}} c : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{C}} \forall^c \beta{:}k.t : \overline{\star}}$.

(1) $\mathrm{D}, \alpha{:}\star, \beta{:}k \vdash_{\mathcal{C}} c : \mathsf{CTL}$ by prem., weak.

(2) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash \alpha : \star$ by rule K-Var

(3) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash t^* : \star$ by IH, weak.

(4) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}} : \star$ by (1–2), Property 1.2

(5) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash t^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}} : \overline{\star}$ by (3–4), rule K-Arr

(6) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash c^* : \mathsf{QUAL}$ by IH, weak.

(7) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash {}^{c^*}(t^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) : \star$ by (5–6), rule K-Type

(8) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}} : \star$ by (1–2), Property 1.2

(9) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash {}^{c^*}(t^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}} : \overline{\star}$ by (7–8), rule K-Arr

(10) $\mathrm{D}^*, \alpha{:}\star, \beta{:}k^* \vdash {}^{\mathsf{L}}({}^{c^*}(t^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}}) : \star$
by (9), rule K-Type

(11) $\mathrm{D}^*, \alpha{:}\star \vdash \forall \beta{:}k^*.{}^{\mathsf{L}}({}^{c^*}(t^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}}) : \star$
by (10), rule K-All

(12) $\mathrm{D}^*, \alpha{:}\star \vdash {}^{\mathsf{L}}\forall \beta{:}k^*.{}^{\mathsf{L}}({}^{c^*}(t^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}}) : \star$
by (11), rule K-Type

(13) $\mathrm{D}^* \vdash \forall \alpha{:}\star\,.\,{}^{\mathsf{L}}\forall \beta{:}k^*.{}^{\mathsf{L}}({}^{c^*}(t^* \multimap \langle\!\langle \alpha, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c \rangle\!\rangle^+_{\mathcal{C}}) : \overline{\star}$
by (12), rule K-All

(14) $\mathrm{D}^* \vdash (\forall^c \beta{:}k.t)^* : \overline{\star}$ by (13), def. $\overline{t}^*$. □

Lemma B.2 (Translation of qualifier judgments)**.**

1. *If* $\mathrm{D} \vdash_{\mathcal{C}} \xi_1 \preceq \xi_2$ *then* $\mathrm{D}^* \vdash \xi_1 \preceq \xi_2$.
2. *If* $\mathrm{D} \vdash_{\mathcal{C}} t \preceq \xi$ *then* $\mathrm{D}^* \vdash t^* \preceq \xi$.
3. *If* $\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \preceq \xi$ *then* $\mathrm{D}^* \vdash \mathrm{G}^* \preceq \xi$.
4. *If* $\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2$ *then* $\mathrm{D}^* \vdash \mathrm{G}^* \rightsquigarrow {\mathrm{G}_1}^* \boxplus {\mathrm{G}_2}^*$

Proof. By simple induction on each derivation. □

Lemma 5.5 (Translation of effect bounds, restated from p. 27)**.**
If $\mathrm{D} \vdash_{\mathcal{C}} c \succeq \xi$ *then* $\mathrm{D}^* \vdash \xi \preceq c^*$.

Proof. By cases on the derivation:

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL}}{\mathrm{D} \vdash_{\mathcal{C}} \bot_{\mathcal{D}} \succeq \xi}$.

Then ${\bot_{\mathcal{C}}}^* = \mathsf{L}$, so by rule QSub-Top.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} c : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{C}} c \succeq \mathsf{U}}$.

By rule QSub-Bot.

Additional cases must be proved for new rules added by specific control effect instances. □

Lemma 5.6 (Translation of effect subsumption, restated from p. 27)**.**
If $\mathrm{D} \vdash_{\mathcal{C}} c_1 \preceq c_2$ *then* $\mathrm{D}^* \vdash {c_2}^* \preceq {c_1}^*$.

Proof. By induction on the derivation:

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} c : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{C}} c \preceq c}$.

By rule QSub-Refl.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} c_1 \preceq c' \qquad \mathrm{D} \vdash_{\mathcal{C}} c' \preceq c_2}{\mathrm{D} \vdash_{\mathcal{C}} c_1 \preceq c_2}$.

By the induction hypothesis twice and Lemma A.1.

Additional cases must be proved for new rules added by specific control effect instances. □

Lemma 5.7 (Translation of term typing, restated from p. 28)**.**
If $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$ *then*

$$\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : \mathsf{L}^{(c^*}(t^* \multimap \langle\!\langle t^*, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle t^*, c \rangle\!\rangle^+_{\mathcal{C}}).$$

Proof. We generalize the lemma to the following induction hypothesis:

> If $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$, then for all τ_0 such that $\mathrm{D}^* \vdash \tau_0 : \star$, and for all ξ_0 such that $\mathrm{D}^* \vdash \xi_0 \preceq c^*$, it is the case that $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : \mathsf{L}^{(\xi_0}(t^* \multimap \langle\!\langle \tau_0, c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau_0, c \rangle\!\rangle^+_{\mathcal{C}})$

We use lexical induction on the pair of: 1) the size of e, using size defined as follows, and 2) the height of the typing derivation for $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c$. The size of an expression is given by:

$$\begin{aligned}
|x| &= 1 & |\lambda x.e'| &= 1 + |e'| \\
|\Lambda.e'| &= 1 + |e'| & |\mathsf{inl}\ v| &= 1 + |v| \\
|\mathsf{inr}\ v| &= 1 + |v| & |[v_1, v_2]| &= |v_1| + |v_2| \\
|\langle v_1, v_2 \rangle| &= |v_1| + |v_2| & |\mathsf{uncurry}\ v| &= \mathbf{3} + |v| \\
|\langle\rangle| &= 1 & |\mathsf{ignore}\ v| &= 1 + |v| \\
|e_1\, e_2| &= |e_1| + |e_2| & |e'\ _| &= 1 + |e'| \\
|\mathsf{new}^{\mathsf{q}}\ e'| &= 1 + |e'| & |\mathsf{free}\ e'| &= 1 + |e'| \\
|\mathsf{read}\ e'| &= 1 + |e'| & |\mathsf{swap}\ e_1\, e_2| &= |e_1| + |e_2| \\
|e_1\ \mathsf{handle}\ \psi \to e_2| &= |e_1| + |e_2| & |\mathsf{raise}\ \psi| &= 1 \\
|\mathsf{reset}\ e| &= 1 + |e| & |\mathsf{shift}\, x\, \mathsf{in}\, e| &= 1 + |e|
\end{aligned}$$

In particular, this means that we can apply the induction hypothesis to any smaller expression than e, or to the same expression e provided that we use a subderivation of the derivation at hand. We proceed by cases on the conclusion of the typing derivation. We start with the cases that apply to values, since those have much in common:

Case v**.**

> By inspection, note that in all rules for typing values, the effect is pure, and thus:
>
> (1) $c = \bot_{\mathcal{C}}$
>
> Note further that by Property 1.1,
>
> (2) $\langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}} = \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}} = \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}$.
>
> Furthermore, by the definition of $[\![v]\!]_{\mathcal{C}}$, we know that

(3) $[\![e]\!]_{\mathcal{C}} = \lambda y.\, y\, v^*$.

Thus, it is sufficient to show that $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.\, y\, v^* : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ (where y is fresh for v). Suppose that (4) $\mathrm{D}^*; \mathrm{G}^* \vdash v^* : t^*$. Then:

(5) $\mathrm{D}^* \vdash \mathsf{L} : \mathsf{QUAL}$ — by rule K-Qual

(6) $\mathrm{D} \vdash_{\mathcal{C}} t : \star$ — by Lemma B.1

(7) $\mathrm{D}^* \vdash t^* : \star$ — by Lemma 5.4

(8) $\mathrm{D}^* \vdash {}^{\xi_0}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) : \star$ — by (5, 7), Property 1.2

(9) $\mathrm{D}^*; \bullet, y{:}{}^{\xi_0}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash y : {}^{\xi_0}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ — by (8)

(10) $\mathrm{D}^* \vdash \mathrm{G}^* \rightsquigarrow \bullet \boxplus \mathrm{G}^*$ — by ind. G^*, rule S-ConsR

(11) $\mathrm{D}^* \vdash \mathrm{G}^*, y{:}{}^{\xi_0}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \rightsquigarrow \bullet, y{:}{}^{\xi_0}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \boxplus \mathrm{G}^*$ — by (10), rule S-ConsL

(12) $\mathrm{D}^*; \mathrm{G}^*, y{:}{}^{\xi_0}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash y\, v^* : \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}$ — by (4, 9, 11)

(13) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.\, y\, v^* : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ — by (5, 12).

Therefore, it is sufficient to show (4) $\mathrm{D}^*; \mathrm{G}^* \vdash v^* : t^*$. We proceed by a nested induction on the structure of v, considering the possible typing derivations:

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} t : \star}{\mathrm{D}; \bullet, x{:}t \vdash_{\mathcal{C}} x : t \,;\, \bot_{\mathcal{C}}}$.

By rule T-Var.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL} \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \preceq \xi \qquad \mathrm{D}; \mathrm{G}, x{:}t_1 \vdash_{\mathcal{C}} e : t_2 \,;\, c'}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \lambda x.\, e : {}^{\xi}(t_1 \overset{c'}{\multimap} t_2) \,;\, \bot_{\mathcal{C}}}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash (\lambda x.\, e)^* : {}^{\xi}((t_1 \overset{c'}{\multimap} t_2)^*)$. Note that

(1) $(t_1 \overset{c'}{\multimap} t_2)^* = \forall \alpha{:}\star.\, {}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}({}^{c'{}^*}(t_2{}^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}}))$ — by def. $\bar{t}^*$ and

(2) $(\lambda x.\, e)^* = \Lambda.\, \lambda x.\, [\![e]\!]_{\mathcal{C}}$ — by def. v^*.

Then

(3) $\mathrm{D}, \alpha{:}\star; \mathrm{G}, x{:}t_1 \vdash_{\mathcal{C}} e : t_2 \,;\, c'$ — by weak.

(4) $\mathrm{D}^*, \alpha{:}\star \vdash \alpha : \star$ — by rule K-Var

(5) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^*, x{:}t_1{}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{c'{}^*}(t_2{}^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}})$ — by IH, (3–4)

(6) $\mathrm{D}^*, \alpha{:}\star \vdash \mathrm{G}^* \preceq \mathsf{L}$ — by Lemma A.4

(7) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^* \vdash \lambda x.\, [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}({}^{c'{}^*}(t_2{}^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}}))$ — by (5–6)

(8) $\mathrm{D}^* \vdash \mathrm{G}^* \preceq \xi$ by Lemma B.2

(9) $\mathrm{D}^*; \mathrm{G}^* \vdash \Lambda.\lambda x.[\![e]\!]_{\mathcal{C}} : {}^{\xi}\forall\alpha{:}\star.{}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}({}^{c'*}(t_2{}^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}}))$ by (7–8)

(10) $\mathrm{D}^*; \mathrm{G}^* \vdash (\lambda x.e)^* : {}^{\xi}((t_1 \xrightarrow{c'} t_2)^*)$ by (1–2, 9).

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL} \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \preceq \xi \qquad \mathrm{D}, \beta{:}k; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c'}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \Lambda.e : {}^{\xi}\forall^{c'}\beta{:}k.t \,;\, \bot_{\mathcal{C}}}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash (\Lambda.e)^* : {}^{\xi}((\forall^{c'}\beta{:}k.t)^*)$. Note that

(1) $(\forall^{c'}\beta{:}k.t)^* = \forall\alpha{:}\star.{}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({}^{c'*}(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}})$ by def. $\bar{t}^*$ and

(2) $(\Lambda.e)^* = \Lambda.\Lambda.[\![e]\!]_{\mathcal{C}}$ by def. v^*.

Then

(3) $\mathrm{D}, \beta{:}k, \alpha{:}\star; \mathrm{G} \vdash_{\mathcal{C}} e : t \,;\, c'$ by weak.

(4) $\mathrm{D}^*, \beta{:}k^*, \alpha{:}\star \vdash \alpha : \star$ by rule K-Var

(5) $\mathrm{D}^*, \beta{:}k^*, \alpha{:}\star; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{c'*}(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}})$ by IH, (3–4)

(6) $\mathrm{D}^*, \beta{:}k^*, \alpha{:}\star \vdash \mathrm{G}^* \preceq \mathsf{L}$ by Lemma A.4

(7) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^* \vdash \Lambda.[\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({}^{c'*}(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}})$ by (5–6)

(8) $\mathrm{D}^* \vdash \mathrm{G}^* \preceq \xi$ by Lemma B.2

(9) $\mathrm{D}^*; \mathrm{G}^* \vdash \Lambda.\Lambda.[\![e]\!]_{\mathcal{C}} : {}^{\xi}\forall\alpha{:}\star.{}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({}^{c'*}(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}})$ by (7–8)

(10) $\mathrm{D}^*; \mathrm{G}^* \vdash (\Lambda.e)^* : {}^{\xi}((\forall^{c'}\beta{:}k.t)^*)$ by (1–2, 9).

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} t_1 \preceq \xi \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 : \star \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} v_1 : t_1 \,;\, \bot_{\mathcal{C}}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{inl}\ v_1 : {}^{\xi}(t_1 \oplus t_2) \,;\, \bot_{\mathcal{C}}}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash (\mathsf{inl}\ v_1)^* : {}^{\xi}((t_1 \oplus t_2)^*)$. Note that

(1) $(t_1 \oplus t_2)^* = t_1{}^* \oplus t_2{}^*$ by def. $\bar{t}^*$ and

(2) $(\mathsf{inl}\ v_1)^* = \mathsf{inl}\ v_1{}^*$ by def. v^*.

Then

(3) $\mathrm{D}^* \vdash t_1{}^* \preceq \xi$ by Lemma B.2

(4) $\mathrm{D}^* \vdash t_2{}^* : \star$ by Lemma 5.4

(5) $\mathrm{D}^*; \mathrm{G}^* \vdash v_1{}^* : t_1{}^*$ by IH (inner)

(6) $\mathrm{D}^*; \mathrm{G}^* \vdash \mathsf{inl}\ v_1{}^* : {}^{\xi}(t_1{}^* \oplus t_2{}^*)$ by (3–5)

(7) $\mathrm{D}^*; \mathrm{G}^* \vdash (\mathsf{inl}\ v_1)^* : ({}^{\xi}(t_1 \oplus t_2))^*$ by (1–2, 6).

Case $\dfrac{\mathrm{D} \vDash_{\mathcal{C}} t_2 \preceq \xi \qquad \mathrm{D} \vDash_{\mathcal{C}} t_1 : \star \qquad \mathrm{D}; \mathrm{G} \vDash_{\mathcal{C}} v_2 : t_2 \,;\, \bot_{\mathcal{C}}}{\mathrm{D}; \mathrm{G} \vDash_{\mathcal{C}} \mathsf{inr}\ v_2 : {}^{\xi}(t_1 \oplus t_2) \,;\, \bot_{\mathcal{C}}}$.

By symmetry.

Case $\dfrac{\begin{array}{c}\mathrm{D} \vDash_{\mathcal{C}} \xi' : \mathsf{QUAL} \qquad \mathrm{D}; \mathrm{G} \vDash_{\mathcal{C}} v_1 : {}^{\xi_1}(t_1 \overset{c'}{\multimap} t) \,;\, \bot_{\mathcal{C}} \\ \mathrm{D} \vDash_{\mathcal{C}} \xi_1 \preceq \xi \qquad \mathrm{D}; \mathrm{G} \vDash_{\mathcal{C}} v_2 : {}^{\xi_2}(t_2 \overset{c'}{\multimap} t) \,;\, \bot_{\mathcal{C}} \qquad \mathrm{D} \vDash_{\mathcal{C}} \xi_2 \preceq \xi\end{array}}{\mathrm{D}; \mathrm{G} \vDash_{\mathcal{C}} [v_1, v_2] : {}^{\xi}({}^{\xi'}(t_1 \oplus t_2) \overset{c'}{\multimap} t) \,;\, \bot_{\mathcal{C}}}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash [v_1, v_2]^* : {}^{\xi}(({}^{\xi'}(t_1 \oplus t_2) \overset{c'}{\multimap} t)^*)$. Note that

(1) $({}^{\xi'}(t_1 \oplus t_2) \overset{c'}{\multimap} t)^* =$
$\forall \alpha{:}\star\,.\,{}^{\mathsf{L}}({}^{\xi'}({t_1}^* \oplus {t_2}^*) \multimap {}^{\mathsf{L}}({}^{c'^*}(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle^+_{\mathcal{C}}))$
by def. $\bar{t}^*$ and

(2) $[v_1, v_2]^* = \Lambda.[\lambda x.{v_1}^*\,\_\,x, \lambda x.{v_2}^*\,\_\,x]$ by def. v^*.

Then:

(3) $\mathrm{D}^*, \alpha{:}\star \vdash \xi' : \mathsf{QUAL}$ by Lemma 5.4, weak.

(4) $\mathrm{D}^*, \alpha{:}\star; \bullet, x{:}{t_1}^* \vdash x : {t_1}^*$ by rule T-Var

(5) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^* \vdash {v_1}^* : {}^{\xi_1}((t_1 \overset{c'}{\multimap} t)^*)$ by IH (inner), weak.

(6) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^* \vdash {v_2}^* : {}^{\xi_2}((t_2 \overset{c'}{\multimap} t)^*)$ by IH (inner), weak.

By Lemma 5.3, there exist some Γ_{11} and Γ_{12} such that

(7) $\mathrm{D}^*, \alpha{:}\star \vdash \mathrm{G}^* \rightsquigarrow \Gamma_{11} \boxplus \Gamma_{12}$,

(8) $\mathrm{D}^*, \alpha{:}\star; \Gamma_{11} \vdash {v_1}^* : {}^{\xi_1}((t_1 \overset{c'}{\multimap} t)^*)$,

(9) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_{12} \preceq \mathsf{A}$, and

(10) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_{11} \preceq \xi_1$,

and likewise, there exist some Γ_{21} and Γ_{22} such that

(11) $\mathrm{D}^*, \alpha{:}\star \vdash \mathrm{G}^* \rightsquigarrow \Gamma_{21} \boxplus \Gamma_{22}$,

(12) $\mathrm{D}^*, \alpha{:}\star; \Gamma_{21} \vdash {v_2}^* : {}^{\xi_2}((t_2 \overset{c'}{\multimap} t)^*)$,

(13) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_{22} \preceq \mathsf{A}$, and

(14) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_{21} \preceq \xi_2$.

Let $\Gamma_1 = \Gamma_{11} \cup \Gamma_{21}$. Then:

(15) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_1 \preceq \xi$ by (10, 14)

(16) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash {v_1}^* : {}^{\xi_1}((t_1 \overset{c'}{\multimap} t)^*)$ by (5), weak.

(17) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash {v_1}^* :$
${}^{\xi_1}\forall\beta{:}\star\,.\,{}^{\mathsf{L}}({t_1}^* \multimap {}^{\mathsf{L}}({}^{c'^*}(t^* \multimap \langle\!\langle \beta, c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \beta, c' \rangle\!\rangle^+_{\mathcal{C}}))$
by def. $\bar{t}^*$, (5)

(18) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash {v_1}^*\_ : {}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}({c'}^*(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^+))$ by (17)

(19) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_1, x{:}t_1{}^* \rightsquigarrow \Gamma_1 \boxplus \bullet, x{:}t_1{}^*$ by rules S-ConsL and S-ConsR

(20) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1, x{:}t_1{}^* \vdash {v_1}^*\_x : {}^{\mathsf{L}}({c'}^*(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^+)$ by (4, 18–19)

(21) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash \lambda x.{v_1}^*\_x : {}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}({c'}^*(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^+))$ by (20)

(22) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash \lambda x.{v_2}^*\_x : {}^{\mathsf{L}}(t_2{}^* \multimap {}^{\mathsf{L}}({c'}^*(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^+))$ by symmetry

(23) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash [\lambda x.{v_1}^*\_x, \lambda x.{v_2}^*\_x] :$ ${}^{\mathsf{L}}({\xi'}(t_1{}^* \oplus t_2{}^*) \multimap {}^{\mathsf{L}}({c'}^*(t^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^+))$ by (3, 21–22)

(24) $\mathrm{D}^*; \Gamma_1 \vdash \Lambda.[\lambda x.{v_1}^*\_x, \lambda x.{v_2}^*\_x] : {}^{\xi}((({\xi'}(t_1 \oplus t_2) \xrightarrow{c'}\!\!\circ\; t)^*)$ by (1, 15, 23), rule T-TAbs

(25) $\mathrm{D}^*; \Gamma_1 \vdash [v_1, v_2]^* : ({}^{\xi}({\xi'}(t_1 \oplus t_2) \xrightarrow{c'}\!\!\circ\; t))^*$ by (2).

Case $\dfrac{\begin{array}{c}\mathrm{D} \vDash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \\ \mathrm{D}; \mathrm{G}_1 \vDash_{\mathcal{C}} v_1 : t_1 ; \bot_{\mathcal{C}} \qquad \mathrm{D} \vDash_{\mathcal{C}} t_1 \preceq \xi \\ \mathrm{D}; \mathrm{G}_2 \vDash_{\mathcal{C}} v_2 : t_2 ; \bot_{\mathcal{C}} \qquad \mathrm{D} \vDash_{\mathcal{C}} t_2 \preceq \xi\end{array}}{\mathrm{D}; \mathrm{G} \vDash_{\mathcal{C}} \langle v_1, v_2 \rangle : {}^{\xi}(t_1 \otimes t_2) ; \bot_{\mathcal{C}}}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash \langle v_1, v_2 \rangle^* : {}^{\xi}((t_1 \otimes t_2)^*)$. Note that

(1) $(t_1 \otimes t_2)^* = t_1{}^* \otimes t_2{}^*$ by def. $\bar{t}^*$ and

(2) $\langle v_1, v_2 \rangle^* = \langle {v_1}^*, {v_2}^* \rangle$ by def. v^*.

Then

(3) $\mathrm{D}^* \vdash \mathrm{G}^* \rightsquigarrow {\mathrm{G}_1}^* \boxplus {\mathrm{G}_2}^*$ by Lemma B.2

(4) $\mathrm{D}^*; {\mathrm{G}_1}^* \vdash {v_1}^* : t_1{}^*$ by IH (inner)

(5) $\mathrm{D}^* \vdash t_1{}^* \preceq \xi$ by Lemma B.2

(6) $\mathrm{D}^*; {\mathrm{G}_2}^* \vdash {v_2}^* : t_2{}^*$ by IH (inner)

(7) $\mathrm{D}^* \vdash t_2{}^* \preceq \xi$ by Lemma B.2

(8) $\mathrm{D}^*; \mathrm{G}^* \vdash \langle {v_1}^*, {v_2}^* \rangle : {}^{\xi}(t_1{}^* \otimes t_2{}^*)$ by (3)–(7)

(9) $\mathrm{D}^*; \mathrm{G}^* \vdash \langle v_1, v_2 \rangle^* : ({}^{\xi}(t_1 \otimes t_2))^*$ by (1–2, 8).

Case $\dfrac{\begin{array}{c}\mathrm{D} \vDash_{\mathcal{C}} \xi' : \mathsf{QUAL} \\ \mathrm{D}; \mathrm{G} \vDash_{\mathcal{C}} v : {}^{\xi}(t_1 \xrightarrow{c_1}\!\!\circ\; {}^{\xi}(t_2 \xrightarrow{c_2}\!\!\circ\; t)) ; \bot_{\mathcal{C}} \qquad \mathrm{D} \vDash_{\mathcal{C}} c_1 \otimes c_2 : \mathsf{CTL}\end{array}}{\mathrm{D}; \mathrm{G} \vDash_{\mathcal{C}} \mathsf{uncurry}\ v : {}^{\xi}({\xi'}(t_1 \otimes t_2) \xrightarrow{c_1 \otimes c_2}\!\!\circ\; t) ; \bot_{\mathcal{C}}}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash (\mathsf{uncurry}\ v)^* : {}^{\xi}((({\xi'}(t_1 \otimes t_2) \xrightarrow{c_1 \otimes c_2}\!\!\circ\; t)^*)$. Then:

(1) $(^{\xi'}(t_1 \otimes t_2) \xrightarrow{c_1 \otimes c_2} t)^* = \forall \alpha{:}\star$
$.\,{}^{\mathsf{L}}(^{\xi'}(t_1{}^* \otimes t_2{}^*) \multimap {}^{\mathsf{L}}(^{(c_1 \otimes c_2)^*}(t^* \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^+_{\mathcal{C}}))$
by def. $\bar{t}^*$ and

(2) $(\mathsf{uncurry}\ v)^* = \Lambda.\ \mathsf{uncurry}\,(\lambda x_1.\lambda x_2.[\![v\,x_1\,x_2]\!]_{\mathcal{C}})$
by def. v^*.

(3) $\mathrm{D}, \alpha{:}\star; \mathrm{G} \vdash_{\mathcal{C}} v : {}^{\xi}(t_1 \xrightarrow{c_1} {}^{\xi}(t_2 \xrightarrow{c_2} t))\,;\,\bot_{\mathcal{C}}$ by prem., weak.

(4) $\mathrm{D}, \alpha{:}\star; \bullet, x_1{:}t_1 \vdash_{\mathcal{C}} x_1 : t_1\,;\,\bot_{\mathcal{C}}$ by rule C-T-Var

(5) $\mathrm{D}, \alpha{:}\star; \bullet, x_2{:}t_2 \vdash_{\mathcal{C}} x_2 : t_2\,;\,\bot_{\mathcal{C}}$ by rule C-T-Var

(6) $\mathrm{D}, \alpha{:}\star; \mathrm{G}, x_1{:}t_1 \vdash_{\mathcal{C}} v\,x_1 : {}^{\xi}(t_2 \xrightarrow{c_2} t)\,;\,c_1$ by (3–4), rule C-T-App

(7) $\mathrm{D}, \alpha{:}\star; \mathrm{G}, x_1{:}t_1, x_2{:}t_2 \vdash_{\mathcal{C}} v\,x_1\,x_2 : t\,;\,c_1 \otimes c_2$ by (5–6), rule C-T-App

(8) $|v\,x_1\,x_2| = 2 + |v|$ by def. $|\cdot|$

(9) $|\mathsf{uncurry}\ v| = 3 + |v|$ by def. $|\cdot|$

(10) $|v\,x_1\,x_2| < |\mathsf{uncurry}\ v|$ by (8–9)

(11) $\mathrm{D}^* \vdash (c_1 \otimes c_2)^* \preceq (c_1 \otimes c_2)^*$ by rule QSub-Refl

(12) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^*, x_1{:}t_1{}^*, x_2{:}t_2{}^* \vdash [\![v\,x_1\,x_2]\!]_{\mathcal{C}} :$
${}^{\mathsf{L}}(^{(c_1 \otimes c_2)^*}(t^* \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^+_{\mathcal{C}})$
by IH (outer), (10–11)

(13) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^* \vdash \lambda x_1.\lambda x_2.[\![v\,x_1\,x_2]\!]_{\mathcal{C}} :$
${}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}(t_2{}^* \multimap {}^{\mathsf{L}}(^{(c_1 \otimes c_2)^*}(t^* \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^+_{\mathcal{C}})))$
by (12), rule T-Abs^2

(14) $\mathrm{D}^*, \alpha{:}\star \vdash \xi' : \mathsf{QUAL}$ by Lemma 5.4, weak.

(15) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^* \vdash \mathsf{uncurry}\,(\lambda x_1.\lambda x_2.[\![v\,x_1\,x_2]\!]_{\mathcal{C}}) :$
${}^{\mathsf{L}}(^{\xi'}(t_1{}^* \otimes t_2{}^*) \multimap {}^{\mathsf{L}}(^{(c_1 \otimes c_2)^*}(t^* \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^+_{\mathcal{C}}))$
by (13–14)

By Lemma 5.3, there exist some Γ_1 and Γ_2 such that

(16) $\mathrm{D}^*, \alpha{:}\star \vdash \mathrm{G}^* \rightsquigarrow \Gamma_1 \boxplus \Gamma_2$,

(17) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash \mathsf{uncurry}\,(\lambda x_1.\lambda x_2.[\![v\,x_1\,x_2]\!]_{\mathcal{C}}) :$
${}^{\mathsf{L}}(^{\xi'}(t_1{}^* \otimes t_2{}^*) \multimap {}^{\mathsf{L}}(^{(c_1 \otimes c_2)^*}(t^* \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^+_{\mathcal{C}}))$

(18) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_2 \preceq \mathsf{A}$, and

(19) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_1 \preceq \xi$.

Then:

(20) $\mathrm{D}^*; \Gamma_1 \vdash \Lambda.\ \mathsf{uncurry}\,(\lambda x_1.\lambda x_2.[\![v\,x_1\,x_2]\!]_{\mathcal{C}}) :$
${}^{\xi}\forall \alpha{:}\star\,.\,{}^{\mathsf{L}}(^{\xi'}(t_1{}^* \otimes t_2{}^*) \multimap {}^{\mathsf{L}}(^{(c_1 \otimes c_2)^*}(t^* \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \alpha, c_1 \otimes c_2 \rangle\!\rangle^+_{\mathcal{C}}))$
by (17, 19)

(21) $\mathrm{D}^*; \Gamma_1 \vdash (\mathsf{uncurry}\ v)^* : (^{\xi}((^{\xi'}(t_1 \otimes t_2) \xrightarrow{c_1 \otimes c_2} t)))^*$
by (1–2, 20)

(22) $\mathrm{D}^*; \mathrm{G}^* \vdash (\mathsf{uncurry}\ v)^* : (^{\xi}((^{\xi'}(t_1 \otimes t_2) \xrightarrow{c_1 \circledcirc c_2}_{\circ} t)))^*$ by weak., (18, 21)

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL}}{\mathrm{D}; \bullet \vdash_{\mathcal{C}} \langle\rangle : {}^{\xi}1 \,;\, \bot_{\mathcal{C}}}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash \langle\rangle^* : {}^{\xi}(1^*)$. Note that

(1) $1^* = 1$ by def. $\bar{t}^*$ and

(2) $\langle\rangle^* = \langle\rangle$ by def. v^*.

Then:

(3) $\mathrm{D}^* \vdash \xi : \mathsf{QUAL}$ by Lemma 5.4

(4) $\mathrm{D}^*; \bullet \vdash \langle\rangle^* : {}^{\xi}1^*$ by rule T-Unit, (1–2).

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{C}} \xi : \mathsf{QUAL} \qquad \mathrm{D} \vdash_{\mathcal{C}} t : \star \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} v : {}^{\xi'}1 \,;\, \bot_{\mathcal{C}}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{ignore}\ v : {}^{\xi}(t \xrightarrow{\bot_{\mathcal{C}}}_{\circ} t) \,;\, \bot_{\mathcal{C}}}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash (\mathsf{ignore}\ v)^* : {}^{\xi}((t \xrightarrow{\bot_{\mathcal{C}}}_{\circ} t)^*)$. Note that

(1) $(t \xrightarrow{\bot_{\mathcal{C}}}_{\circ} t)^* = \forall \alpha{:}\star\,.\,{}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}})$ by def. $\bar{t}^*$ and

(2) $(\mathsf{ignore}\ v)^* = \Lambda.\,\lambda x.\ \mathsf{ignore}\ v^*\ [\![x]\!]_{\mathcal{C}}$ by def. v^*.

Then:

(3) $\mathrm{D}^* \vdash \xi : \mathsf{QUAL}$ by Lemma 5.4

(4) $\mathrm{D}^* \vdash t^* : \star$ by Lemma 5.4

(5) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^* \vdash v^* : {}^{\xi'}1$ by IH (inner), def. $\bar{t}^*$

(6) $\mathrm{D}^*, \alpha{:}\star \vdash {}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) : \star$ by (4), rule K-Var, Property 1.2, ...

(7) $\mathrm{D}^*, \alpha{:}\star; \mathrm{G}^* \vdash \mathsf{ignore}\ v^* : {}^{\xi}({}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap {}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}))$ by (5) and rule T-UnitE

By Lemma 5.3, there exist some Γ_1 and Γ_2 such that

(8) $\mathrm{D}^*, \alpha{:}\star \vdash \mathrm{G}^* \rightsquigarrow \Gamma_1 \boxplus \Gamma_2$,

(9) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash \mathsf{ignore}\ v^* : {}^{\xi}({}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap {}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}))$

(10) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_2 \preceq \mathsf{A}$, and

(11) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_1 \preceq \xi$.

Then:

(12) $\mathrm{D}, \alpha{:}\star; \bullet, x{:}t \vdash_{\mathcal{C}} x : t \,;\, \bot_{\mathcal{C}}$ by rule C-T-Var

(13) $\mathrm{D}^*, \alpha{:}\star \vdash \alpha : \star$ by rule K-Type
(14) $|x| = 1 < 1 + |v| = |\mathsf{ignore}\ v|$ by def. $|\cdot|$
(15) $\mathrm{D}^* \vdash \mathsf{L} \preceq \bot_{\mathcal{C}}{}^*$ by rule QSub-Refl
(16) $\mathrm{D}^*, \alpha{:}\star; \bullet, x{:}t^* \vdash [\![x]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}})$ by IH (outer), (12–15)
(17) $\mathrm{D}^*, \alpha{:}\star \vdash \Gamma_1, x{:}t^* \rightsquigarrow \Gamma_1 \boxplus \bullet, x{:}t^*$ by rules S-ConsR and S-ConsL, ...
(18) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1, x{:}t^* \vdash \mathsf{ignore}\ v^*\ [\![x]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}})$ by (7, 16–17)
(19) $\mathrm{D}^*, \alpha{:}\star; \Gamma_1 \vdash \lambda x.\ \mathsf{ignore}\ v^*\ [\![x]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}}))$ by (18)
(20) $\mathrm{D}^*; \Gamma_1 \vdash \Lambda.\lambda x.\ \mathsf{ignore}\ v^*\ [\![x]\!]_{\mathcal{C}} : {}^{\xi}\forall \alpha{:}\star.{}^{\mathsf{L}}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}}))$ by (11, 20)
(21) $\mathrm{D}^*; \Gamma_1 \vdash (\mathsf{ignore}\ v)^* : ({}^{\xi}(t \xrightarrow{\bot_{\mathcal{C}}} t))^*$ by (1–2)
(22) $\mathrm{D}^*; \mathrm{G}^* \vdash (\mathsf{ignore}\ v)^* : ({}^{\xi}(t \xrightarrow{\bot_{\mathcal{C}}} t))^*$ by weak. and (10).

That completes the value case. We continue with the non-value expressions, again considering type derivations:

$$\textbf{Case}\ \frac{\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \qquad \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e : t\ ;\ c \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \mathsf{A}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t\ ;\ c}.$$

By Lemma B.2 and IH.

$$\textbf{Case}\ \frac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t\ ;\ c' \qquad \mathrm{D} \vdash_{\mathcal{C}} c' \preceq c}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t\ ;\ c}.$$

(1) $\mathrm{D}^* \vdash c^* \preceq c'^*$ by Property 5.3
(2) $\mathrm{D}^* \vdash \xi \preceq c^*$ by antecedent
(3) $\mathrm{D}^* \vdash \xi \preceq c'^*$ by (1–2), Lemma A.1
(4) $\exists \tau''.\langle\!\langle \tau'', c' \rangle\!\rangle^-_{\mathcal{C}} = \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}$ and $\langle\!\langle \tau'', c' \rangle\!\rangle^+_{\mathcal{C}} = \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}}$ by (1), Property 1.4
(5) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\xi}(t^* \multimap \langle\!\langle \tau'', c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau'', c' \rangle\!\rangle^+_{\mathcal{C}})$ by IH(τ''), (3)
(6) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\xi}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by (4–5).

$$\textbf{Case}\ \frac{\begin{array}{c}\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \qquad \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e_1 : {}^{\xi_1}(t_1 \xrightarrow{c'} t_2)\ ;\ c'_1 \qquad \mathrm{D}; \mathrm{G}_2 \vdash_{\mathcal{C}} e_2 : t_1\ ;\ c'_2 \\ \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \xi_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c'_1 \succeq \xi_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c'_2 \succeq \xi_1 \qquad \mathrm{D} \vdash_{\mathcal{C}} c'_1 \otimes c'_2 \otimes c' : \mathsf{CTL}\end{array}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e_1\ e_2 : t_2\ ;\ c'_1 \otimes c'_2 \otimes c'}.$$

We want to show that

$$\mathrm{D}^*; \mathrm{G}^* \vdash [\![e_1\ e_2]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\xi_0}({t_2}^* \multimap \langle\!\langle \tau', c'_1 \otimes c'_2 \otimes c' \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c'_1 \otimes c'_2 \otimes c' \rangle\!\rangle^+_{\mathcal{C}}).$$

Consider whether the term $e_1\, e_2$ has a control effect:

Case $c_1' \otimes c_2' \otimes c' \neq \bot_{\mathcal{C}}$.

By Property 4.2, there exist some $c_1 \neq \bot_{\mathcal{C}}$, $c_2 \neq \bot_{\mathcal{C}}$, and $c \neq \bot_{\mathcal{C}}$ such that

(1) $\mathrm{D} \vdash_{\mathcal{C}} c_1' \preceq c_1$,
(2) $\mathrm{D} \vdash_{\mathcal{C}} c_2' \preceq c_2$,
(3) $\mathrm{D} \vdash_{\mathcal{C}} c' \preceq c$,
(4) $c_1 \otimes c_2 \otimes c = c_1' \otimes c_2' \otimes c'$, and
(5) $\mathrm{D} \vdash_{\mathcal{C}} c_1 \otimes c_2 \otimes c : \mathsf{CTL}$.

From the antecedent of the lemma to be proved,

(6) $\mathrm{D}^* \vdash \xi_0 \preceq (c_1' \otimes c_2' \otimes c')^*$.

Then:

(7) $[\![e_1\, e_2]\!]_{\mathcal{C}} = \lambda y.\, [\![e_1]\!]_{\mathcal{C}}\, (\lambda x_1.\, [\![e_2]\!]_{\mathcal{C}}\, (\lambda x_2.\, x_1\, \_\, x_2\, (\lambda x.\, y\, x)))$
by def. $[\![e]\!]_{\mathcal{C}}$

(8) $\exists \tau_1''.\langle\!\langle \tau_1'', c_1' \rangle\!\rangle_{\mathcal{C}}^- = \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^-$ and $\langle\!\langle \tau_1'', c_1' \rangle\!\rangle_{\mathcal{C}}^+ = \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^+$
by (1), Property 1.4

(9) $\exists \tau_2''.\langle\!\langle \tau_2'', c_2' \rangle\!\rangle_{\mathcal{C}}^- = \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^-$ and $\langle\!\langle \tau_2'', c_2' \rangle\!\rangle_{\mathcal{C}}^+ = \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^+$
by (2), Property 1.4

(10) $\exists \tau''.\langle\!\langle \tau'', c' \rangle\!\rangle_{\mathcal{C}}^- = \langle\!\langle \tau', c \rangle\!\rangle_{\mathcal{C}}^-$ and $\langle\!\langle \tau'', c' \rangle\!\rangle_{\mathcal{C}}^+ = \langle\!\langle \tau', c \rangle\!\rangle_{\mathcal{C}}^+$
by (3), Property 1.4

(11) $\mathrm{D}^*; \mathrm{G}_1{}^* \vdash [\![e_1]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(c_1'{}^*\, ({}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*) \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^+)$
by IH(τ_1''), (8)

(12) $\mathrm{D}^*; \mathrm{G}_2{}^* \vdash [\![e_2]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(c_2'{}^*\, (t_1{}^* \multimap \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^+)$
by IH(τ_2''), (9)

(13) $\mathrm{D}^*; \bullet, x_1{:}{}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*) \vdash x_1 : {}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*)$ by rule T-Var

(14) $\mathrm{D}^*; \bullet, x_1{:}{}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*) \vdash x_1 :$
${}^{\xi_1}(\forall \alpha{:}\star.\, {}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}(c'{}^*\, (t_2{}^* \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \alpha, c' \rangle\!\rangle_{\mathcal{C}}^+)))$
by (13), def. $\bar{t}^*$

(15) $\mathrm{D}^*; \bullet, x_1{:}{}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*) \vdash x_1\, \_ :$
${}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}(c'{}^*\, (t_2{}^* \multimap \langle\!\langle \tau'', c' \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau'', c' \rangle\!\rangle_{\mathcal{C}}^+))$
by (14), rule T-TApp

(16) $\mathrm{D}^*; \bullet, x_1{:}{}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*) \vdash x_1\, \_ :$
${}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}(c'{}^*\, (t_2{}^* \multimap \langle\!\langle \tau', c \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau', c \rangle\!\rangle_{\mathcal{C}}^+))$ by (10, 15)

(17) $\mathrm{D}^*; \bullet, x_2{:}t_1{}^* \vdash x_2 : t_1{}^*$ by rule T-Var

(18) $\mathrm{D}^*; \bullet, x_1{:}{}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*), x_2{:}t_1{}^* \vdash x_1\, \_\, x_2 :$
${}^{\mathsf{L}}(c'{}^*\, (t_2{}^* \multimap \langle\!\langle \tau', c \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau', c \rangle\!\rangle_{\mathcal{C}}^+)$ by (16–17), rule T-App

(19) $\mathrm{D}^*; \bullet, y{:}{}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathcal{C}}^-) \vdash y :$
${}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathcal{C}}^-)$ by rule T-Var

(20) $\mathrm{D}^* \vdash (c_1' \otimes c_2' \otimes c')^* \preceq c'^*$ by Property 3

(21) $\mathrm{D}^* \vdash \xi_0 \preceq c'^*$ by (6, 20), Lemma A.1

(22) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash \lambda x.\, y\, x :$
$^{c'^*}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-)$ by (19, 21), Lemma 5.2

(23) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash \lambda x.\, y\, x : {}^{c'^*}(t_2{}^* \multimap \langle\!\langle \tau', c \rangle\!\rangle_{\mathrm{e}}^-)$ by (22), Property 1.3a

(24) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*), x_2{:}t_1{}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash$
$x_1\ \_\ x_2\ (\lambda x.\, y\, x) : \langle\!\langle \tau', c \rangle\!\rangle_{\mathrm{e}}^+$ by (18, 23), rule T-App

(25) $\mathrm{D}^* \vdash \xi_1 \preceq c_2'{}^*$ by Property 5.2

(26) $\mathrm{D}^* \vdash \xi_0 \preceq c_2'{}^*$ by (6), Property 3, Lemma A.1

(27) $\mathrm{D}^* \vdash \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*), y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \preceq c_2'{}^*$ by (25–26)

(28) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*), y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash$
$\lambda x_2.\, x_1\ \_\ x_2\ (\lambda x.\, y\, x) : {}^{c_2'^*}(t_1{}^* \multimap \langle\!\langle \tau', c \rangle\!\rangle_{\mathrm{e}}^+)$ by (24, 27), rule T-Abs

(29) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*), y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash$
$\lambda x_2.\, x_1\ \_\ x_2\ (\lambda x.\, y\, x) : {}^{c_2'^*}(t_1{}^* \multimap \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathrm{e}}^-)$ by (28), Property 1.3c

(30) $\mathrm{D}^*; \mathrm{G}_2{}^*, x_1{:}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*), y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash$
$[\![e_2]\!]_{\mathrm{e}}\ (\lambda x_2.\, x_1\ \_\ x_2\ (\lambda x.\, y\, x)) : \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathrm{e}}^+$ by (12, 29), rule T-App

(31) $\mathrm{D}^* \vdash \xi_2 \preceq c_1'{}^*$ by Property 5.2

(32) $\mathrm{D}^* \vdash \mathrm{G}_2{}^* \preceq c_1'{}^*$ by (31), Lemma B.2

(33) $\mathrm{D}^* \vdash \xi_0 \preceq c_1'{}^*$ by Property 3, Lemma A.1

(34) $\mathrm{D}^* \vdash \mathrm{G}_2{}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \preceq c_1'{}^*$ by (32–33)

(35) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash$
$\lambda x_1.\, [\![e_2]\!]_{\mathrm{e}}\ (\lambda x_2.\, x_1\ \_\ x_2\ (\lambda x.\, y\, x)) : {}^{c_1'^*}({}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*) \multimap \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathrm{e}}^+)$ by (30, 34), rule T-Abs

(36) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash$
$\lambda x_1.\, [\![e_2]\!]_{\mathrm{e}}\ (\lambda x_2.\, x_1\ \_\ x_2\ (\lambda x.\, y\, x)) : {}^{c_1'^*}({}^{\xi_1}((t_1 \xrightarrow{c'} t_2)^*) \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathrm{e}}^-)$ by (35), Property 1.3c

(37) $\mathrm{D}^* \vdash \mathrm{G}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \rightsquigarrow$
$\mathrm{G}_1{}^* \boxplus \mathrm{G}_2{}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-)$ by Lemma B.2, rule S-ConsR

(38) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau', c_1 \otimes c_2 \otimes c \rangle\!\rangle_{\mathrm{e}}^-) \vdash$
$[\![e_1]\!]_{\mathrm{e}}\ (\lambda x_1.\, [\![e_2]\!]_{\mathrm{e}}\ (\lambda x_2.\, x_1\ \_\ x_2\ (\lambda x.\, y\, x))) : \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathrm{e}}^+$ by (11, 36–37), rule T-App

(39) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.[\![e_1]\!]_{\mathcal{C}}\,(\lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.x_1\_x_2\,(\lambda x.y\,x))) :$
$\mathsf{L}(\xi_0(t_2{}^* \multimap \langle\!|\tau', c_1 \otimes c_2 \otimes c|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau', c_1|\!\rangle_{\mathcal{C}}^+)$ by (38), rule T-Abs

(40) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e_1\,e_2]\!]_{\mathcal{C}} :$
$\mathsf{L}(\xi_0(t_2{}^* \multimap \langle\!|\tau', c_1 \otimes c_2 \otimes c|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau', c_1 \otimes c_2 \otimes c|\!\rangle_{\mathcal{C}}^+)$
by (39), Property 1.3b, (7).

(41) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e_1\,e_2]\!]_{\mathcal{C}} :$
$\mathsf{L}(\xi_0(t_2{}^* \multimap \langle\!|\tau', c_1' \otimes c_2' \otimes c'|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau', c_1' \otimes c_2' \otimes c'|\!\rangle_{\mathcal{C}}^+)$
by (4, 40).

Case $c_1' \otimes c_2' \otimes c' = \bot_{\mathcal{C}}$.

By Property 4.1, $c_1 = c_2 = c = \bot_{\mathcal{C}}$. This is similar to the previous case, but much of the effect management goes away:

(1) $[\![e_1\,e_2]\!]_{\mathcal{C}} = \lambda y.[\![e_1]\!]_{\mathcal{C}}\,(\lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.x_1\_x_2\,(\lambda x.y\,x)))$
by def. $[\![e]\!]_{\mathcal{C}}$

(2) $\mathrm{D}^*; \mathrm{G}_1{}^* \vdash [\![e_1]\!]_{\mathcal{C}} : \mathsf{L}(\mathsf{L}(\xi_1((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*) \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}})$
by IH

(3) $\mathrm{D}^*; \mathrm{G}_2{}^* \vdash [\![e_2]\!]_{\mathcal{C}} : \mathsf{L}(\mathsf{L}(t_1{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}})$
by IH

(4) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*) \vdash x_1 : {}^{\xi_1}((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*)$
by rule T-Var

(5) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*) \vdash x_1 :$
${}^{\xi_1}(\forall\alpha{:}\star.\mathsf{L}(t_1{}^* \multimap \mathsf{L}(\mathsf{L}(t_2{}^* \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\alpha|\!\rangle_{\mathcal{C}})))$
by (4), def. $\bar{t}^*$

(6) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*) \vdash x_1\_ :$
$\mathsf{L}(t_1{}^* \multimap \mathsf{L}(\mathsf{L}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}))$ by (5), rule T-TApp

(7) $\mathrm{D}^*; \bullet, x_2{:}t_1{}^* \vdash x_2 : t_1{}^*$ by rule T-Var

(8) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*), x_2{:}t_1{}^* \vdash x_1\_x_2 :$
$\mathsf{L}(\mathsf{L}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}})$ by (6–7), rule T-App

(9) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \vdash y : {}^{\xi_0}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}})$
by rule T-Var

(10) $\mathrm{D}^* \vdash \xi_0 \preceq \mathsf{L}$ by rule QSub-Top

(11) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \vdash \lambda x.y\,x : \mathsf{L}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}})$
by (9–10), Lemma 5.2

(12) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*), x_2{:}t_1{}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \vdash$
$x_1\_x_2\,(\lambda x.y\,x) : \langle\!|\tau'|\!\rangle_{\mathcal{C}}$ by (8, 11), rule T-App

(13) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*), y{:}^{\xi_0}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \vdash \lambda x_2.x_1\_x_2\,(\lambda x.y\,x) :$
$\mathsf{L}(t_1{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}})$ by (12), rule T-Abs

(14) $\mathrm{D}^*; \mathrm{G}_2{}^*, x_1{:}^{\xi_1}((t_1 \xrightarrow{\bot_{\mathcal{C}}}\!\circ\; t_2)^*), y{:}^{\xi_0}(t_2{}^* \multimap \langle\!|\tau'|\!\rangle_{\mathcal{C}}) \vdash$
$[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.x_1\_x_2\,(\lambda x.y\,x)) : \langle\!|\tau'|\!\rangle_{\mathcal{C}}$ by (3, 13), rule T-App

(15) $\mathrm{D}^*; \mathrm{G}_2^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash \lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.x_1\,\_\,x_2\,(\lambda x.y\,x)) :$
$\mathsf{L}^{(\xi_1}((t_1 \xrightarrow{\perp_{\mathcal{C}}} t_2)^*) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by (14), rule T-Abs

(16) $\mathrm{D}^* \vdash \mathrm{G}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \rightsquigarrow \mathrm{G}_1{}^* \boxplus \mathrm{G}_2^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$
by Lemma B.2, rule S-ConsR

(17) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t_2{}^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash$
$[\![e_1]\!]_{\mathcal{C}}\,(\lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.x_1\,\_\,x_2\,(\lambda x.y\,x))) : \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}$
by (2, 15–16), rule T-App

(18) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.[\![e_1]\!]_{\mathcal{C}}\,(\lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.x_1\,\_\,x_2\,(\lambda x.y\,x))) :$
$\mathsf{L}^{(\xi_0}(t_2{}^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by (17), rule T-Abs

(19) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e_1\,e_2]\!]_{\mathcal{C}} : \mathsf{L}^{(\xi_0}(t_2{}^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$
by (18), (1).

Case $\dfrac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : {}^{\xi}\forall^{c_2'}\beta{:}k.t\,;\,c_1' \qquad \mathrm{D} \vdash_{\mathcal{C}} i : k \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1' \otimes c_2' : \mathsf{CTL}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e\,\_ : \{i/\beta\}t\,;\,c_1' \otimes c_2'}$.

We want to show that

$$\mathrm{D}^*; \mathrm{G}^* \vdash [\![e\,\_]\!]_{\mathcal{C}} : \mathsf{L}^{(\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau', c_1' \otimes c_2' \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau', c_1' \otimes c_2' \rangle\!\rangle_{\mathcal{C}}^+).$$

Consider whether the term $e\,\_$ has a control effect:

Case $c_1' \otimes c_2' \neq \perp_{\mathcal{C}}$.

By Property 4.2, there exist some $c_1' \neq \perp_{\mathcal{C}}$ and $c_2' \neq \perp_{\mathcal{C}}$ such that

(1) $\mathrm{D} \vdash_{\mathcal{C}} c_1' \preceq c_1$,
(2) $\mathrm{D} \vdash_{\mathcal{C}} c_2' \preceq c_2$,
(3) $c_1 \otimes c_2 = c_1' \otimes c_2'$, and
(4) $\mathrm{D} \vdash_{\mathcal{C}} c_1 \otimes c_2 : \mathsf{CTL}$.

From the antecedent of the lemma to be proved,

(5) $\mathrm{D}^* \vdash \xi_0 \preceq (c_1' \otimes c_2')^*$.

Then:

(6) $[\![e\,\_]\!]_{\mathcal{C}} = \lambda y.[\![e]\!]_{\mathcal{C}}\,(\lambda x_1.x_1\,\_\,\_\,(\lambda x.y\,x))$ by def. $[\![e]\!]_{\mathcal{C}}$

(7) $\exists\tau_1''.\langle\!\langle \tau_1'', c_1' \rangle\!\rangle_{\mathcal{C}}^- = \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^-$ and $\langle\!\langle \tau_1'', c_1' \rangle\!\rangle_{\mathcal{C}}^+ = \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^+$
by (1), Property 1.4

(8) $\exists\tau_2''.\langle\!\langle \tau_2'', c_2' \rangle\!\rangle_{\mathcal{C}}^- = \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^-$ and $\langle\!\langle \tau_2'', c_2' \rangle\!\rangle_{\mathcal{C}}^+ = \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^+$
by (2), Property 1.4

(9) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : \mathsf{L}^{(c_1'{}^*}({}^{\xi}((\forall^{c_2'}\beta{:}k.t)^*) \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^+)$
by IH(τ_1''), (7)

(10) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{c_2'}\beta{:}k.t)^*) \vdash x_1 : {}^{\xi}((\forall^{c_2'}\beta{:}k.t)^*)$ by rule T-Var

(11) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{c_2'}\beta{:}k.t)^*) \vdash x_1 :$
$^{\xi}\forall\alpha{:}\star.{}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({c_2'}^*(t^* \multimap \langle\!|\alpha, c_2'|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\alpha, c_2'|\!\rangle_{\mathcal{C}}^+)$
by def. $\bar{t}^*$

(12) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{c_2'}\beta{:}k.t)^*) \vdash x_1 :$
${}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({c_2'}^*(t^* \multimap \langle\!|\tau_2'', c_2'|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau_2'', c_2'|\!\rangle_{\mathcal{C}}^+)$ by (11), rule T-TApp

(13) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{c_2'}\beta{:}k.t)^*) \vdash x_1\ \_ :$
${}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({c_2'}^*(t^* \multimap \langle\!|\tau', c_2|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau', c_2|\!\rangle_{\mathcal{C}}^+)$ by (8, 12)

(14) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{c_2'}\beta{:}k.t)^*) \vdash x_1\ \_\ \_ :$
${}^{\mathsf{L}}({c_2'}^*(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_2|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau', c_2|\!\rangle_{\mathcal{C}}^+)$ by (13), rule T-TApp

(15) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \vdash y :$
$^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-)$ by rule T-Var

(16) $\mathrm{D}^* \vdash \xi_0 \preceq {c_2'}^*$ by (5), Property 3

(17) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \vdash \lambda x.\, y\, x :$
${c_2'}^*(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-)$ by (15–16), Lemma 5.2

(18) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \vdash \lambda x.\, y\, x :$
${c_2'}^*(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_2|\!\rangle_{\mathcal{C}}^-)$ by (17), Property 1.3a

(19) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{c_2'}\beta{:}k.t)^*), y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \vdash$
$x_1\ \_\ \_\ (\lambda x.\, y\, x) : \langle\!|\tau', c_2|\!\rangle_{\mathcal{C}}^+$ by (14, 18), rule T-App

(20) $\mathrm{D}^* \vdash \xi_0 \preceq {c_1'}^*$ by (5), Property 3

(21) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \vdash \lambda x_1.\, x_1\ \_\ \_\ (\lambda x.\, y\, x) :$
${c_1'}^*(^{\xi}((\forall^{c_2'}\beta{:}k.t)^*) \multimap \langle\!|\tau', c_2|\!\rangle_{\mathcal{C}}^+)$ by (19–20), rule T-Abs

(22) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \vdash \lambda x_1.\, x_1\ \_\ \_\ (\lambda x.\, y\, x) :$
${c_1'}^*(^{\xi}((\forall^{c_2'}\beta{:}k.t)^*) \multimap \langle\!|\tau', c_1|\!\rangle_{\mathcal{C}}^-)$ by Property 1.3c

(23) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \vdash$
$[\![e]\!]_{\mathcal{C}}\ (\lambda x_1.\, x_1\ \_\ \_\ (\lambda x.\, y\, x)) : \langle\!|\tau', c_1|\!\rangle_{\mathcal{C}}^+$ by (9, 22), rule T-App

(24) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.\, [\![e]\!]_{\mathcal{C}}\ (\lambda x_1.\, x_1\ \_\ \_\ (\lambda x.\, y\, x)) :$
${}^{\mathsf{L}}(^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau', c_1|\!\rangle_{\mathcal{C}}^+)$
by (23), rule T-Abs

(25) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e\ \_]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau', c_1 \otimes c_2|\!\rangle_{\mathcal{C}}^+)$
by (6, 24), Property 1.3b

(26) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e\ \_]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!|\tau', c_1' \otimes c_2'|\!\rangle_{\mathcal{C}}^-) \multimap \langle\!|\tau', c_1' \otimes c_2'|\!\rangle_{\mathcal{C}}^+)$
by (3, 25)

Case $c_1' \otimes c_2' = \bot_{\mathcal{C}}$.

By Property 4.1, $c_1' = c_2' = \bot_{\mathcal{C}}$. This is similar to the previous case, but much of the effect management goes away:

(1) $[\![e\ \_]\!]_{\mathcal{C}} = \lambda y.\, [\![e]\!]_{\mathcal{C}}\ (\lambda x_1.\, x_1\ \_\ \_\ (\lambda x.\, y\, x))$ by def. $[\![e]\!]_{\mathcal{C}}$

(2) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\mathsf{L}}({}^{\xi}((\forall^{\perp_{\mathcal{C}}}\beta{:}k.t)^*) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by IH

(3) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{\perp_{\mathcal{C}}}\beta{:}k.t)^*) \vdash x_1 : {}^{\xi}((\forall^{\perp_{\mathcal{C}}}\beta{:}k.t)^*)$ by rule T-Var

(4) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{\perp_{\mathcal{C}}}\beta{:}k.t)^*) \vdash x_1 :$
${}^{\xi}\forall\alpha{:}\star.{}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \alpha \rangle\!\rangle_{\mathcal{C}})$ by def. $\bar{t}^*$

(5) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{\perp_{\mathcal{C}}}\beta{:}k.t)^*) \vdash x_1\,\_ : {}^{\mathsf{L}}\forall\beta{:}k^*.{}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by (4), rule T-TApp

(6) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{\perp_{\mathcal{C}}}\beta{:}k.t)^*) \vdash x_1\,\_\,\_ : {}^{\mathsf{L}}({}^{\mathsf{L}}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by (5), rule T-TApp

(7) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash y : {}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by rule T-Var

(8) $\mathrm{D}^* \vdash \xi_0 \preceq \mathsf{L}$ by rule QSub-Top

(9) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash \lambda x.\, y\, x : {}^{\mathsf{L}}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by (7–8), Lemma 5.2

(10) $\mathrm{D}^*; \bullet, x_1{:}^{\xi}((\forall^{\perp_{\mathcal{C}}}\beta{:}k.t)^*), y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash x_1\,\_\,\_\,(\lambda x.\, y\, x) :$
$\langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}$ by (6, 9), rule T-App

(11) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash \lambda x_1.\, x_1\,\_\,\_\,(\lambda x.\, y\, x) :$
${}^{\mathsf{L}}({}^{\xi}((\forall^{\perp_{\mathcal{C}}}\beta{:}k.t)^*) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by (8, 10), rule T-Abs

(12) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \vdash [\![e]\!]_{\mathcal{C}}\,(\lambda x_1.\, x_1\,\_\,\_\,(\lambda x.\, y\, x)) : \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}$ by (2, 11), rule T-App

(13) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.\, [\![e]\!]_{\mathcal{C}}\,(\lambda x_1.\, x_1\,\_\,\_\,(\lambda x.\, y\, x)) :$
${}^{\mathsf{L}}({}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by (12), rule T-Abs

(14) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e\,\_]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\xi_0}(\{i^*/\beta\}t^* \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}}) \multimap \langle\!\langle \tau' \rangle\!\rangle_{\mathcal{C}})$ by (1, 13)

Case $\dfrac{\mathsf{q} \preceq \mathsf{A} \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t\,;c \qquad \mathrm{D} \vdash_{\mathcal{C}} t \preceq \mathsf{A}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{new}^{\mathsf{q}}\, e : {}^{\mathsf{q}}\mathsf{ref}\ t\,;c}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{new}^{\mathsf{q}}\, e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\xi_0}({}^{\mathsf{q}}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^{-}_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^{+}_{\mathcal{C}})$. Then:

(1) $[\![\mathsf{new}^{\mathsf{q}}\, e]\!]_{\mathcal{C}} = \lambda y.\, [\![e]\!]_{\mathcal{C}}\,(\lambda x.\, y\,(\mathsf{new}^{\mathsf{q}}\, x))$ by def. $[\![e]\!]_{\mathcal{C}}$

(2) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^{-}_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^{+}_{\mathcal{C}})$ by IH

(3) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}({}^{\mathsf{q}}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^{-}_{\mathcal{C}}) \vdash y : {}^{\xi_0}({}^{\mathsf{q}}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^{-}_{\mathcal{C}})$ by rule T-Var

(4) $\mathrm{D}^*; \bullet, x{:}t^* \vdash x : t^*$ by rule T-Var

(5) $\mathrm{D}^* \vdash t^* \preceq \mathsf{A}$ by Lemma B.2

(6) $\mathrm{D}^*; \bullet, x{:}t^* \vdash \mathsf{new}^{\mathsf{q}}\, x : {}^{\xi}\mathsf{ref}\ t^*$ by (4–5), rule T-NewUA

(7) $\mathrm{D}^*; \bullet, x{:}t^*, y{:}^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash y\,(\mathsf{new}^{\mathfrak{q}}\,x) : \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}$ by (3, 6), rule T-App

(8) $\mathrm{D}^* \vdash \bullet, y{:}^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \preceq \xi_0$ by rule B-Type

(9) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash \lambda x.y\,(\mathsf{new}^{\mathfrak{q}}\,x) : {}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}})$ by (7–8), rule T-Abs

(10) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash [\![e]\!]_{\mathcal{C}}\,(\lambda x.y\,(\mathsf{new}^{\mathfrak{q}}\,x)) : \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}}$ by (2, 9), rule T-App

(11) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.[\![e]\!]_{\mathcal{C}}\,(\lambda x.y\,(\mathsf{new}^{\mathfrak{q}}\,x)) : {}^{\mathsf{L}}(^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by (10), rule T-Abs

(12) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{new}^{\mathfrak{q}}\,e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by (1, 11).

Case $\dfrac{\mathsf{R} \preceq \mathfrak{q} \qquad \mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : t\,;\, c}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{new}^{\mathfrak{q}}\,e : {}^{\mathfrak{q}}\mathsf{ref}\ t\,;\, c}$.

As in the previous case.

Case $\dfrac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : {}^{\xi}\mathsf{ref}\ t\,;\, c \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathsf{A} \preceq \xi}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{free}\ e : t\,;\, c}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{free}\ e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$. Then:

(1) $[\![\mathsf{free}\ e]\!]_{\mathcal{C}} = \lambda y.[\![e]\!]_{\mathcal{C}}\,(\lambda x.y\,(\mathsf{free}\ x))$ by def. $[\![e]\!]_{\mathcal{C}}$

(2) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by IH

(3) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash y : {}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}})$ by rule T-Var

(4) $\mathrm{D}^*; \bullet, x{:}^{\xi}\mathsf{ref}\ t^* \vdash x : {}^{\xi}\mathsf{ref}\ t^*$ by rule T-Var

(5) $\mathrm{D}^* \vdash \mathsf{A} \preceq \xi$ by Lemma B.2

(6) $\mathrm{D}^*; \bullet, x{:}^{\xi}\mathsf{ref}\ t^* \vdash \mathsf{free}\ x : t^*$ by (4–5), rule T-Delete

(7) $\mathrm{D}^*; \bullet, x{:}^{\xi}\mathsf{ref}\ t^*, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash y\,(\mathsf{free}\ x) : \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}$ by (3, 6), rule T-App

(8) $\mathrm{D}^* \vdash \bullet, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \preceq \xi_0$ by rule B-Type

(9) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash \lambda x.y\,(\mathsf{free}\ x) : {}^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}})$ by (7–8), rule T-Abs

(10) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash [\![e]\!]_{\mathcal{C}}\,(\lambda x.y\,(\mathsf{free}\ x)) : \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}}$ by (2, 9), rule T-App

(11) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.[\![e]\!]_{\mathcal{C}}\,(\lambda x.y\,(\mathsf{free}\ x)) : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by (10), rule T-Abs

(12) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{free}\ e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by (1, 11).

Case $\dfrac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} e : {}^{\xi}\mathsf{ref}\ t\ ;\ c \qquad \mathrm{D} \vdash_{\mathcal{C}} t \preceq \mathsf{R}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{read}\ e : t\ ;\ c}$.

We want to show that $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{read}\ e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$. Then:

(1) $[\![\mathsf{read}\ e]\!]_{\mathcal{C}} = \lambda y.[\![e]\!]_{\mathcal{C}}\,(\lambda x.y\,(\mathsf{read}\ x))$ by def. $[\![e]\!]_{\mathcal{C}}$

(2) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by IH

(3) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash y : {}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}})$ by rule T-Var

(4) $\mathrm{D}^*; \bullet, x{:}^{\xi}\mathsf{ref}\ t^* \vdash x : {}^{\xi}\mathsf{ref}\ t^*$ by rule T-Var

(5) $\mathrm{D}^* \vdash t^* \preceq \mathsf{R}$ by Lemma B.2

(6) $\mathrm{D}^*; \bullet, x{:}^{\xi}\mathsf{ref}\ t^* \vdash \mathsf{read}\ x : t^*$ by (4–5), rule T-Read

(7) $\mathrm{D}^*; \bullet, x{:}^{\xi}\mathsf{ref}\ t^*, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash y\,(\mathsf{read}\ x) : \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}$ by (3, 6), rule T-App

(8) $\mathrm{D}^* \vdash \bullet, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \preceq \xi_0$ by rule B-Type

(9) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash \lambda x.y\,(\mathsf{read}\ x) : {}^{\xi_0}(^{\xi}\mathsf{ref}\ t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}})$ by (7–8), rule T-Abs

(10) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \vdash [\![e]\!]_{\mathcal{C}}\,(\lambda x.y\,(\mathsf{read}\ x)) : \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}}$ by (2, 9), rule T-App

(11) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.[\![e]\!]_{\mathcal{C}}\,(\lambda x.y\,(\mathsf{read}\ x)) : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by (10), rule T-Abs

(12) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{read}\ e]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap \langle\!\langle \tau', c \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c \rangle\!\rangle^+_{\mathcal{C}})$ by (1, 11).

Case $\dfrac{\begin{array}{c}\mathrm{D} \vdash_{\mathcal{C}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \qquad \mathrm{D}; \mathrm{G}_1 \vdash_{\mathcal{C}} e_1 : {}^{\xi_1}\mathsf{ref}\ t_1\ ;\ c_1' \\ \mathrm{D}; \mathrm{G}_2 \vdash_{\mathcal{C}} e_2 : t_2\ ;\ c_2' \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathrm{G}_2 \preceq \xi_2 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1' \succeq \xi_2 \\ \mathrm{D} \vdash_{\mathcal{C}} c_2' \succeq \xi_1 \qquad \mathrm{D} \vdash_{\mathcal{C}} \mathsf{A} \preceq \xi_1 \qquad \mathrm{D} \vdash_{\mathcal{C}} t_2 \preceq \xi_1 \qquad \mathrm{D} \vdash_{\mathcal{C}} c_1' \otimes c_2' : \mathsf{CTL}\end{array}}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{C}} \mathsf{swap}\ e_1\, e_2 : {}^{\mathsf{L}}(^{\xi}\mathsf{ref}\ t_2 \otimes t_1)\ ;\ c_1' \otimes c_2'}$.

We want to show that

$$\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{swap}\ e_1\, e_2]\!]_{\mathcal{C}} : {}^{\mathsf{L}}(^{\xi_0}({}^{\mathsf{L}}(^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \otimes c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c_1 \otimes c_2 \rangle\!\rangle^+_{\mathcal{C}}).$$

Consider whether the term $\mathsf{swap}\ e_1\ e_2$ has a control effect:

Case $c_1' \oslash c_2' \neq \bot_{\mathcal{C}}$.

By Property 4.2, there exist some $c_1 \neq \bot_{\mathcal{C}}$ and $c_2 \neq \bot_{\mathcal{C}}$ such that

(1) $\mathrm{D} \vdash_{\mathcal{C}} c_1' \preceq c_1$,
(2) $\mathrm{D} \vdash_{\mathcal{C}} c_2' \preceq c_2$,
(3) $c_1 \oslash c_2 = c_1' \oslash c_2'$, and
(4) $\mathrm{D} \vdash_{\mathcal{C}} c_1 \oslash c_2 : \mathsf{CTL}$.

From the antecedent of the lemma to be proved,

(5) $\mathrm{D}^* \vdash \xi_0 \preceq (c_1' \oslash c_2')^*$.

Then:

(6) $[\![\mathsf{swap}\ e_1\ e_2]\!]_{\mathcal{C}} = \lambda y.[\![e_1]\!]_{\mathcal{C}}\ (\lambda x_1.[\![e_2]\!]_{\mathcal{C}}\ (\lambda x_2.y\ (\mathsf{swap}\ x_1\ x_2)))$ by def. $[\![e]\!]_{\mathcal{C}}$

(7) $\exists \tau_1''.\langle\!\langle \tau_1'', c_1' \rangle\!\rangle_{\mathcal{C}}^- = \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^-$ and $\langle\!\langle \tau_1'', c_1' \rangle\!\rangle_{\mathcal{C}}^+ = \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^+$ by (1), Property 1.4

(8) $\exists \tau_2''.\langle\!\langle \tau_2'', c_2' \rangle\!\rangle_{\mathcal{C}}^- = \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^-$ and $\langle\!\langle \tau_2'', c_2' \rangle\!\rangle_{\mathcal{C}}^+ = \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^+$ by (2), Property 1.4

(9) $\mathrm{D}^*; \mathrm{G}_1{}^* \vdash [\![e_1]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{c_1'{}^*}({}^{\xi_1}\mathsf{ref}\ t_1{}^* \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle_{\mathcal{C}}^+)$ by IH(τ_1''), (7)

(10) $\mathrm{D}^*; \mathrm{G}_2{}^* \vdash [\![e_2]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{c_2'{}^*}(t_2{}^* \multimap \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^-) \multimap \langle\!\langle \tau', c_2 \rangle\!\rangle_{\mathcal{C}}^+)$ by IH(τ_1''), (8)

(11) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}\mathsf{ref}\ t_1{}^* \vdash x_1 : {}^{\xi_1}\mathsf{ref}\ t_1{}^*$ by rule T-Var

(12) $\mathrm{D}^*; \bullet, x_2{:}t_2{}^* \vdash x_2 : t_2{}^*$ by rule T-Var

(13) $\mathrm{D}^* \vdash \mathsf{A} \preceq \xi_1$ by Lemma B.2

(14) $\mathrm{D}^* \vdash t_2{}^* \preceq \xi_1$ by Lemma B.2

(15) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}\mathsf{ref}\ t_1{}^*, x_2{:}t_2{}^* \vdash \mathsf{swap}\ x_1\ x_2 : {}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ t_2{}^* \otimes t_1{}^*)$ by (11–13), (14), rule T-SwapStrong

(16) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ t_2{}^* \otimes t_1{}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle_{\mathcal{C}}^-) \vdash y : {}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ t_2{}^* \otimes t_1{}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle_{\mathcal{C}}^-)$ by rule T-Var

(17) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}\mathsf{ref}\ t_1{}^*, x_2{:}t_2{}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ t_2{}^* \otimes t_1{}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle_{\mathcal{C}}^-) \vdash y\ (\mathsf{swap}\ x_1\ x_2) : \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle_{\mathcal{C}}^-$ by (15–16), rule T-App

(18) $\mathrm{D}^* \vdash \xi_1 \preceq c_2'{}^*$ by Property 5.2

(19) $\mathrm{D}^* \vdash \xi_0 \preceq c_2'{}^*$ by (5), Property 3

(20) $\mathrm{D}^* \vdash \bullet, x_1{:}^{\xi_1}\mathsf{ref}\ t_1{}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ t_2{}^* \otimes t_1{}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle_{\mathcal{C}}^-) \preceq c_2'{}^*$ by rule B-Cons, (18–19), ...

(21) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}\mathsf{ref}\ {t_1}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \vdash \lambda x_2.y\,(\mathsf{swap}\ x_1\ x_2) : {c'_2}^*({t_2}^* \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}})$ by (17, 20), rule T-Abs

(22) $\mathrm{D}^*; \bullet, x_1{:}^{\xi_1}\mathsf{ref}\ {t_1}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \vdash \lambda x_2.y\,(\mathsf{swap}\ x_1\ x_2) : {c'_2}^*({t_2}^* \multimap \langle\!\langle \tau', c_2 \rangle\!\rangle^-_{\mathcal{C}})$ by (21), Property 1.3a

(23) $\mathrm{D}^*; {\mathrm{G}_2}^*, x_1{:}^{\xi_1}\mathsf{ref}\ {t_1}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \vdash [\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.y\,(\mathsf{swap}\ x_1\ x_2)) : \langle\!\langle \tau', c_2 \rangle\!\rangle^+_{\mathcal{C}}$ by (10, 22), rule T-App

(24) $\mathrm{D}^* \vdash \xi_2 \preceq {c'_1}^*$ by Property 5.2

(25) $\mathrm{D}^* \vdash {\mathrm{G}_2}^* \preceq \xi_2$ by Lemma B.2

(26) $\mathrm{D}^* \vdash {\mathrm{G}_2}^* \preceq {c'_1}^*$ by (24–25), ind. ${\mathrm{G}_2}^*$, ...

(27) $\mathrm{D}^* \vdash \xi_0 \preceq {c'_1}^*$ by (5), Property 3

(28) $\mathrm{D}^* \vdash {\mathrm{G}_2}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \preceq {c'_1}^*$ by rule B-Cons, (24, 27), ...

(29) $\mathrm{D}^*; {\mathrm{G}_2}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \vdash \lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.y\,(\mathsf{swap}\ x_1\ x_2)) : {c'_1}^*({}^{\xi_1}\mathsf{ref}\ {t_1}^* \multimap \langle\!\langle \tau', c_2 \rangle\!\rangle^+_{\mathcal{C}})$ by (23, 28), rule T-Abs

(30) $\mathrm{D}^*; {\mathrm{G}_2}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \vdash \lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.y\,(\mathsf{swap}\ x_1\ x_2)) : {c'_1}^*({}^{\xi_1}\mathsf{ref}\ {t_1}^* \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle^-_{\mathcal{C}})$ by (29), Property 1.3c

(31) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \vdash [\![e_1]\!]_{\mathcal{C}}\,(\lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.y\,(\mathsf{swap}\ x_1\ x_2))) : \langle\!\langle \tau', c_1 \rangle\!\rangle^+_{\mathcal{C}}$ by (9, 30), rule T-App

(32) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.[\![e_1]\!]_{\mathcal{C}}\,(\lambda x_1.[\![e_2]\!]_{\mathcal{C}}\,(\lambda x_2.y\,(\mathsf{swap}\ x_1\ x_2))) : {}^{\mathsf{L}}({}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c_1 \rangle\!\rangle^+_{\mathcal{C}})$ by (31), rule T-Abs

(33) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{swap}\ e_1\ e_2]\!]_{\mathcal{C}} : {}^{\mathsf{L}}({}^{\xi_0}({}^{\mathsf{L}}({}^{\xi}\mathsf{ref}\ {t_2}^* \otimes {t_1}^*) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^-_{\mathcal{C}}) \multimap \langle\!\langle \tau', c_1 \oslash c_2 \rangle\!\rangle^+_{\mathcal{C}})$ by (32), Property 1.3b, (6).

Case $c'_1 \oslash c'_2 = \bot_{\mathcal{C}}$**.**

As in the previous case, but with all the control effect manipulation elided, because all the control effects are pure. This follows from the impure case just as the pure cases for application and type application from from their impure cases.

$$\textbf{Case}\ \frac{\mathsf{D} \vdash_{\mathsf{e}} \mathsf{G} \rightsquigarrow \mathsf{G}_1 \boxplus \mathsf{G}_2 \qquad \mathsf{D}; \mathsf{G}_1 \vdash_{\mathsf{e}} e_1 : {}^{\xi_1}\mathsf{ref}\ t\ ;\ c_1 \qquad \mathsf{D}; \mathsf{G}_2 \vdash_{\mathsf{e}} e_2 : t\ ;\ c_2 \qquad \mathsf{D} \vdash_{\mathsf{e}} \mathsf{G}_2 \preceq \xi_2 \qquad \mathsf{D} \vdash_{\mathsf{e}} c_1 \succeq \xi_2 \qquad \mathsf{D} \vdash_{\mathsf{e}} c_2 \succeq \xi_1 \qquad \mathsf{D} \vdash_{\mathsf{e}} c_1 \oslash c_2 : \mathsf{CTL}}{\mathsf{D}; \mathsf{G} \vdash_{\mathsf{e}} \mathsf{swap}\ e_1\ e_2 : {}^{\mathsf{L}}(^{\xi}\mathsf{ref}\ t \otimes t)\ ;\ c_1 \oslash c_2}.$$

As in the previous case. □

Corollary 5.8 (Translation of program typing, restated from p. 29)**.**

If $\mathsf{D}; \mathsf{G} \vdash_{\mathsf{e}} e : t\ ;\ \bot_{\mathsf{e}}$ *where* $\mathsf{D} \vdash_{\mathsf{e}} t \preceq \mathsf{A}$*, then*

$$\mathsf{D}^*; \mathsf{G}^* \vdash [\![e]\!]_{\mathsf{e}}\ \mathsf{done}_{\mathsf{e}} : \langle\!\langle t^* \rangle\!\rangle_{\mathsf{e}}.$$

Proof. Then:

(1)	$\mathsf{D}^* \vdash t^* \preceq \mathsf{A}$	by Lemma B.2
(2)	$\mathsf{D}^* \vdash t^* : \star$	by Lemma B.1, Lemma 5.4
(3)	$\mathsf{D} \vdash_{\mathsf{e}} \mathsf{L} \preceq \bot_{\mathsf{e}}{}^*$	by rule QSub-Refl
(4)	$\mathsf{D}^*; \mathsf{G}^* \vdash [\![e]\!]_{\mathsf{e}} : {}^{\mathsf{L}}({}^{\mathsf{L}}(t^* \multimap \langle\!\langle t^* \rangle\!\rangle_{\mathsf{e}}) \multimap \langle\!\langle t^* \rangle\!\rangle_{\mathsf{e}})$	by prem., (2–3), Lemma 5.7
(5)	$\mathsf{D}^*; \bullet \vdash \mathsf{done}_{\mathsf{e}} : {}^{\mathsf{L}}(t^* \multimap \langle\!\langle t^* \rangle\!\rangle_{\mathsf{e}})$	by (1–2), Property 2
(6)	$\mathsf{D}^* \vdash \mathsf{G}^* \rightsquigarrow \mathsf{G}^* \boxplus \bullet$	by ind. G^*, rule S-ConsL
(7)	$\mathsf{D}^*; \mathsf{G}^* \vdash [\![e]\!]_{\mathsf{e}}\ \mathsf{done}_{\mathsf{e}} : \langle\!\langle t^* \rangle\!\rangle_{\mathsf{e}}$	by (4–6), rule T-App. □

C Proofs for Example Control Effects

In this section, we prove that each of the control effects in §6 meets the control effect parameter soundness criteria.

C.1 Delimited Continuation Effects

In this section, we consider the delimited continuation effects from §6.1.

Lemma C.1 (Top)**.**
If $d^* = \mathsf{U}$ *then* $(d \sqcup d')^* = \mathsf{U}$.

Proof. Assuming $d^* = \mathsf{U}$, we proceed by cases on $d \sqcup d'$:

Case α.

Then $(d \sqcup d')^* = \alpha$. By the quotienting of d, $d \sqcup d' = \alpha$ only if d is α or $\bot_{\mathcal{D}}$. If $d = \alpha$ then $d^* = \alpha$, which contradicts our assumption. If $d = \bot_{\mathcal{D}}$ then $d^* = \mathsf{L}$, which also contradicts our assumption.

Case $\bot_{\mathcal{D}}$.

Then $(d \sqcup d')^* = \mathsf{L}$. By the quotienting of d, $d \sqcup d' = \bot_{\mathcal{D}}$ only if $d = \bot_{\mathcal{D}}$, which means that $d^* = \mathsf{L}$, which contradicts our assumption.

Case $\overline{\xi}$.

Then $(d \sqcup d')^* = \xi$. By the quotienting of d, we can have $d \sqcup d' = \overline{\xi}$ in one of two ways:

Case $d = \overline{\xi_1}$ and $d' = \overline{\xi_2}$ and $\xi_1 \sqcap \xi_2 = \xi$.

That is, $(d \sqcup d')^* = \xi_1 \sqcap \xi_2$ and $d^* = \xi_1$. Then $\xi_1 = \mathsf{U}$, which means that $(d \sqcup d')^* = \xi_1 \sqcap \xi_2 = \mathsf{U} \sqcap \xi_2 = \mathsf{U}$.

Case $d = d' = \overline{\xi}$.

This is subsumed by the previous case.

Otherwise.

Then $(d \sqcup d')^* = \mathsf{U}$, which is the desired conclusion. □

Theorem 6.2 (Delimited continuation properties, restated from p. 32)**.**

Delimited continuation effects $(\mathcal{D}, \bot_{\mathcal{D}}, \sqcup)$ *satisfy Properties 1–5.*

Proof.

Property 1 (Answer types). All properties here are trivial because $\langle\!\langle \tau, d \rangle\!\rangle^-_{\mathcal{D}} = \langle\!\langle \tau, d \rangle\!\rangle^+_{\mathcal{D}} = {}^{\mathsf{U}}1$.

Property 2 (Done). Then:

(1)	$\Delta; \bullet \vdash \langle\rangle : {}^{\mathsf{U}}1$	by rules T-Unit and K-Qual
(2)	$\Delta \vdash \bullet, x{:}\tau \preceq \mathsf{A}$	by rules B-Nil and B-Cons
(3)	$\Delta \vdash \bullet, x{:}\tau \rightsquigarrow \bullet \boxplus \bullet, x{:}\tau$	by rules S-Nil and S-ConsR
(4)	$\Delta; \bullet, x{:}\tau \vdash \langle\rangle : {}^{\mathsf{U}}1$	by (1–3), rule T-Weak
(5)	$\Delta; \bullet \vdash \lambda x.\langle\rangle : {}^{\mathsf{L}}(\tau \multimap {}^{\mathsf{U}}1)$	by (4), rule T-Abs
(6)	$\langle\!\langle \tau \rangle\!\rangle_{\mathcal{D}} = {}^{\mathsf{U}}1$	by def. $\langle\!\langle \tau \rangle\!\rangle_{\mathcal{D}}$

(7) $\Delta; \bullet \vdash \lambda x.\langle\rangle : {}^{\mathsf{L}}(\tau \multimap \langle\!\langle \tau \rangle\!\rangle_{\mathcal{D}})$ by (5–6).

Property 3 (Effect sequencing). Let $\mathrm{D} \vDash_{\mathfrak{E}} d_1 \sqcup d_2 : \mathsf{CTL}$. We must show that $\mathrm{D}^* \vdash (d_1 \sqcup d_2)^* \preceq {d_1}^*$ and $\mathrm{D}^* \vdash (d_1 \sqcup d_2)^* \preceq {d_2}^*$. By symmetry, it suffices to show the former. Then:

(1) $\mathrm{D} \vDash_{\mathcal{D}} d_1 \preceq d_1$ by rule CSub-Trans

(2) $\mathrm{D} \vDash_{\mathcal{D}} \bot_{\mathcal{D}} \preceq d_2$ by rule DSub-Bot

(3) $\mathrm{D} \vDash_{\mathcal{D}} d_1 \sqcup \bot_{\mathcal{D}} \preceq d_1 \sqcup d_2$ by (1–2), rule DSub-Join

(4) $\mathrm{D} \vDash_{\mathcal{D}} d_1 \preceq d_1 \sqcup d_2$ by (3), quotient

(5) $\mathrm{D}^* \vdash (d_1 \sqcup d_2)^* \preceq {d_1}^*$ by Lemma 5.6.

Property 4 (Bottom and lifting).

1. To show that $d_1 \sqcup d_2 = \bot_{\mathcal{D}}$ if and only if $d_1 = d_2 = \bot_{\mathcal{D}}$, we consider the quotienting of $\mathcal{D}$.
2. We must also show that if $\mathrm{D} \vDash_{\mathcal{D}} d_1 \sqcup d_2 : \mathsf{CTL}$ and $d_1 \sqcup d_2 \neq \bot_{\mathcal{D}}$, then there exist some $d_1' \neq \bot_{\mathcal{D}}$ and $d_2' \neq \bot_{\mathcal{D}}$ with particular properties. For each ${d_i}^{(i \in \{1,2\})}$, if $d_i = \bot_{\mathcal{D}}$ then let $d_i' = \overline{\mathsf{L}}$; otherwise, let $d_i' = d_i$. This ensures that 1–2) each $\mathrm{D} \vDash_{\mathcal{D}} d_i \preceq d_i'$, 3) $d_1 \sqcup d_2 = d_1' \sqcup d_2'$, and 4) $d_1' \sqcup d_2'$ is well formed.

Property 5 (New rules).

1. For translation of effect bounds, let $\mathrm{D} \vDash_{\mathcal{D}} d \succeq \xi$; we need to show that $\mathrm{D}^* \vdash \xi \preceq d^*$. We proceed by induction on the derivation of $\mathrm{D} \vDash_{\mathcal{D}} d \succeq \xi$, which has two new cases to consider:

   Case $\dfrac{\mathrm{D} \vDash_{\mathfrak{E}} \xi \preceq \xi'}{\mathrm{D} \vDash_{\mathcal{D}} \overline{\xi'} \succeq \xi}$.

   Then $\overline{\xi'}^* = \xi'$, and by Lemma B.2, $\mathrm{D}^* \vdash \xi \preceq \xi'$.

   Case $\dfrac{\mathrm{D} \vDash_{\mathcal{D}} d_1 \succeq \xi \qquad \mathrm{D} \vDash_{\mathcal{D}} d_2 \succeq \xi}{\mathrm{D} \vDash_{\mathcal{D}} d_1 \sqcup d_2 \succeq \xi}$.

   By the induction hypothesis,

   (1) $\mathrm{D}^* \vdash \xi \preceq {d_1}^*$ and

   (2) $\mathrm{D}^* \vdash \xi \preceq {d_2}^*$.

   Now we consider several possibilities for $d_1 \sqcup d_2$ in light of the quotienting of $\mathcal{D}$:

   – If $d_1 = d_2$ then $d_1 \sqcup d_2 = d_1$, so we have $\mathrm{D}^* \vdash \xi \preceq (d_1 \sqcup d_2)^*$ by (1).

- If $d_1 = \bot_{\mathcal{D}}$, then $d_1 \sqcup d_2 = d_2$, and thus $\mathrm{D}^* \vdash \xi \preceq (d_1 \sqcup d_2)^*$ by (2).
- By symmetry with the previous case if $d_2 = \bot_{\mathcal{D}}$.
- If $d_1 = \overline{\xi_1}$ and $d_2 = \overline{\xi_2}$ where $\xi_1 \sqcap \xi_2$ is defined, then
  (3) $d_1 \sqcup d_2 = \overline{(\xi_1 \sqcap \xi_2)}$.

  Then:

  (4) $\mathrm{D}^* \vdash \xi \preceq \xi_1$ by (1), def. $\overline{\xi_1}^*$
  (5) $\mathrm{D}^* \vdash \xi \preceq \xi_2$ by (2), def. $\overline{\xi_2}^*$
  (6) $\mathrm{D}^* \vdash \xi \preceq \xi_1 \sqcap \xi_2$ by (4–5), Lemma A.2
  (7) $\mathrm{D}^* \vdash \xi \preceq (d_1 \sqcup d_2)^*$ by (3, 6), def. $\overline{\xi}^*$
- If $d_1 = \overline{\xi_1}$ and $d_2 = \overline{\xi_2}$ where $\xi_1 \sqcap \xi_2$ is not defined, then by Lemma A.3 and (1–2), $\xi = \mathsf{U}$. Then by rule QSub-Bot.
- If $d_1 = \alpha$, then since $\mathrm{D} \vdash_{\mathcal{D}} \alpha \succeq \xi$, we know by inversion that $\xi = \mathsf{U}$. Then by rule QSub-Bot.
- If $d_2 = \alpha$, then by symmetry with the previous case.

2. For translation of effect subsumption, let $\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_2$; we must show that $\mathrm{D}^* \vdash d_2^* \preceq d_1^*$. We proceed by induction on the derivation of $\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_2$, which has several cases to consider:

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{D}} d : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{D}} \bot_{\mathcal{D}} \preceq d}$.
Then $\mathrm{D}^* \vdash d^* \preceq \mathsf{L}$ by rule QSub-Top.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{E}} \xi : \mathsf{QUAL}}{\mathrm{D} \vdash_{\mathcal{D}} \overline{\mathsf{L}} \preceq \overline{\xi}}$.
Then $\mathrm{D}^* \vdash \xi \preceq \mathsf{L}$ by rule QSub-Top.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{D}} d : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{D}} d \preceq \overline{\mathsf{U}}}$.
Then $\mathrm{D}^* \vdash \mathsf{U} \preceq d^*$ by rule QSub-Bot.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_1' \quad \mathrm{D} \vdash_{\mathcal{D}} d_2 \preceq d_2' \qquad \mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 : \mathsf{CTL} \quad \mathrm{D} \vdash_{\mathcal{D}} d_1' \sqcup d_2' : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{D}} d_1 \sqcup d_2 \preceq d_1' \sqcup d_2'}$.
Without loss of generality, let

(1) $\mathrm{D} = \bullet, \alpha{:}\mathsf{QUAL}, \beta{:}\mathsf{QUAL}, \alpha'{:}\mathsf{CTL}, \beta'{:}\mathsf{CTL}$.

By cases on the possibilities for d_1, d_2, d_1', and d_2', we construct a table showing all the cases in Figure 19. We label the rows with instances of the judgment (i) $\mathrm{D} \vdash_{\mathcal{D}} d_1 \preceq d_1'$ and the columns with instances of the judgment (ii) $\mathrm{D} \vdash_{\mathcal{D}} d_2 \preceq d_2'$. Using Lemma C.1, we fill the cells with instances of $\mathrm{D}^* \vdash (d_1' \sqcup d_2')^* \preceq (d_1 \sqcup d_2)^*$.

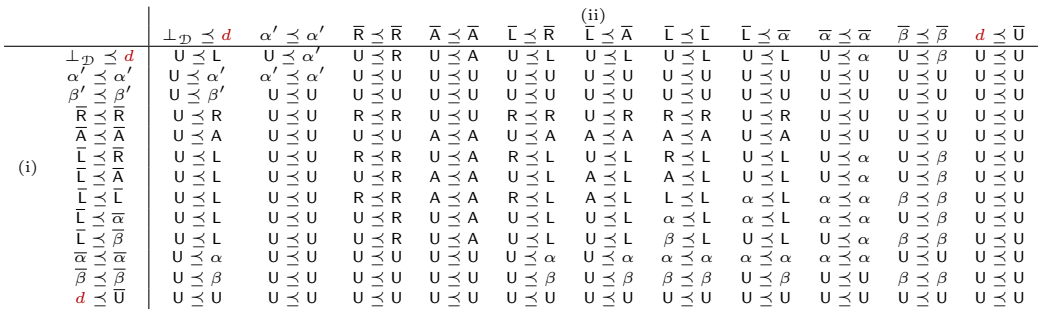

Figure 19: Exhaustive proof for final case in translation subsumption

3. For translation of kinding, let $\mathrm{D} \vdash_{\mathfrak{e}} d : \mathsf{CTL}$. We must show that $\mathrm{D}^* \vdash d^* : \mathsf{QUAL}$. We proceed by induction on the derivation, considering the two new kinding rules:

   Case $\dfrac{\mathrm{D} \vdash_{\mathfrak{e}} \xi : \mathsf{QUAL}}{\mathrm{D} \vdash_{\mathfrak{D}} \overline{\xi} : \mathsf{CTL}}$**.**

   By the induction hypothesis, $\mathrm{D}^* \vdash \xi : \mathsf{QUAL}$, noting that $(\overline{\xi})^* = \xi$ and $\mathsf{CTL}^* = \mathsf{QUAL}$.

   Case $\dfrac{\mathrm{D} \vdash_{\mathfrak{e}} d_1 : \mathsf{CTL} \qquad \mathrm{D} \vdash_{\mathfrak{e}} d_2 : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathfrak{D}} d_1 \sqcup d_2 : \mathsf{CTL}}$**.**

   By cases on $d_1 \sqcup d_2$:

   Case α**.**

   Then $(d_1 \sqcup d_2)^* = \alpha$. By the quotienting of d, $d_1 \sqcup d_2 = \alpha$ only if at least one of d_1 or d_2 is α. From the premises, we know that $\mathrm{D} \vdash_{\mathfrak{e}} \alpha : \mathsf{CTL}$, which means that α:CTL $\in \mathrm{D}$. This means that α:QUAL $\in \mathrm{D}^*$, so $\mathrm{D}^* \vdash \alpha : \mathsf{QUAL}$.

   Case $\bot_{\mathfrak{D}}$**.**

   Then $(d_1 \sqcup d_2)^* = \mathsf{L}$, so $\mathrm{D}^* \vdash \mathsf{L} : \mathsf{QUAL}$ by rule K-Qual.

   Case $\overline{\xi}$**.**

   Then $(d_1 \sqcup d_2)^* = \xi$. By the quotienting of d, $d_1 \sqcup d_2 = \overline{\xi}$ in one of two ways:

   Case $d_1 = \overline{\xi_1}$ and $d_2 = \overline{\xi_2}$ and $\xi_1 \sqcap \xi_2 = \xi$**.**

   That is,

   (1) $(d_1 \sqcup d_2)^* = \xi_1 \sqcap \xi_2$,

   (2) ${d_1}^* = \xi_1$, and

   (3) ${d_2}^* = \xi_2$.

   By the induction hypothesis, twice, we have that

   (4) $\mathrm{D}^* \vdash {d_1}^* : \mathsf{QUAL}$ and

   (5) $\mathrm{D}^* \vdash {d_2}^* : \mathsf{QUAL}$,

   and by substitution (2–3).

(6) $\mathrm{D}^* \vdash \xi_1 : \mathsf{QUAL}$ and

(7) $\mathrm{D}^* \vdash \xi_2 : \mathsf{QUAL}$,

Since $\xi_1 \sqcap \xi_2 = \xi$, the meet is defined. By the definition of meet, $\xi_1 \sqcap \xi_2$ is either a constant qualifier $\mathfrak{q}$, which has a kind by rule K-Qual, or it is a identical to at least one of ξ_1 or ξ_2, both of which are well-kinded under D^*.

Case $d_1 = d_2 = \overline{\xi}$**.**

This is subsumed by the previous case.

Otherwise.

Then $(d_1 \sqcup d_2)^* = \mathsf{U}$, so $\mathrm{D}^* \vdash \mathsf{U} : \mathsf{QUAL}$ by rule K-Qual.

4. For translation of typing, let
   - $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{D}} e : t \,;\, d$.
   - $\mathrm{D}^* \vdash \xi_0 \preceq d^*$, and
   - $\mathrm{D}^* \vdash \tau' : \star$.

   We must show that $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{D}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$. We proceed by induction on the typing derivation, with two cases to consider:

   Case $\dfrac{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{D}} e' : {}^{\mathsf{U}}1 \,;\, d'}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{D}} \mathsf{reset}\ e' : {}^{\mathsf{U}}1 \,;\, \bot_{\mathcal{D}}}$**.**

   We must show that $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e']\!]_{\mathcal{D}} : {}^{\mathsf{L}}(^{\xi_0}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$. Then,

   (1) $[\![\mathsf{reset}\ e']\!]_{\mathcal{D}} = \lambda y.\, y\,([\![e']\!]_{\mathcal{D}}\,(\lambda x.x))$ by def. $[\![e]\!]_{\mathcal{D}}$

   (2) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e']\!]_{\mathcal{D}} : {}^{\mathsf{L}}(^{d'^*}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$ by IH

   (3) $\mathrm{D}^*; \bullet, x{:}{}^{\mathsf{U}}1 \vdash x : {}^{\mathsf{U}}1$ by rule T-Var

   (4) $\mathrm{D}^*; \bullet \vdash \lambda x.x : {}^{d'^*}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1)$ by (3), rule T-Abs

   (5) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e']\!]_{\mathcal{D}}\,(\lambda x.x) : {}^{\mathsf{U}}1$ by (2, 4), rule T-App

   (6) $\mathrm{D}^*; \bullet, y{:}{}^{\xi_0}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \vdash y : {}^{\xi_0}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1)$ by rule T-Var

   (7) $\mathrm{D}^*; \mathrm{G}^*, y{:}{}^{\xi_0}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \vdash y\,([\![e']\!]_{\mathcal{D}}\,(\lambda x.x)) : {}^{\mathsf{U}}1$ by (5–6), rule T-App

   (8) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.\, y\,([\![e']\!]_{\mathcal{D}}\,(\lambda x.x)) : {}^{\mathsf{L}}(^{\xi_0}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$ by (7),rule T-Abs

   (9) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{reset}\ e']\!]_{\mathcal{D}} : {}^{\mathsf{L}}(^{\xi_0}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$ by (1, 8).

   Case $\dfrac{\mathrm{D}; \mathrm{G}, y'{:}{}^{\xi}(t \xrightarrow{\bot_{\mathcal{D}}} {}^{\mathsf{U}}1) \vdash_{\mathcal{D}} e' : {}^{\mathsf{U}}1 \,;\, d'}{\mathrm{D}; \mathrm{G} \vdash_{\mathcal{D}} \mathsf{shift}\ y'\ \mathsf{in}\ e' : t \,;\, d' \sqcup \overline{\xi}}$**.**

   We must show that $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{shift}\ y'\ \mathsf{in}\ e']\!]_{\mathcal{D}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$. By our premises, we know that

   (1) $\mathrm{D}^* \vdash \xi_0 \preceq (d' \sqcup \overline{\xi})^*$

Now, consider whether $d'^* \sqcap \xi$ is defined: If so, then $\mathrm{D}^* \vdash \xi_0 \preceq d'^* \sqcap \xi$, which means that $\mathrm{D}^* \vdash \xi_0 \preceq \xi$ by Lemma A.2. If the meet is not defined, then $(d' \sqcup \overline{\xi})^* = \mathsf{U}$, which means that $\mathrm{D}^* \vdash \xi_0 \preceq \mathsf{U}$. In either case, we have that

(2) $\mathrm{D}^* \vdash \xi_0 \preceq \xi$.

Then,

(3) $[\![\mathsf{shift}\, y' \,\mathsf{in}\, e']\!]_{\mathcal{D}} = \lambda y.\, (\lambda y'.\, [\![e']\!]_{\mathcal{D}}\, (\lambda x.x))\, (\Lambda.\lambda x.\lambda y''.y''\,(y\,x))$ by def. $[\![e]\!]_{\mathcal{D}}$

(4) $\mathrm{D}^*; \mathrm{G}^*, y'{:}({}^{\xi}(t \xrightarrow{\perp_{\mathcal{D}}}\!\circ\ {}^{\mathsf{U}}1))^* \vdash [\![e']\!]_{\mathcal{D}} : {}^{\mathsf{L}}({}^{d'^*}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$ by IH

(5) $\mathrm{D}^*; \bullet \vdash \lambda x.x : {}^{d'^*}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1)$ by rules T-Var and T-Abs

(6) $\mathrm{D}^*; \mathrm{G}^*, y'{:}({}^{\xi}(t \xrightarrow{\perp_{\mathcal{D}}}\!\circ\ {}^{\mathsf{U}}1))^* \vdash [\![e']\!]_{\mathcal{D}}\, (\lambda x.x) : {}^{\mathsf{U}}1$ by (4–5), rule T-App

(7) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y'.\, [\![e']\!]_{\mathcal{D}}\, (\lambda x.x) : {}^{\mathsf{L}}(({}^{\xi}(t \xrightarrow{\perp_{\mathcal{D}}}\!\circ\ {}^{\mathsf{U}}1))^* \multimap {}^{\mathsf{U}}1)$ by (6), rule T-Abs

(8) $\mathrm{D}^*, \alpha{:}\star; \bullet, y''{:}{}^{\mathsf{L}}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \vdash y'' : {}^{\mathsf{L}}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1)$ by rule T-Var

(9) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \vdash y : {}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1)$ by rule T-Var

(10) $\mathrm{D}^*, \alpha{:}\star; \bullet, x{:}t^* \vdash x : t^*$ by rule T-Var

(11) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1), x{:}t^* \vdash y\,x : {}^{\mathsf{U}}1$ by (9–10), rule T-App

(12) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1), x{:}t^*, y''{:}{}^{\mathsf{L}}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \vdash y''\,(y\,x) : {}^{\mathsf{U}}1$ by (8, 11), rule T-App

(13) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1), x{:}t^* \vdash \lambda y''.y''\,(y\,x) : {}^{\mathsf{L}}({}^{\mathsf{L}}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$ by (12), rule T-Abs

(14) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \vdash \lambda x.\lambda y''.y''\,(y\,x) : {}^{\mathsf{L}}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{L}}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1))$ by (13), rule T-Abs

(15) $\mathrm{D}^* \vdash \bullet, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \preceq \xi$ by (2)

(16) $\mathrm{D}^*; \bullet, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \vdash \Lambda.\lambda x.\lambda y''.y''\,(y\,x) : {}^{\xi}\forall\alpha{:}\star.{}^{\mathsf{L}}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{L}}({}^{\mathsf{U}}1 \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1))$ by (14–15), rule T-TAbs

(17) $\mathrm{D}^*; \bullet, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \vdash \Lambda.\lambda x.\lambda y''.y''\,(y\,x) : ({}^{\xi}(t \xrightarrow{\perp_{\mathcal{D}}}\!\circ\ {}^{\mathsf{U}}1))^*$ by (16), def. $\overline{t}^*$

(18) $\mathrm{D}^*; \mathrm{G}^*, y{:}{}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \vdash (\lambda y'.\, [\![e']\!]_{\mathcal{D}}\, (\lambda x.x))\, (\Lambda.\lambda x.\lambda y''.y''\,(y\,x)) : {}^{\mathsf{U}}1$ by (7, 17), rule T-App

(19) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.\, (\lambda y'.\, [\![e']\!]_{\mathcal{D}}\, (\lambda x.x))\, (\Lambda.\lambda x.\lambda y''.y''\,(y\,x)) : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$ by (18), rule T-Abs

(20) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{shift}\, y' \,\mathsf{in}\, e']\!]_{\mathcal{D}} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{U}}1) \multimap {}^{\mathsf{U}}1)$ by (3, 19). □

C.2 Answer-Type Modification Effects

In this section, we consider the delimited continuations with answer-type modification effects from §6.2.

Definition C.2 ($\lambda^{\mathsf{URAL}}(\mathcal{A})$ effects to $\lambda^{\mathsf{URAL}}(\mathcal{D})$ effects)**.**
We define a translation from answer-type modification effects to fixed-answer type effects:

$$\mathcal{D}(\bot_{\mathcal{A}}) = \bot_{\mathcal{D}}$$
$$\mathcal{D}({}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2)) = \overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j}$$

Lemma C.3 (Effect translation)**.**
For all a and a':

1. $a^* = \mathcal{D}(a)^*$.
2. $\mathcal{D}(a \oslash a') = \mathcal{D}(a) \sqcup \mathcal{D}(a')$ *when* $a \oslash a'$ *is defined.*
3. *If* $\mathrm{D} \vdash_{\mathcal{A}} a : \mathsf{CTL}$ *then* $\mathrm{D} \vdash_{\mathcal{D}} \mathcal{D}(a) : \mathsf{CTL}$.
4. *If* $\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a_2$ *then* $\mathrm{D} \vdash_{\mathcal{D}} \mathcal{D}(a_1) \preceq \mathcal{D}(a_2)$.

Proof.

1. By cases on a:

   Case $\bot_{\mathcal{A}}$**.**
   Then ${\bot_{\mathcal{A}}}^* = \mathsf{L} = {\bot_{\mathcal{D}}}^* = \mathcal{D}(\bot_{\mathcal{A}})^*$.

   Case ${}^{\Xi}(t_1 \rightarrowtail t_2)$**.**
   By cases on Ξ:

   Case ξ**.**
   Then ${}^{\xi}(t_1 \rightarrowtail t_2)^* = \xi = \overline{\xi}^* = \mathcal{D}({}^{\xi}(t_1 \rightarrowtail t_2))^*$.

   Case $\xi_1, \ldots, \xi_j$**.**
   Then $({}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2))^* = \mathsf{U} = (\overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j})^* = \mathcal{D}(({}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2)))^*$.

2. By cases on a:

   Case $\bot_{\mathcal{A}}$**.**
   Then $\mathcal{D}(\bot_{\mathcal{A}} \oslash a') = \mathcal{D}(a_2) = \bot_{\mathcal{D}} \sqcup \mathcal{D}(a') = \mathcal{D}(\bot_{\mathcal{A}}) \sqcup \mathcal{D}(a')$.

   Case ${}^{\Xi}(t_1 \rightarrowtail t_2)$**.**
   By cases on a':

Case $\bot_{\mathcal{A}}$.

By symmetry with the $a = \bot_{\mathcal{A}}$ case above.

Case ${}^{\Xi'}(t_1' \rightarrowtail t_2')$.

Then ${}^{\Xi}(t_1 \rightarrowtail t_2) \otimes {}^{\Xi'}(t_1' \rightarrowtail t_2')$ is well formed only if $t_1 = t_2'$. Let $\xi_1, \ldots, \xi_j = \Xi$ and $\xi_1', \ldots, \xi_k' = \Xi'$. Then,

(1) $\mathcal{D}({}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2) \otimes {}^{\xi_1',\ldots,\xi_k'}(t_1' \rightarrowtail t_2'))$

(2) $= \mathcal{D}({}^{\xi_1,\ldots,\xi_j,\xi_1',\ldots,\xi_k'}(t_1' \rightarrowtail t_2))$

(3) $= \overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j} \sqcup \overline{\xi_1'} \sqcup \ldots \sqcup \overline{\xi_k'}$

(4) $= \mathcal{D}({}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2)) \sqcup \mathcal{D}({}^{\xi_1',\ldots,\xi_k'}(t_1' \rightarrowtail t_2'))$.

3. By induction on the derivation of $\mathrm{D} \vdash_{\mathcal{A}} a : \mathsf{CTL}$:

Case $\dfrac{}{\mathrm{D} \vdash_{\mathcal{A}} \bot_{\mathcal{A}} : \mathsf{CTL}}$.

Then by rule C-K-Bot, $\mathrm{D} \vdash_{\mathcal{D}} \bot_{\mathcal{D}} : \mathsf{CTL}$.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} \xi : \mathsf{QUAL} \qquad \mathrm{D} \vdash_{\mathcal{A}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{A}} t_2 : \star}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\xi}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}$.

Then by rule D-K-Qual, $\mathrm{D} \vdash_{\mathcal{D}} \overline{\xi} : \mathsf{CTL}$.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2) : \mathsf{CTL} \qquad \mathrm{D} \vdash_{\mathcal{A}} {}^{\xi_1',\ldots,\xi_k'}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j,\xi_1',\ldots,\xi_k'}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}$.

By the induction hypothesis, twice,

(1) $\mathrm{D} \vdash_{\mathcal{D}} \overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j} : \mathsf{CTL}$ and

(2) $\mathrm{D} \vdash_{\mathcal{D}} \overline{\xi_1'} \sqcup \ldots \sqcup \overline{\xi_k'} : \mathsf{CTL}$.

Then by rule D-K-Join,

(3) $\mathrm{D} \vdash_{\mathcal{D}} \overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j} \sqcup \overline{\xi_1'} \sqcup \ldots \sqcup \overline{\xi_k'} : \mathsf{CTL}$,

or equivalently,

(4) $\mathrm{D} \vdash_{\mathcal{D}} \mathcal{D}({}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2)) \otimes \mathcal{D}({}^{\xi_1',\ldots,\xi_k'}(t_1 \rightarrowtail t_2)) : \mathsf{CTL}$.

Otherwise.

No other cases assign kind CTL to a type.

4. By induction on the derivation of $\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a_2$:

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} a : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} a \preceq a}$.

Then $\mathrm{D} \vdash_{\mathcal{D}} \mathcal{D}(a) \preceq \mathcal{D}(a)$ by rule CSub-Refl.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a' \qquad \mathrm{D} \vdash_{\mathcal{A}} a' \preceq a_2}{\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a_2}$.

By the induction hypothesis twice and rule CSub-Trans.

Case $\dfrac{\mathsf{D} \vdash_{\mathcal{A}} {}^{\Xi}(t \rightarrowtail t) : \mathsf{CTL}}{\mathsf{D} \vdash_{\mathcal{A}} \bot_{\mathcal{A}} \preceq {}^{\Xi}(t \rightarrowtail t)}$.

Then $\mathsf{D} \vdash_{\mathcal{D}} \bot_{\mathcal{D}} \preceq \mathcal{D}({}^{\Xi}(t \rightarrowtail t))$ by rule DSub-Bot.

Case $\dfrac{\mathsf{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}{\mathsf{D} \vdash_{\mathcal{A}} {}^{\mathsf{L}}(t_1 \rightarrowtail t_2) \preceq {}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2)}$.

For each ξ_k in $\xi_1, \ldots, \xi_j$, by rule DSub-Lin, we have that $\mathsf{D} \vdash_{\mathcal{D}} \overline{\mathsf{L}} \preceq \overline{\xi_k}$. By induction on the length of $\overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j}$ and repeated application of rule DSub-Join, we have that $\mathsf{D} \vdash_{\mathcal{D}} \overline{\mathsf{L}} \preceq \overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j}$. Then note that ${}^{\mathsf{L}}(t_1 \rightarrowtail t_2) = {}^{\mathsf{L}}(t_1 \rightarrowtail t_2)$ by the quotienting of Ξ.

Case $\dfrac{\mathsf{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}{\mathsf{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2) \preceq {}^{\mathsf{U}}(t_1 \rightarrowtail t_2)}$.

As in the previous case, but using rule DSub-Top for each $\mathsf{D} \vdash_{\mathcal{D}} \overline{\xi_j} \preceq \overline{\mathsf{U}}$.

Case $\dfrac{\mathsf{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2) \preceq {}^{\xi''_1,\ldots,\xi''_k}(t_1 \rightarrowtail t_2) \quad \mathsf{D} \vdash_{\mathcal{A}} {}^{\xi'_1,\ldots,\xi'_m}(t_1 \rightarrowtail t_2) \preceq {}^{\xi'''_1,\ldots,\xi'''_n}(t_1 \rightarrowtail t_2)}{\mathsf{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j,\xi''_1,\ldots,\xi''_k}(t_1 \rightarrowtail t_2) \preceq {}^{\xi'_1,\ldots,\xi'_m,\xi'''_1,\ldots,\xi'''_n}(t_1 \rightarrowtail t_2)}$.

By the induction hypothesis,

(1) $\mathsf{D} \vdash_{\mathcal{D}} \overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j} \preceq \overline{\xi''_1} \sqcup \ldots \sqcup \overline{\xi''_k}$ and
(2) $\mathsf{D} \vdash_{\mathcal{D}} \overline{\xi'_1} \sqcup \ldots \sqcup \overline{\xi'_m} \preceq \overline{\xi'''_1} \sqcup \ldots \sqcup \overline{\xi'''_n}$.

Then by rule DSub-Join,

(3) $\mathsf{D} \vdash_{\mathcal{D}} \overline{\xi_1} \sqcup \ldots \sqcup \overline{\xi_j} \sqcup \overline{\xi'_1} \sqcup \ldots \sqcup \overline{\xi'_m} \preceq \overline{\xi''_1} \sqcup \ldots \sqcup \overline{\xi''_k} \sqcup \overline{\xi'''_1} \sqcup \ldots \sqcup \overline{\xi'''_n}$. □

Lemma C.4 (No information)**.**
If $a^* = \mathsf{U}$ *then* $(a \circledcirc a')^* = \mathsf{U}$.

Proof. Then:

(1)	$a^* = \mathsf{U}$	by antecedent
(2)	$\mathcal{D}(a)^* = \mathsf{U}$	by (1), Lemma C.3.1
(3)	$(a \circledcirc a')^* = \mathcal{D}(a \circledcirc a')^*$	by Lemma C.3.1
(4)	$= (\mathcal{D}(a) \sqcup \mathcal{D}(a'))^*$	by Lemma C.3.2
(5)	$= \mathsf{U}$	by (2), Lemma C.1. □

Theorem 6.3 (Answer-type effect properties, restated from p. 36)**.**

Answer-type modification effects $(\mathcal{A}, \bot_{\mathcal{A}}, \circ)$ *satisfy Properties 1–5.*

Proof.

Property 1 (Answer types).

1. By the definition, $\langle\!\langle \tau, \bot_{\mathcal{A}} \rangle\!\rangle^{-}_{\mathcal{A}} = \tau = \langle\!\langle \tau, \bot_{\mathcal{A}} \rangle\!\rangle^{+}_{\mathcal{A}}$. Thus, $\langle\!\langle \tau \rangle\!\rangle_{\mathcal{A}} = \tau$.
2. Let $\mathrm{D}^* \vdash \tau : \star$ and $\mathrm{D} \vdash_{\mathcal{A}} a : \mathsf{CTL}$. We need to show that $\mathrm{D}^* \vdash \langle\!\langle \tau, a \rangle\!\rangle^{-}_{\mathcal{A}} : \star$ and $\mathrm{D}^* \vdash \langle\!\langle \tau, a \rangle\!\rangle^{+}_{\mathcal{A}} : \star$. By induction on the latter derivation:

   Case $\dfrac{}{\mathrm{D} \vdash_{\mathcal{A}} \bot_{\mathcal{A}} : \mathsf{CTL}}$.
   Then $\langle\!\langle \tau, a \rangle\!\rangle^{-}_{\mathcal{A}} = \langle\!\langle \tau, a \rangle\!\rangle^{+}_{\mathcal{A}} = \langle\!\langle \tau \rangle\!\rangle_{\mathcal{A}} = \tau$, and $\mathrm{D}^* \vdash \tau : \star$ by our assumption.

   Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} \xi : \mathsf{QUAL} \qquad \mathrm{D} \vdash_{\mathcal{A}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{A}} t_2 : \star}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\xi}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}$.
   Then $\langle\!\langle \tau, a \rangle\!\rangle^{-}_{\mathcal{A}} = {t_1}^*$ and $\langle\!\langle \tau, a \rangle\!\rangle^{+}_{\mathcal{A}} = {t_2}^*$. By Lemma 5.4, $\mathrm{D}^* \vdash {t_1}^* : \star$ and $\mathrm{D}^* \vdash {t_1}^* : \star$.

   Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_1}(t_1 \rightarrowtail t_2) : \mathsf{CTL} \qquad \mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_2}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_1,\Xi_2}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}$.
   Then $\langle\!\langle \tau, a \rangle\!\rangle^{-}_{\mathcal{A}} = {t_1}^* = \langle\!\langle \tau, {}^{\Xi_1}(t_1 \rightarrowtail t_2) \rangle\!\rangle^{-}_{\mathcal{A}}$ and $\langle\!\langle \tau, a \rangle\!\rangle^{+}_{\mathcal{A}} = {t_2}^* = \langle\!\langle \tau, {}^{\Xi_1}(t_1 \rightarrowtail t_2) \rangle\!\rangle^{+}_{\mathcal{A}}$. Then by the induction hypothesis.

   Otherwise.
   No other cases assign kind CTL to a type.

3. Let $c_1 \neq \bot_{\mathcal{A}}$, $c_2 \neq \bot_{\mathcal{A}}$, and $\mathrm{D} \vdash_{\mathcal{A}} a_1 \circledcirc a_2 : \mathsf{CTL}$. By the definition of effect sequencing, $a_1 \circledcirc a_2$ is defined only if one of the effects is $\bot_{\mathcal{A}}$, which is rules out by the assumption, or if the sequenced effect is of the form

$$a_1 \circledcirc a_2 = {}^{\Xi}(t' \rightarrowtail t_2) \circledcirc {}^{\Xi'}(t_1 \rightarrowtail t') = {}^{\Xi,\Xi'}(t_1 \rightarrowtail t_2).$$

   Then:

   (a) $\langle\!\langle \tau, a_1 \circledcirc a_2 \rangle\!\rangle^{-}_{\mathcal{A}} = \langle\!\langle \tau, {}^{\Xi,\Xi'}(t_1 \rightarrowtail t_2) \rangle\!\rangle^{-}_{\mathcal{A}} = {t_1}^* = \langle\!\langle \tau, {}^{\Xi'}(t_1 \rightarrowtail t') \rangle\!\rangle^{-}_{\mathcal{A}} = \langle\!\langle \tau, a_2 \rangle\!\rangle^{-}_{\mathcal{A}}$.
   (b) $\langle\!\langle \tau, a_1 \circledcirc a_2 \rangle\!\rangle^{+}_{\mathcal{A}} = \langle\!\langle \tau, {}^{\Xi,\Xi'}(t_1 \rightarrowtail t_2) \rangle\!\rangle^{+}_{\mathcal{A}} = {t_2}^* = \langle\!\langle \tau, {}^{\Xi}(t' \rightarrowtail t_2) \rangle\!\rangle^{+}_{\mathcal{A}} = \langle\!\langle \tau, a_1 \rangle\!\rangle^{+}_{\mathcal{A}}$.
   (c) $\langle\!\langle \tau, a_1 \rangle\!\rangle^{-}_{\mathcal{A}} = \langle\!\langle \tau, {}^{\Xi}(t' \rightarrowtail t_2) \rangle\!\rangle^{-}_{\mathcal{A}} = {t'}^* = \langle\!\langle \tau, {}^{\Xi'}(t_1 \rightarrowtail t') \rangle\!\rangle^{+}_{\mathcal{A}} = \langle\!\langle \tau, a_2 \rangle\!\rangle^{+}_{\mathcal{A}}$.

4. Let $\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a_2$. Given an arbitrary type τ, we must find a type τ' such that $\langle\!\langle \tau', a_1 \rangle\!\rangle^{-}_{\mathcal{A}} = \langle\!\langle \tau, a_2 \rangle\!\rangle^{-}_{\mathcal{A}}$ and $\langle\!\langle \tau', a_1 \rangle\!\rangle^{+}_{\mathcal{A}} = \langle\!\langle \tau, a_2 \rangle\!\rangle^{+}_{\mathcal{A}}$. By induction on the effect subsumption derivation:

   Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} a : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} a \preceq a}$.
   Then $a_1 = a_2$, so by substitution of one for the other.

   Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a' \qquad \mathrm{D} \vdash_{\mathcal{A}} a' \preceq a_2}{\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a_2}$.
   By the induction hypothesis, twice, and transitivity of equality.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi}(t \rightarrowtail t) : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} \bot_{\mathcal{A}} \preceq {}^{\Xi}(t \rightarrowtail t)}$.

Let $\tau' = t^*$. Then $\langle\!\langle t^*, a_1 \rangle\!\rangle^-_{\mathcal{A}} = \langle\!\langle t^* \rangle\!\rangle_{\mathcal{A}} = t^* = \langle\!\langle \tau, {}^{\Xi}(t \rightarrowtail t) \rangle\!\rangle^-_{\mathcal{A}}$, and likewise $\langle\!\langle t^*, a_1 \rangle\!\rangle^+_{\mathcal{A}} = \langle\!\langle t^* \rangle\!\rangle_{\mathcal{A}} = t^* = \langle\!\langle \tau, {}^{\Xi}(t \rightarrowtail t) \rangle\!\rangle^+_{\mathcal{A}}$.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\mathsf{L}}(t_1 \rightarrowtail t_2) \preceq {}^{\Xi}(t_1 \rightarrowtail t_2)}$.

It does not matter what τ' we choose, so let $\tau' = \tau$. Then $\langle\!\langle \tau, a_1 \rangle\!\rangle^-_{\mathcal{A}} = {t_1}^* = \langle\!\langle \tau, a_2 \rangle\!\rangle^-_{\mathcal{A}}$ and $\langle\!\langle \tau, a_1 \rangle\!\rangle^+_{\mathcal{A}} = {t_2}^* = \langle\!\langle \tau, a_2 \rangle\!\rangle^+_{\mathcal{A}}$.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi}(t_1 \rightarrowtail t_2) : \mathsf{CTL}}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi}(t_1 \rightarrowtail t_2) \preceq {}^{\mathsf{U}}(t_1 \rightarrowtail t_2)}$.

As in the previous case, let $\tau' = \tau$.

Case $\dfrac{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_1}(t_1 \rightarrowtail t_2) \preceq {}^{\Xi'_1}(t_1 \rightarrowtail t_2) \qquad \mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_2}(t_1 \rightarrowtail t_2) \preceq {}^{\Xi'_2}(t_1 \rightarrowtail t_2)}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\Xi_1, \Xi_2}(t_1 \rightarrowtail t_2) \preceq {}^{\Xi'_1, \Xi'_2}(t_1 \rightarrowtail t_2)}$.

As in the previous case, let $\tau' = \tau$.

Property 2 (Done).

(1) $\Delta; \bullet, x{:}\tau \vdash x : \tau$ by rule T-Var

(2) $\Delta; \bullet \vdash \lambda x.x : {}^{\mathsf{L}}(\tau \multimap \tau)$ by (1), rule T-Abs

(3) $\langle\!\langle \tau \rangle\!\rangle_{\mathcal{A}} = \tau$ by def. $\langle\!\langle \tau \rangle\!\rangle_{\mathcal{A}}$

(4) $\Delta; \bullet \vdash \lambda x.x : {}^{\mathsf{L}}(\tau \multimap \langle\!\langle \tau \rangle\!\rangle_{\mathcal{A}})$ by (2–3).

Property 3 (Effect sequencing). Let $\mathrm{D} \vdash_{\mathcal{A}} a_1 \circledcirc a_2 : \mathsf{CTL}$. By Lemma C.3,

(1) $(a_1 \circledcirc a_2)^* = \mathcal{D}(a_1 \circledcirc a_2)^* = \mathcal{D}(a_1)^* \sqcup \mathcal{D}(a_2)^*$,

(2) ${a_1}^* = \mathcal{D}(a_1)^*$,

(3) ${a_2}^* = \mathcal{D}(a_2)^*$,

(4) $\mathrm{D} \vdash_{\mathcal{D}} \mathcal{D}(a_1) \sqcup \mathcal{D}(a_2) : \mathsf{CTL}$.

By this same property for $\lambda^{\mathsf{URAL}}(\mathcal{D})$, we have that

(5) $\mathrm{D}^* \vdash \mathcal{D}(a_1)^* \sqcup \mathcal{D}(a_2)^* \preceq \mathcal{D}(a_1)^*$ and

(6) $\mathrm{D}^* \vdash \mathcal{D}(a_1)^* \sqcup \mathcal{D}(a_2)^* \preceq \mathcal{D}(a_2)^*$.

Property 4 (Bottom and lifting).

1. By the definition of $a_1 \circledcirc a_2$.
2. By cases on a_1 and a_2:

Case $\bot_{\mathcal{A}}$ and $\bot_{\mathcal{A}}$.
Contradicts the assumption that $a_1 \otimes a_2 \neq \bot_{\mathcal{A}}$.
Case $\bot_{\mathcal{A}}$ and $^{\Xi_2}(t_1 \rightarrowtail t')$.
Let $c_1' = {}^{\mathsf{L}}(t' \rightarrowtail t')$ and $c_2' = c_2$.
Case $^{\Xi_1}(t' \rightarrowtail t_2)$ and $\bot_{\mathcal{A}}$.
Let $c_1' = c_1$ and $c_2' = {}^{\mathsf{L}}(t' \rightarrowtail t')$.
Case $^{\Xi_1}(t' \rightarrowtail t_2)$ and $^{\Xi_2}(t_1 \rightarrowtail t')$.
Let $c_1' = c_1$ and $c_2' = c_2$.

Property 5 (New rules).

1. For translation of effect bounds, let $\mathrm{D} \vdash_{\mathcal{A}} a \succeq \xi$; we need to show that $\mathrm{D}^* \vdash \xi \preceq a^*$. We proceed by induction on the derivation of $\mathrm{D} \vdash_{\mathcal{A}} a \succeq \xi$, which has but one new case to consider:

   Case $$\frac{\mathrm{D} \vdash_{\mathcal{A}} \xi \preceq \xi_1 \quad \cdots \quad \mathrm{D} \vdash_{\mathcal{A}} \xi \preceq \xi_j \qquad \mathrm{D} \vdash_{\mathcal{A}} t_1 : \star \qquad \mathrm{D} \vdash_{\mathcal{A}} t_2 : \star}{\mathrm{D} \vdash_{\mathcal{A}} {}^{\xi_1,\ldots,\xi_j}(t_1 \rightarrowtail t_2) \succeq \xi}.$$

   If $\xi_1, \ldots, \xi_j$ is equivalent to a single qualifier ξ', then ${}^{\xi'}(t_1 \rightarrowtail t_2)^* = \xi'$. By the premises, $\mathrm{D} \vdash_{\mathcal{A}} \xi \preceq \xi'$, and by Lemma B.2, $\mathrm{D}^* \vdash \xi \preceq \xi'$.

   Otherwise, $\xi_1, \ldots, \xi_j$ is not equivalent to a single qualifier. Based on the quotienting of $\mathcal{A}$, means that there are some qualifiers ξ_i and ξ_i' in the collection of qualifiers such that $\xi_i \sqcap \xi_i'$ is undefined. By Lemma A.3, we know that $\xi = \mathsf{U}$. Then by rule QSub-Bot.

2. For translation of effect subsumption, let $\mathrm{D} \vdash_{\mathcal{A}} a_1 \preceq a_2$; we must show that $\mathrm{D}^* \vdash {a_2}^* \preceq {a_1}^*$.

(1)	$\mathrm{D} \vdash_{\mathcal{D}} \mathcal{D}(a_1) \preceq \mathcal{D}(a_2)$	by Lemma C.3.4
(2)	$\mathrm{D}^* \vdash \mathcal{D}(a_2)^* \preceq \mathcal{D}(a_1)^*$	by $\lambda^{\mathsf{URAL}}(\mathcal{D})$ Property 5
(3)	$\mathrm{D}^* \vdash {a_2}^* \preceq {a_1}^*$	by Lemma C.3.1.

3. For translation of kinding, let $\mathrm{D} \vdash_{\mathcal{A}} a : \mathsf{CTL}$. By Lemma C.3.3, $\mathrm{D} \vdash_{\mathcal{D}} \mathcal{D}(a) : \mathsf{CTL}$. Then by Lemma 5.4, $\mathrm{D}^* \vdash \mathcal{D}(a)^* : \mathsf{QUAL}$. Note, finally, that $\mathcal{D}(a)^* = a^*$ by Lemma C.3.1.

4. For translation of typing, let
   - $\mathrm{D}; \mathrm{G} \vdash_{\mathcal{A}} e : t \,;\, a$.
   - $\mathrm{D}^* \vdash \xi_0 \preceq a^*$, and
   - $\mathrm{D}^* \vdash \tau' : \star$.

   We must show that $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]_{\mathcal{A}} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap \langle\!\langle \tau', a \rangle\!\rangle_{\mathcal{A}}^{-}) \multimap \langle\!\langle \tau', a \rangle\!\rangle_{\mathcal{A}}^{-})$. We proceed by induction on the typing derivation, with two cases to consider:

Case $\dfrac{\mathrm{D};\mathrm{G} \vdash_{\mathcal{A}} e' : t_0 \,;\, {}^{\Xi}(t_0 \rightarrowtail t)}{\mathrm{D};\mathrm{G} \vdash_{\mathcal{A}} \mathsf{reset}\ e' : t \,;\, \bot_{\mathcal{A}}}$.

We must show that $\mathrm{D}^*;\mathrm{G}^* \vdash [\![e']\!]_{\mathcal{A}} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap \tau') \multimap \tau')$. Let $a' = {}^{\Xi}(t_0 \rightarrowtail t)$. Then,

(1) $[\![\mathsf{reset}\ e']\!]_{\mathcal{A}} = \lambda y.\, y\,([\![e']\!]_{\mathcal{D}}\,(\lambda x.x))$ by def. $[\![e]\!]_{\mathcal{A}}$

(2) $\mathrm{D}^*;\mathrm{G}^* \vdash [\![e']\!]_{\mathcal{A}} : {}^{\mathsf{L}}({}^{a'^*}({t_0}^* \multimap {t_0}^*) \multimap t^*)$ by IH

(3) $\mathrm{D}^*;\bullet, x{:}{t_0}^* \vdash x : {t_0}^*$ by rule T-Var

(4) $\mathrm{D}^*;\bullet \vdash \lambda x.x : {}^{a'^*}({t_0}^* \multimap {t_0}^*)$ by (3), rule T-Abs

(5) $\mathrm{D}^*;\mathrm{G}^* \vdash [\![e']\!]_{\mathcal{A}}\,(\lambda x.x) : t^*$ by (2, 4), rule T-App

(6) $\mathrm{D}^*;\bullet, y{:}{}^{\xi_0}(t^* \multimap \tau') \vdash y : {}^{\xi_0}(t^* \multimap \tau')$ by rule T-Var

(7) $\mathrm{D}^*;\mathrm{G}^*, y{:}{}^{\xi_0}(t^* \multimap \tau') \vdash y\,([\![e']\!]_{\mathcal{A}}\,(\lambda x.x)) : \tau'$ by (5–6), rule T-App

(8) $\mathrm{D}^*;\mathrm{G}^* \vdash \lambda y.\, y\,([\![e']\!]_{\mathcal{A}}\,(\lambda x.x)) : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap \tau') \multimap \tau')$ by (7), rule T-Abs

(9) $\mathrm{D}^*;\mathrm{G}^* \vdash [\![\mathsf{reset}\ e']\!]_{\mathcal{A}} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap \tau') \multimap \tau')$ by (1, 8).

Case $\dfrac{\mathrm{D};\mathrm{G}, y'{:}{}^{\xi}(t_1 \overset{\bot_{\mathcal{A}}}{\multimap} t_2) \vdash_{\mathcal{A}} e' : t_0 \,;\, {}^{\Xi}(t_0 \rightarrowtail t)}{\mathrm{D};\mathrm{G} \vdash_{\mathcal{A}} \mathsf{shift}\ y'\ \mathsf{in}\ e' : t_1 \,;\, {}^{\Xi,\xi}(t_2 \rightarrowtail t)}$.

We must show that $\mathrm{D}^*;\mathrm{G}^* \vdash [\![\mathsf{shift}\ y'\ \mathsf{in}\ e']\!]_{\mathcal{A}} : {}^{\mathsf{L}}({}^{\xi_0}({t_1}^* \multimap {t_2}^*) \multimap t^*)$. Let $a' = {}^{\Xi}(t_0 \rightarrowtail t)$.

(1) $[\![\mathsf{shift}\ y'\ \mathsf{in}\ e']\!]_{\mathcal{A}} = \lambda y.\,(\lambda y'.\,[\![e']\!]_{\mathcal{A}}\,(\lambda x.x))\,(\Lambda.\lambda x.\lambda y''.y''\,(y\,x))$ by def. $[\![e]\!]_{\mathcal{A}}$

(2) $\mathrm{D}^* \vdash \xi_0 \preceq ({}^{\Xi,\xi}(t_2 \rightarrowtail t))^*$ by lem. assumption

(3) $\mathrm{D}^* \vdash ({}^{\Xi}(t_0 \rightarrowtail t) \otimes {}^{\xi}(t_2 \rightarrowtail t_0))^* \preceq {}^{\xi}(t_2 \rightarrowtail t_0)^*$ by Property 3

(4) $\mathrm{D}^* \vdash ({}^{\Xi,\xi}(t_2 \rightarrowtail t))^* \preceq \xi$ by (3), def a^*

(5) $\mathrm{D}^* \vdash \xi_0 \preceq \xi$ by (2, 4), trans.

(6) $\mathrm{D}^* \vdash \bullet, y{:}{}^{\xi_0}({t_1}^* \multimap {t_2}^*) \preceq \xi$ by (5)

(7) $\mathrm{D}^*;\mathrm{G}^*, y'{:}({}^{\xi}(t_1 \overset{\bot_{\mathcal{A}}}{\multimap} t_2))^* \vdash [\![e']\!]_{\mathcal{A}} : {}^{\mathsf{L}}({}^{a'^*}({t_0}^* \multimap {t_0}^*) \multimap t^*)$ by IH

(8) $\mathrm{D}^*;\bullet \vdash \lambda x.x : {}^{a'^*}({t_0}^* \multimap {t_0}^*)$ by rules T-Var and T-Abs

(9) $\mathrm{D}^*;\mathrm{G}^*, y'{:}({}^{\xi}(t_1 \overset{\bot_{\mathcal{A}}}{\multimap} t_2))^* \vdash [\![e']\!]_{\mathcal{A}}\,(\lambda x.x) : t^*$ by (7–8), rule T-App

(10) $\mathrm{D}^*;\mathrm{G}^* \vdash \lambda y'.\,[\![e']\!]_{\mathcal{A}}\,(\lambda x.x) : {}^{\mathsf{L}}(({}^{\xi}(t_1 \overset{\bot_{\mathcal{A}}}{\multimap} t_2))^* \multimap t^*)$ by (9), rule T-Abs

(11) $\mathrm{D}^*, \alpha{:}\star;\bullet, y''{:}{}^{\mathsf{L}}({t_2}^* \multimap \alpha) \vdash y'' : {}^{\mathsf{L}}({t_2}^* \multimap \alpha)$ by rule T-Var

(12) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}^{\xi_0}(t_1{}^* \multimap t_2{}^*) \vdash y : {}^{\xi_0}(t_1{}^* \multimap t_2{}^*)$ by rule T-Var

(13) $\mathrm{D}^*, \alpha{:}\star; \bullet, x{:}t_1{}^* \vdash x : t_1{}^*$ by rule T-Var

(14) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}^{\xi_0}(t_1{}^* \multimap t_2{}^*), x{:}t_1{}^* \vdash y\,x : t_2{}^*$ by (12–13), rule T-App

(15) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}^{\xi_0}(t_1{}^* \multimap t_2{}^*), x{:}t_1{}^*, y''{:}^{\mathsf{L}}(t_2{}^* \multimap \alpha) \vdash y''\,(y\,x) : \alpha$ by (11, 14), rule T-App

(16) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}^{\xi_0}(t_1{}^* \multimap t_2{}^*), x{:}t_1{}^* \vdash \lambda y''.y''\,(y\,x) : {}^{\mathsf{L}}({}^{\mathsf{L}}(t_2{}^* \multimap \alpha) \multimap \alpha)$ by (15), rule T-Abs

(17) $\mathrm{D}^*, \alpha{:}\star; \bullet, y{:}^{\xi_0}(t_1{}^* \multimap t_2{}^*) \vdash \lambda x.\lambda y''.y''\,(y\,x) : {}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}({}^{\mathsf{L}}(t_2{}^* \multimap \alpha) \multimap \alpha))$ by (16), rule T-Abs

(18) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t_1{}^* \multimap t_2{}^*) \vdash \Lambda.\lambda x.\lambda y''.y''\,(y\,x) : {}^{\xi}\forall\alpha{:}\star.{}^{\mathsf{L}}(t_1{}^* \multimap {}^{\mathsf{L}}({}^{\mathsf{L}}(t_2{}^* \multimap \alpha) \multimap \alpha))$ by (6, 17), rule T-TAbs

(19) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t_1{}^* \multimap t_2{}^*) \vdash \Lambda.\lambda x.\lambda y''.y''\,(y\,x) : ({}^{\xi}(t_1 \xrightarrow{\perp_{\mathcal{A}}} t_2))^*$ by (18), def. $\overline{t}^*$

(20) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t_1{}^* \multimap t_2{}^*) \vdash (\lambda y'.[\![e']\!]_{\mathcal{A}}\,(\lambda x.x))\,(\Lambda.\lambda x.\lambda y''.y''\,(y\,x)) : t^*$ by (10, 19), rule T-App

(21) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.(\lambda y'.[\![e']\!]_{\mathcal{A}}\,(\lambda x.x))\,(\Lambda.\lambda x.\lambda y''.y''\,(y\,x)) : {}^{\mathsf{L}}({}^{\xi_0}(t_1{}^* \multimap t_2{}^*) \multimap t^*)$ by (20), rule T-Abs

(22) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![\mathsf{shift}\,y'\,\mathsf{in}\,e']\!]_{\mathcal{A}} : {}^{\mathsf{L}}({}^{\xi_0}(t_1{}^* \multimap t_2{}^*) \multimap t^*)$ by (1, 21). □

C.3 Exception Effects

In this section, we consider the exception effects from §6.3.

Theorem 6.4 (Exception effect properties, restated from p. 40)**.**

Exception effects $(\mathcal{X}, \varnothing, \cup)$ *satisfy Properties 1–5.*

Proof.

Property 1 (Answer types).

1. Trivial, because $\langle\!\langle \tau, \Psi \rangle\!\rangle_{\mathcal{X}}^- = {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau) = \langle\!\langle \tau, \Psi \rangle\!\rangle_{\mathcal{X}}^+$ for all τ and Ψ.
2. Then,
   (1) $\mathrm{D}^* \vdash \tau : \star$ by lemma assumption
   (2) $\mathrm{D}^* \vdash {}^{\mathsf{U}}\mathsf{exn} : \star$ by def. exn, rule K-Type

(3) $\mathrm{D}^* \vdash {}^{\mathsf{U}}\mathsf{exn} \oplus \tau : \overline{\star}$ by (1–2), rule K-Sum
(4) $\mathrm{D}^* \vdash {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau) : \star$ by (3), rule K-Type
(5) $\mathrm{D}^* \vdash \langle\!\langle \tau, \Psi \rangle\!\rangle_x^- : \star$ and $\mathrm{D}^* \vdash \langle\!\langle \tau, \Psi \rangle\!\rangle_x^+ : \star$ by (4), defs.

3. Trivial, as in the first case.
4. Let $\tau' = \tau$.

Property 2 (Done).

(1) $\Delta \vdash \tau \preceq \mathsf{L}$ by rule B-Var or rules B-Type and QSub-Top
(2) $\Delta \vdash {}^{\mathsf{U}}\mathsf{exn} : \star$ by rule K-Type
(3) $\Delta; \bullet, x{:}\tau \vdash x : \tau$ by rule T-Var
(4) $\Delta; \bullet, x{:}\tau \vdash \mathsf{inr}\ x : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau)$ by (1–3), rule T-Inr
(5) $\Delta; \bullet \vdash \lambda x.\ \mathsf{inr}\ x : {}^{\mathsf{L}}(\tau \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau))$ by (4), rule T-Abs
(6) $\Delta; \bullet \vdash \mathsf{done}_x : {}^{\mathsf{L}}(\tau \multimap \langle\!\langle \tau \rangle\!\rangle_x)$ by (5), def. done_x, def. $\langle\!\langle \tau \rangle\!\rangle_x$.

Property 3 (Effect sequencing). Let $\mathrm{D} \vdash_{\mathcal{A}} \Psi_1 \cup \Psi_2 : \mathsf{CTL}$. We need that $\mathrm{D}^* \vdash (\Psi_1 \cup \Psi_2)^* \preceq {\Psi_1}^*$ and $\mathrm{D}^* \vdash (\Psi_1 \cup \Psi_2)^* \preceq {\Psi_2}^*$. By cases on Ψ_1 and Ψ_2:

Case $\Psi_1 = \varnothing$ and $\Psi_2 = \varnothing$.
Then $(\Psi_1 \cup \Psi_2)^* = \mathsf{L}$, ${\Psi_1}^* = \mathsf{L}$, and ${\Psi_2}^* = \mathsf{L}$, so by rule QSub-Top.

Case $\Psi_1 = \varnothing$ and $\Psi_2 \neq \varnothing$.
Then $(\Psi_1 \cup \Psi_2)^* = \mathsf{A}$, ${\Psi_1}^* = \mathsf{L}$, and ${\Psi_2}^* = \mathsf{A}$, so by rules QSub-Top and QSub-Refl.

Case $\Psi_1 \neq \varnothing$ and $\Psi_2 = \varnothing$.
By symmetry.

Case $\Psi_1 \neq \varnothing$ and $\Psi_2 \neq \varnothing$.
Then $(\Psi_1 \cup \Psi_2)^* = \mathsf{A}$, ${\Psi_1}^* = \mathsf{A}$, and ${\Psi_2}^* = \mathsf{A}$, so by rule QSub-Refl.

Property 4 (Bottom and lifting).

1. By the definition of set union.
2. Let $c_1' = c_1$ and $c_2' = c_2$. □

Property 5 (New rules).

1. For translation of effect bounds, let $\mathrm{D} \vdash_{\overline{\mathrm{x}}} \Psi \succeq \xi$; we need to show that $\mathrm{D}^* \vdash \xi \preceq \Psi^*$. We proceed by induction on the derivation of $\mathrm{D} \vdash_{\overline{\mathrm{x}}} \Psi \succeq \xi$, which has but one new case to consider:

   Case $\dfrac{\mathrm{D} \vdash_{\overline{\mathrm{x}}} \Psi : \mathsf{CTL}}{\mathrm{D} \vdash_{\overline{\mathrm{x}}} \Psi \succeq \mathsf{A}}$**.**

   By cases on Ψ:

   Case $\varnothing$**.**
   Then $\Psi^* = \mathsf{L}$, so by rule QSub-Top.

   Otherwise.
   Then $\Psi^* = \mathsf{A}$, so by rule QSub-Refl.

2. For translation of effect subsumption, let $\mathrm{D} \vdash_{\overline{\mathrm{x}}} \Psi_1 \preceq \Psi_2$; we must show that $\mathrm{D}^* \vdash {\Psi_2}^* \preceq {\Psi_1}^*$. By cases on Ψ_1 and Ψ_2:

   Case $\Psi_1 = \varnothing$ **and** $\Psi_2 = \varnothing$**.**
   Then ${\Psi_1}^* = \mathsf{L}$ and ${\Psi_2}^* = \mathsf{L}$, so by rule QSub-Top.

   Case $\Psi_1 = \varnothing$ **and** $\Psi_2 \neq \varnothing$**.**
   Then ${\Psi_1}^* = \mathsf{L}$ and ${\Psi_2}^* = \mathsf{A}$, so by rule QSub-Top.

   Case $\Psi_1 \neq \varnothing$ **and** $\Psi_2 = \varnothing$**.**
   Vacuous, because $\Psi_1 \not\subseteq \Psi_2$.

   Case $\Psi_1 \neq \varnothing$ **and** $\Psi_2 \neq \varnothing$**.**
   Then ${\Psi_1}^* = \mathsf{A}$ and ${\Psi_2}^* = \mathsf{A}$, so by rule QSub-Refl.

3. By definition $\Psi^* = \mathsf{L}$ or $\Psi^* = \mathsf{A}$, so $\mathrm{D}^* \vdash \Psi^* : \mathsf{QUAL}$ by rule K-Qual.

4. For translation of typing, let

   - $\mathrm{D}; \mathrm{G} \vdash_{\overline{\mathrm{x}}} e : t \,;\, \Psi$.
   - $\mathrm{D}^* \vdash \xi_0 \preceq \Psi^*$, and
   - $\mathrm{D}^* \vdash \tau' : \star$.

   We must show that

   $$\mathrm{D}^*; \mathrm{G}^* \vdash [\![e]\!]^{\Psi}_{\overline{\mathrm{x}}} : {}^{\mathsf{L}}(^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')).$$

   We proceed by induction on the typing derivation, with two cases to consider:

   Case $\dfrac{\mathrm{D} \vdash_{\mathrm{e}} t : \star}{\mathrm{D}; \bullet \vdash_{\overline{\mathrm{x}}} \mathsf{raise}\ \psi : t \,;\, \{\psi\}}$**.**

   (1) $[\![\mathsf{raise}\ \psi]\!]^{\{\psi\}}_{\overline{\mathrm{x}}} = \lambda x.\, \mathsf{inl}\ \psi^*$ — by def. $[\![e]\!]^{\Psi}_{\overline{\mathrm{x}}}$

   (2) $\mathrm{D}^* \vdash {}^{\mathsf{U}}\mathsf{exn} \preceq \mathsf{L}$ — by rules B-Type and QSub-Top

   (3) $\mathrm{D}^* \vdash \bullet, x{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \rightsquigarrow \bullet \boxplus \bullet, x{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ — by rules S-Nil and S-ConsR

(4) $\mathrm{D}^*; \bullet \vdash \psi^* : {}^{\mathsf{U}}\mathsf{exn}$ by def. exn
(5) $\Psi^* = \mathsf{A}$ by def. $\{\psi\}^*$
(6) $\mathrm{D}^* \vdash \xi_0 \preceq \mathsf{A}$ by lemma assumption, (5)
(7) $\mathrm{D}^* \vdash \bullet, x{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \preceq \mathsf{A}$ by (6), rules B-Type and B-Cons
(8) $\mathrm{D}^*; \bullet, x{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash \psi^* : {}^{\mathsf{U}}\mathsf{exn}$ by (3–4, 7), rule T-Weak
(9) $\mathrm{D}^*; \bullet, x{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash \mathsf{inl}\ \psi^* : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')$ by (2, 8), rule T-Inl
(10) $\mathrm{D}^*; \bullet \vdash \lambda x.\ \mathsf{inl}\ \psi^* : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$
(11) $\mathrm{D}^*; \bullet \vdash [\![\mathsf{raise}\ \psi]\!]_{\mathrm{x}}^{\{\psi\}} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (1, 10).

Case $\dfrac{\mathrm{D} \vdash_{\mathrm{e}} \mathrm{G} \rightsquigarrow \mathrm{G}_1 \boxplus \mathrm{G}_2 \quad \mathrm{D}; \mathrm{G}_1 \vdash_{\mathrm{x}} e_1 : t\ ;\ \{\psi\} \cup \Psi' \quad \mathrm{D}; \mathrm{G}_2 \vdash_{\mathrm{x}} e_2 : t\ ;\ \Psi' \quad \mathrm{D} \vdash_{\mathrm{e}} \mathrm{G}_2 \preceq \mathsf{A}}{\mathrm{D}; \mathrm{G} \vdash_{\mathrm{x}} e_1\ \mathsf{handle}\ \psi \to e_2 : t\ ;\ \Psi'}$.

By cases on Ψ':
Case $\Psi' = \varnothing$**.**

(1) $[\![e_1\ \mathsf{handle}\ \psi \to e_2]\!]_{\mathrm{x}}^{\varnothing} = \lambda y.\, [\lambda\_.\, [\![e_2]\!]_{\mathrm{x}}^{\varnothing}\, y, y]\, ([\![e_1]\!]_{\mathrm{x}}^{\{\psi\}}\, (\lambda x.\ \mathsf{inr}\ x))$ by def. $[\![e]\!]_{\mathrm{x}}^{\Psi}$
(2) $\mathrm{D}^*; \mathrm{G}_2{}^* \vdash [\![e_2]\!]_{\mathrm{x}}^{\varnothing} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by IH, $\xi_0 \preceq \varnothing^*$
(3) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash y : {}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by rule T-Var
(4) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash [\![e_2]\!]_{\mathrm{x}}^{\varnothing}\, y : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')$ by (2–3), rule T-App
(5) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')), x'{:}^{\mathsf{U}}\mathsf{exn} \vdash [\![e_2]\!]_{\mathrm{x}}^{\varnothing}\, y : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')$ by (4), rule T-Weak
(6) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash \lambda\_.\, [\![e_2]\!]_{\mathrm{x}}^{\varnothing}\, y : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (5), rule T-Abs
(7) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash y : {}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (3), rule T-Weak
(8) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash [\lambda\_.\, [\![e_2]\!]_{\mathrm{x}}^{\varnothing}\, y, y] : {}^{\mathsf{L}}({}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (6–7), rule T-SumE
(9) $\mathrm{D}^*; \mathrm{G}_1{}^* \vdash [\![e_1]\!]_{\mathrm{x}}^{\{\psi\}} : {}^{\mathsf{L}}({}^{\mathsf{A}}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*)) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*))$ by IH, $\mathsf{A} \preceq \{\psi\}^*$

(10) $\mathrm{D}^*; \bullet, x{:}t^* \vdash x : t^*$ by rule T-Var

(11) $\mathrm{D}^*; \bullet, x{:}t^* \vdash \mathsf{inr}\ x : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*)$ by (10), rule T-Inr

(12) $\mathrm{D}^*; \bullet \vdash \lambda x.\ \mathsf{inr}\ x : {}^{\mathsf{A}}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*))$ by (11), rule T-Abs

(13) $\mathrm{D}^*; \mathrm{G}_1{}^* \vdash [\![e_1]\!]_x^{\{\psi\}}\ (\lambda x.\ \mathsf{inr}\ x) : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*)$ by (9, 12), rule T-App

(14) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash [\lambda\_.\, [\![e_2]\!]_x^{\varnothing}\ y, y]\ ([\![e_1]\!]_x^{\{\psi\}}\ (\lambda x.\ \mathsf{inr}\ x)) : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')$ by (8, 13), rule T-App

(15) $\mathrm{D}^*; \mathrm{G}^* \vdash \lambda y.\, [\lambda\_.\, [\![e_2]\!]_x^{\varnothing}\ y, y]\ ([\![e_1]\!]_x^{\{\psi\}}\ (\lambda x.\ \mathsf{inr}\ x)) : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (14), rule T-Abs

(16) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e_1\ \mathsf{handle}\ \psi \to e_2]\!]_x^{\varnothing} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (1, 15).

Case $\Psi' \neq \varnothing$.

This means that $\Psi'^* = \mathsf{A}$, so we know that

(1) $\mathrm{D}^* \vdash \xi_0 \preceq \mathsf{A}$.

Then:

(2) $[\![e_1\ \mathsf{handle}\ \psi \to e_2]\!]_x^{\Psi'} = \lambda y.\, [[\lambda\_.\, [\![e_2]\!]_x^{\Psi'}\ y, \lambda x.\ \mathsf{inl}\ x]_\psi, y]\ ([\![e_1]\!]_x^{\{\psi\}\cup\Psi'}\ (\lambda x.\ \mathsf{inr}\ x))$ by def. $[\![e_1]\!]_x^{\Psi}$

(3) $\mathrm{D}^*; \mathrm{G}_2{}^* \vdash [\![e_2]\!]_x^{\Psi'} : {}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by IH, $\xi_0 \preceq \varnothing^*$

(4) $\mathrm{D}^*; \bullet, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash y : {}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by rule T-Var

(5) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash [\![e_2]\!]_x^{\Psi'}\ y : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')$ by (3–4), rule T-App

(6) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')), x'{:}^{\mathsf{U}}\mathsf{exn} \vdash [\![e_2]\!]_x^{\Psi'}\ y : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')$ by (5), rule T-Weak

(7) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash \lambda\_.\, [\![e_2]\!]_x^{\Psi'}\ y : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (6), rule T-Abs

(8) $\mathrm{D}^*; \bullet \vdash \lambda x.\ \mathsf{inl}\ x : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by rules T-Var, T-Inl, and T-Abs

(9) $\mathrm{D}^* \vdash \mathrm{G}_2{}^* \preceq \mathsf{A}$ by assumption, Lemma B.2

(10) $\mathrm{D}^* \vdash \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \preceq \mathsf{A}$ by (1, 9)

(11) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash \lambda x.\ \mathsf{inl}\ x :$
${}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (8, 10), rule T-Weak

(12) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash [\lambda\_.[\![e_2]\!]_x^{\Psi'}\ y, \lambda x.\ \mathsf{inl}\ x]_\psi :$
${}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (7, 11), def. $[v_1, v_2]_\psi$

(13) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash y : {}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (4), rule T-Weak

(14) $\mathrm{D}^*; \mathrm{G}_2{}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash$
$[[\lambda\_.[\![e_2]\!]_x^{\Psi'}\ y, \lambda x.\ \mathsf{inl}\ x]_\psi, y] : {}^{\mathsf{L}}({}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (12–13), rule T-SumE

(15) $\mathrm{D}^*; \mathrm{G}_1{}^* \vdash [\![e_1]\!]_x^{\{\psi\}\cup\Psi'} :$
${}^{\mathsf{L}}({}^{\mathsf{A}}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*)) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*))$ by IH, $\mathsf{A} \preceq (\{\psi\} \cup \Psi')^*$

(16) $\mathrm{D}^*; \bullet \vdash \lambda x.\ \mathsf{inr}\ x : {}^{\mathsf{A}}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*))$ by rules T-Var, T-Inr, and T-Abs

(17) $\mathrm{D}^*; \mathrm{G}_1{}^* \vdash [\![e_1]\!]_x^{\{\psi\}\cup\Psi'}\ (\lambda x.\ \mathsf{inr}\ x) : {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus t^*)$ by (15–16), rule T-App

(18) $\mathrm{D}^*; \mathrm{G}^*, y{:}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \vdash$
$[[\lambda\_.[\![e_2]\!]_x^{\Psi'}\ y, \lambda x.\ \mathsf{inl}\ x]_\psi, y]\ ([\![e_1]\!]_x^{\{\psi\}\cup\Psi'}\ (\lambda x.\ \mathsf{inr}\ x)) :$
${}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')$ by (14, 17), rule T-App

(19) $\mathrm{D}^*; \mathrm{G}^* \vdash$
$\lambda y.[[\lambda\_.[\![e_2]\!]_x^{\Psi'}\ y, \lambda x.\ \mathsf{inl}\ x]_\psi, y]\ ([\![e_1]\!]_x^{\{\psi\}\cup\Psi'}\ (\lambda x.\ \mathsf{inr}\ x)) :$
${}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (18), rule T-Abs

(20) $\mathrm{D}^*; \mathrm{G}^* \vdash [\![e_1\ \mathsf{handle}\ \psi \to e_2]\!]_x^{\Psi'} :$
${}^{\mathsf{L}}({}^{\xi_0}(t^* \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau')) \multimap {}^{\mathsf{L}}({}^{\mathsf{U}}\mathsf{exn} \oplus \tau'))$ by (2, 19).