Reducibility Among Fractional Stability Problems

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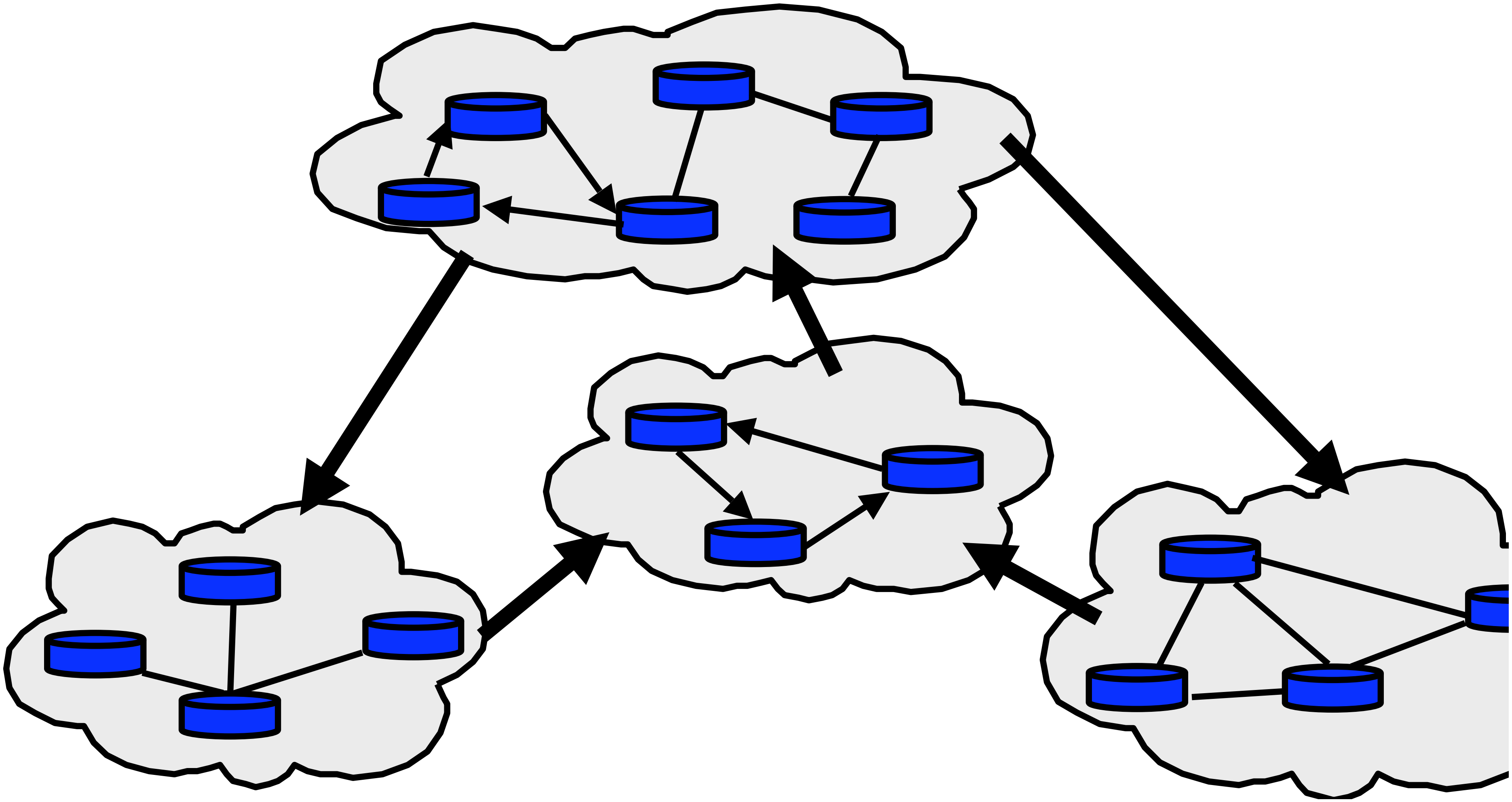
Shang-Hua Teng (Boston U./USC)

(Slides courtesy of Laura Poplawski)

Border Gateway Protocol

- Rehkter, Li. A Border Gateway Protocol (BGP version 4). RFC 1771, 1995.



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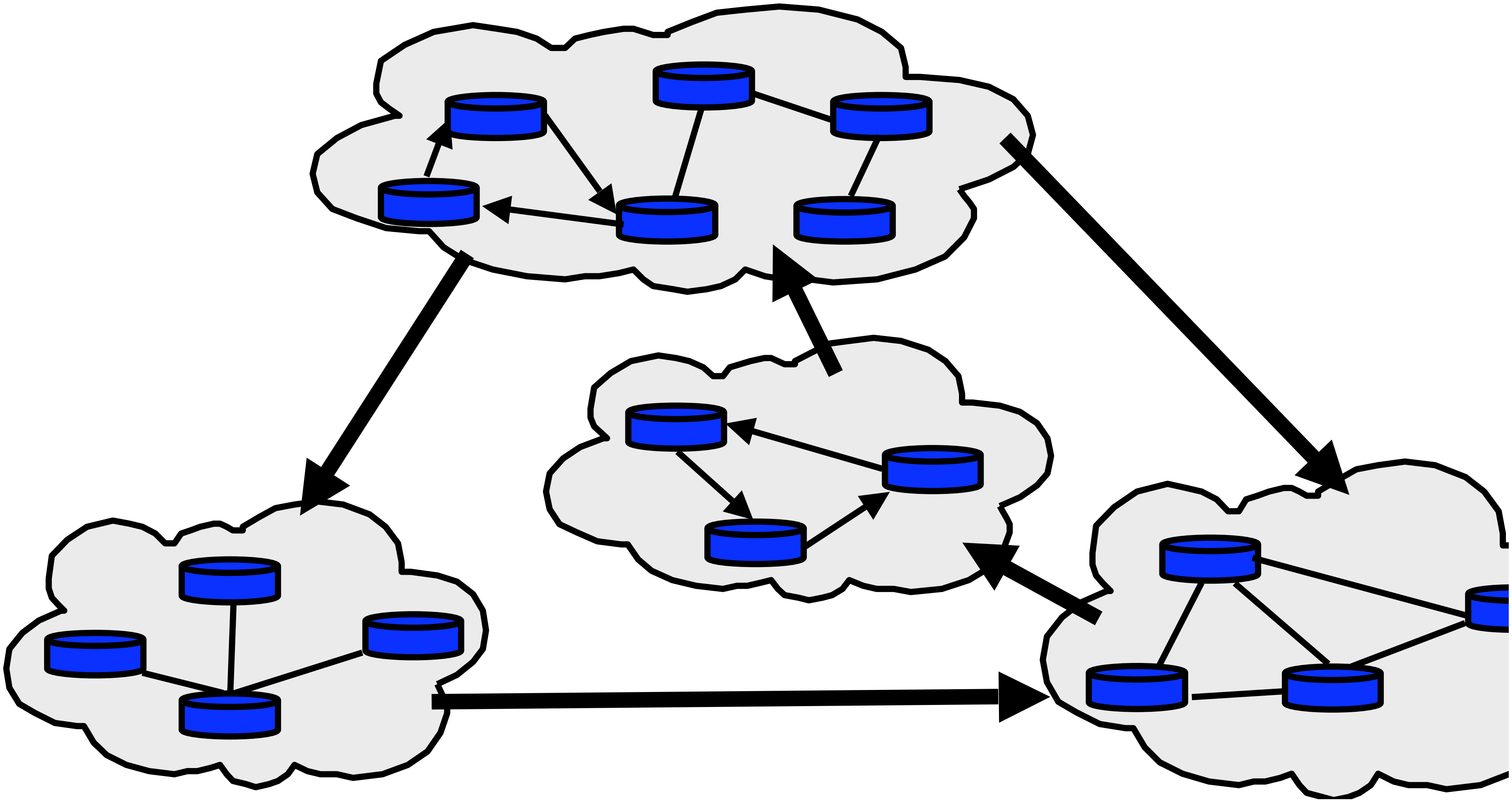
- Varadhan, Govindan, and Estrin, 1996. Persistent Route Oscillations in Inter-Domain Routing.



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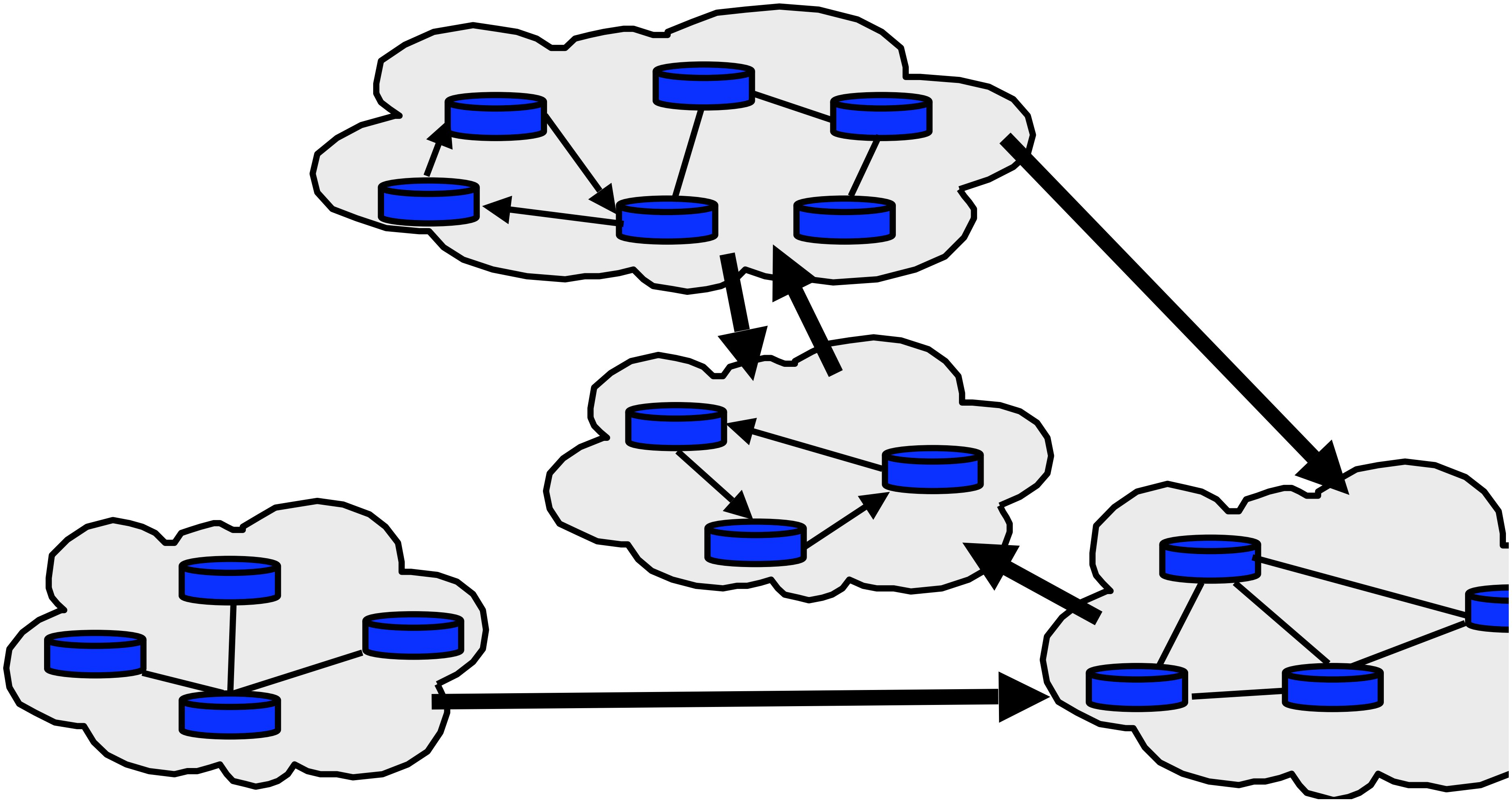

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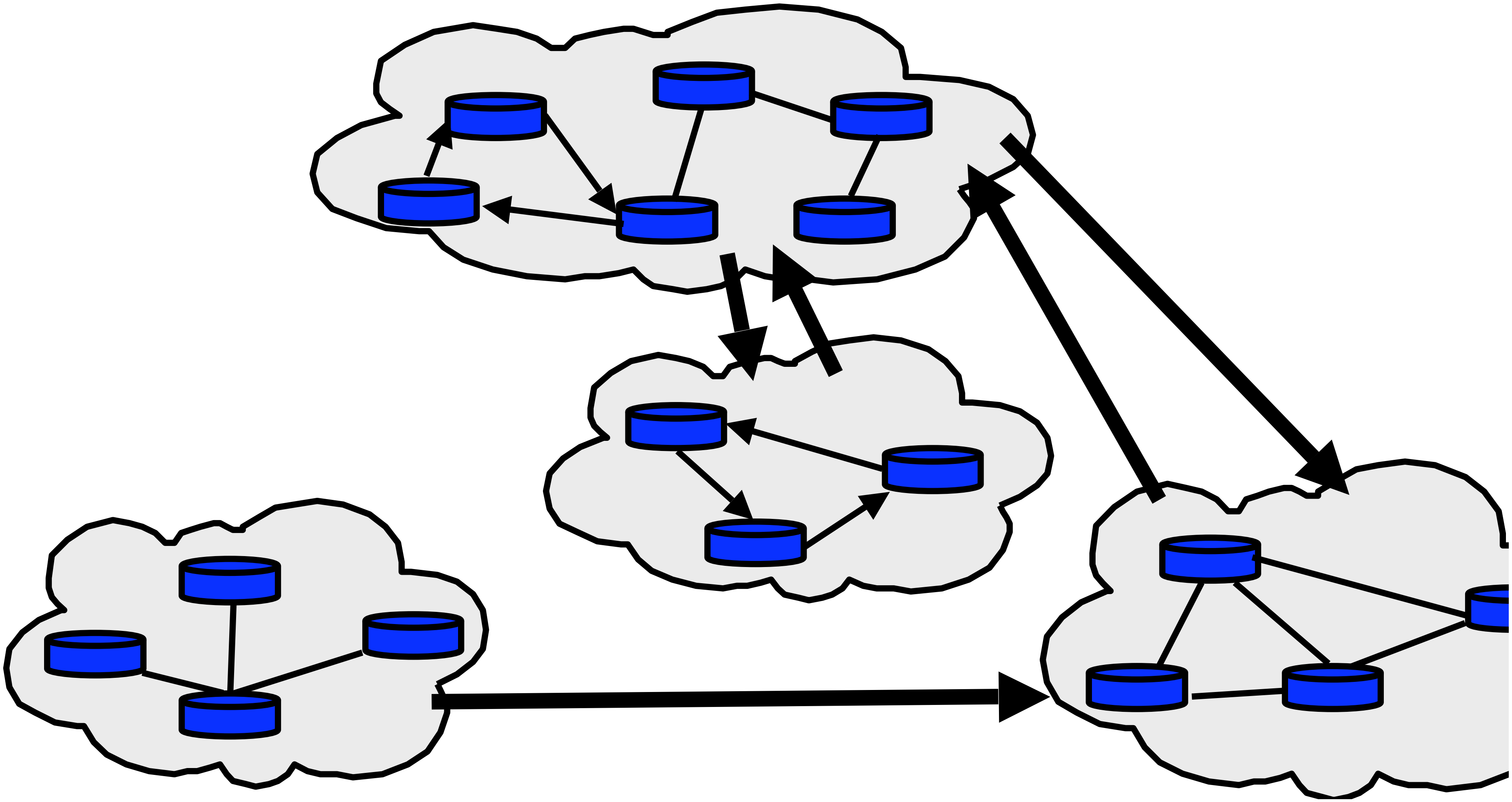
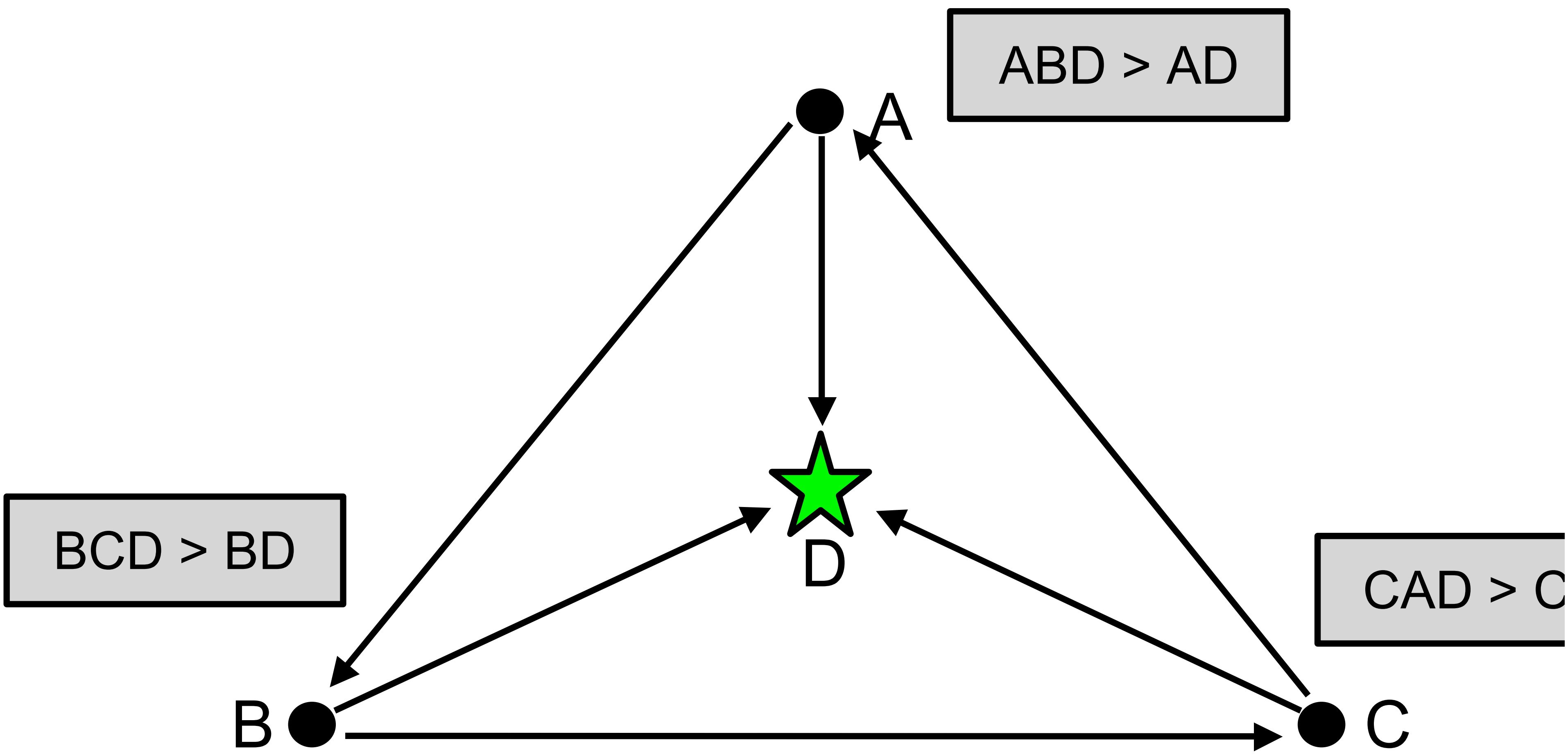


Stable Paths Problem

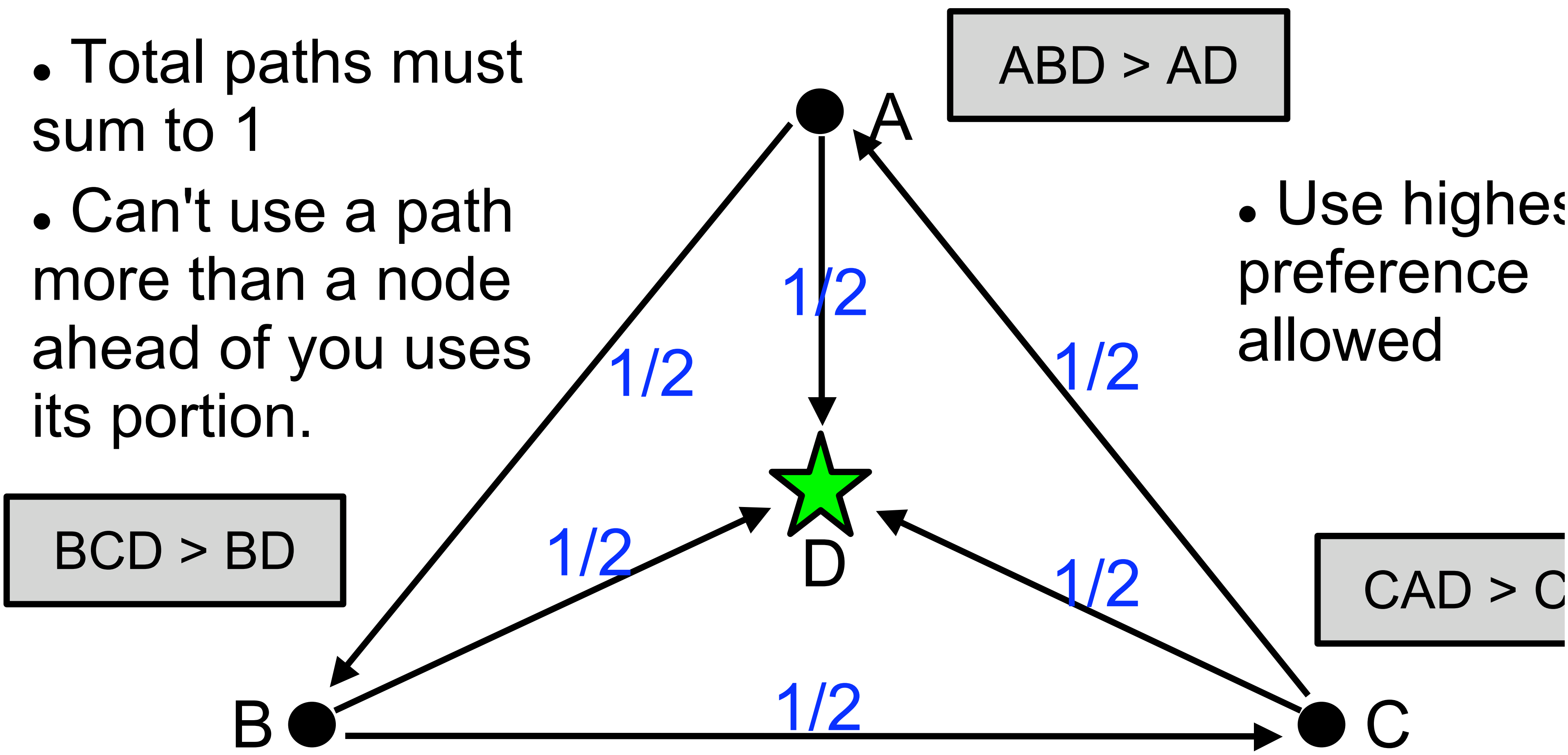
- Griffin, Shepherd, and Wilfong. The stable paths problem and interdomain routing. Transactions on Networking, 2002.



Fractional Stable Paths Problem

- Haxell and Wilfong. A fractional model of the border gateway protocol (BGP). SODA, 2008.

- Total paths must sum to 1
- Can't use a path more than a node ahead of you uses its portion.



- Use highes preference allowed

Stable Paths Problem as a Gam

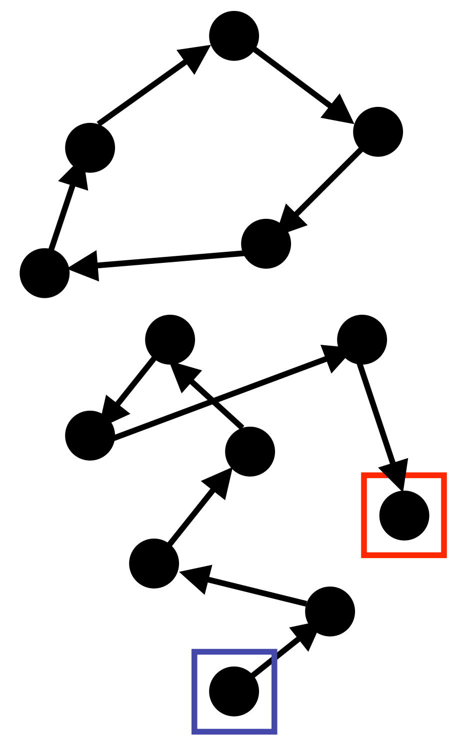
- Node's strategy set is collection of paths to destination
- Preference relation among strategies: Strategy *P* is preferred at least as much as strategy *P'* if
  - either *P'* is not feasible, or
  - both are feasible and path *P* is more preferred than path *P'*
- Utility for feasible path given by its preference
- A stable solution is precisely a pure Nash equilibrium
- NP-complete to determine whether a given SPP instance has a stable paths solution [Griffin,Shepherd,Wilfong 02]
- In every FSPP instance, there exists a stable solution [Haxell-Wilfong 08]

Fractional Hypergraph Matching

- Hypergraphic Preference System:
  - A hypergraph G = (V,E)
  - Each vertex has a linear order over its incident edges
- Stable Matching:
  - Each vertex is in at most one edge
  - For each edge e, there exists a vertex v in e and an edge m matching such that v prefers m over e
- Stable fractional matching: $w: E \rightarrow R$
  - For each vertex, total weight of incident edges at most 1
  - For each edge e, there exists v in e such that sum of weights edges that v prefers over e equals 1.
- A stable fractional matching always exists [Aharoni-Fleiner 03]

Complexity class PPAD

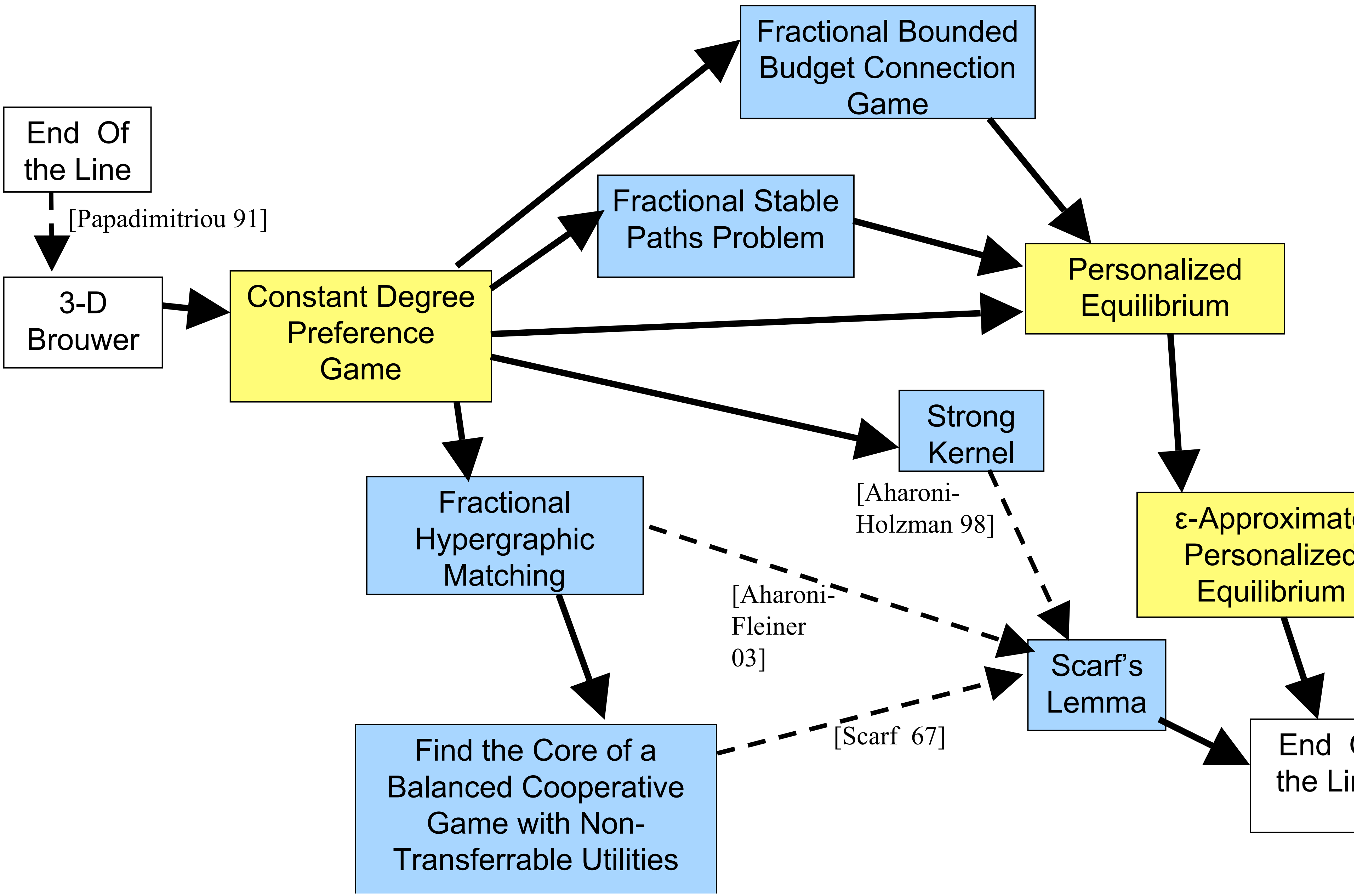
- Search problems for which existence proofs are based on parity arguments
  - Polynomial Parity Argument in a Directed graph [Papadimitriou 94]
- All problems poly-reducible to END OF THE LINE
  - Number of vertices 2^n
  - Given poly-size predecessor and successor circuits and source vertex label
  - In-degree and out-degree at most one
  - At least one source implies at least one sink



PPAD-Hard Problems

- PPAD-complete problems:
  - Sperner's Lemma, discrete versions of Brouwer's Fixed Point Theorem, Borsuk-Ulam Theorem
  - Nash equilibria in matrix games
- Every matrix game has a mixed Nash equilibrium [Nash 51]
- There exist 3-player games with rational inputs in which every Nash equilibrium is irrational [Nash 51]
- For 4-player games, ε-Nash is PPAD-complete [Daskalakis,Goldberg,Papadimitriou 06]
- PPAD-completeness for 3- and 2-player games [Chen,Deng 06; Chen,Deng,Teng 06]

Main Results:A Slew of Reductio



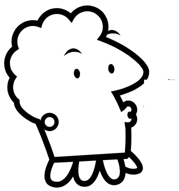
Structural and Hardness Results

- Preference Game: A very simple new game that captures the complexity of several stability problems
  - Easily reduces to FSPP, Stable Fractional Matching, Core of Balanced Games, Computational version of Scarf's Lemma.
- Reduce Brouwer's fixed point problem to Preference Game:
  - No fully polynomial time approximation scheme for Preference Games, unless FP = PPAD.
- Personalized Equilibrium: A new notion for matrix games that generalizes several stability problems
- The set of stable solutions can be expressed as the union of (an exponential number of) linear programs
  - Rational solutions always exist
  - Also useful in placing all the above problems in PPAD.

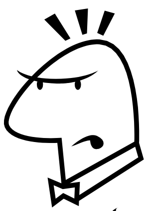
Outline of Talk

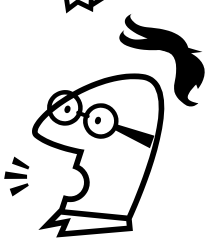
- The Preference Game
- Preference Game reduces to Fractional Stable Paths Problem (FSPP)
- PPAD-hardness:
  - Exact and ε-approximate equilibria
- Other Stability problems

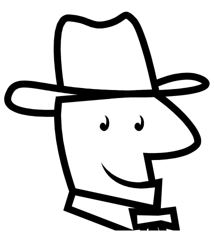
The Preference Game - given

Alice 

Bob 

Charlie 

David 

Ed 

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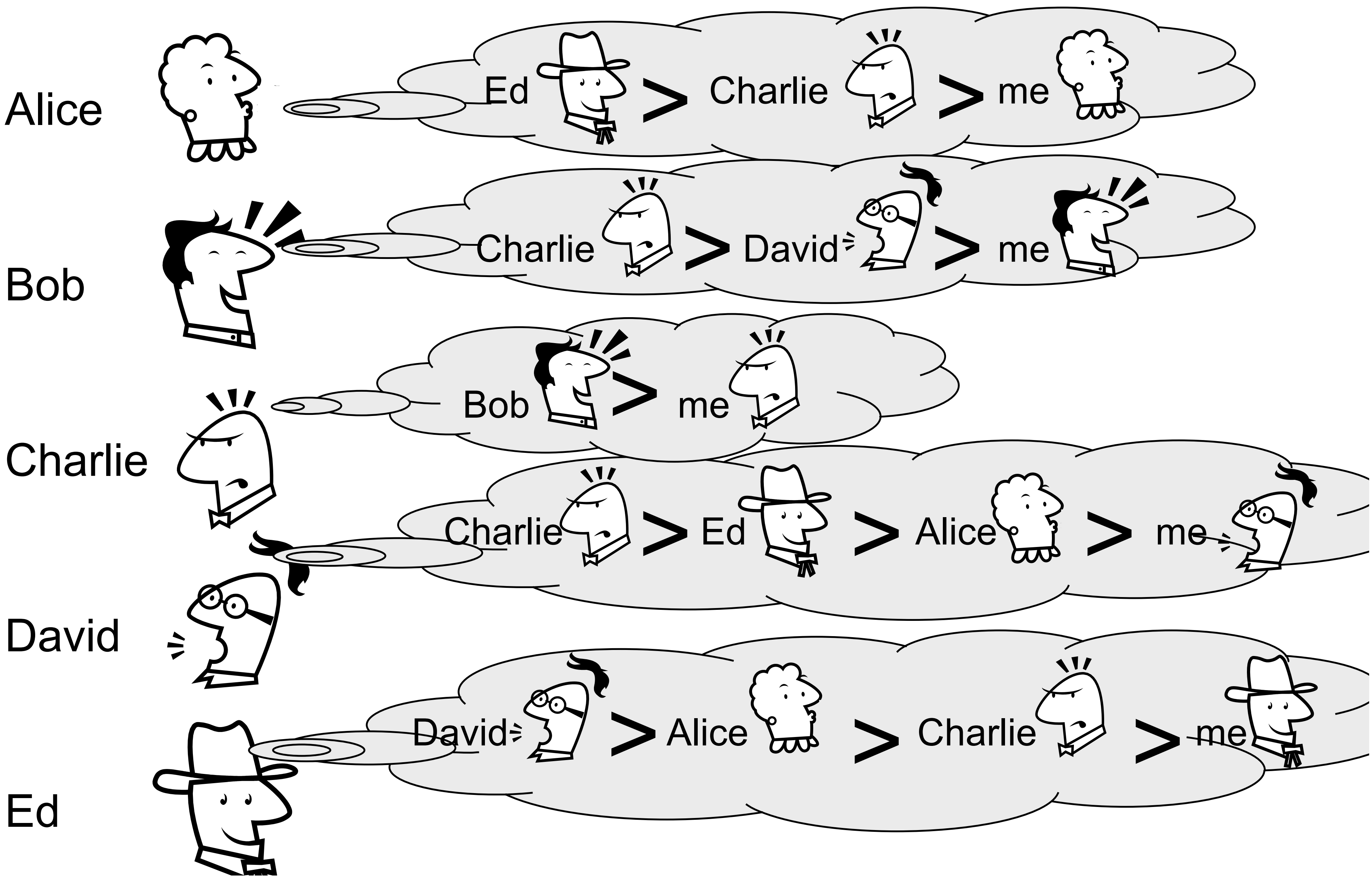
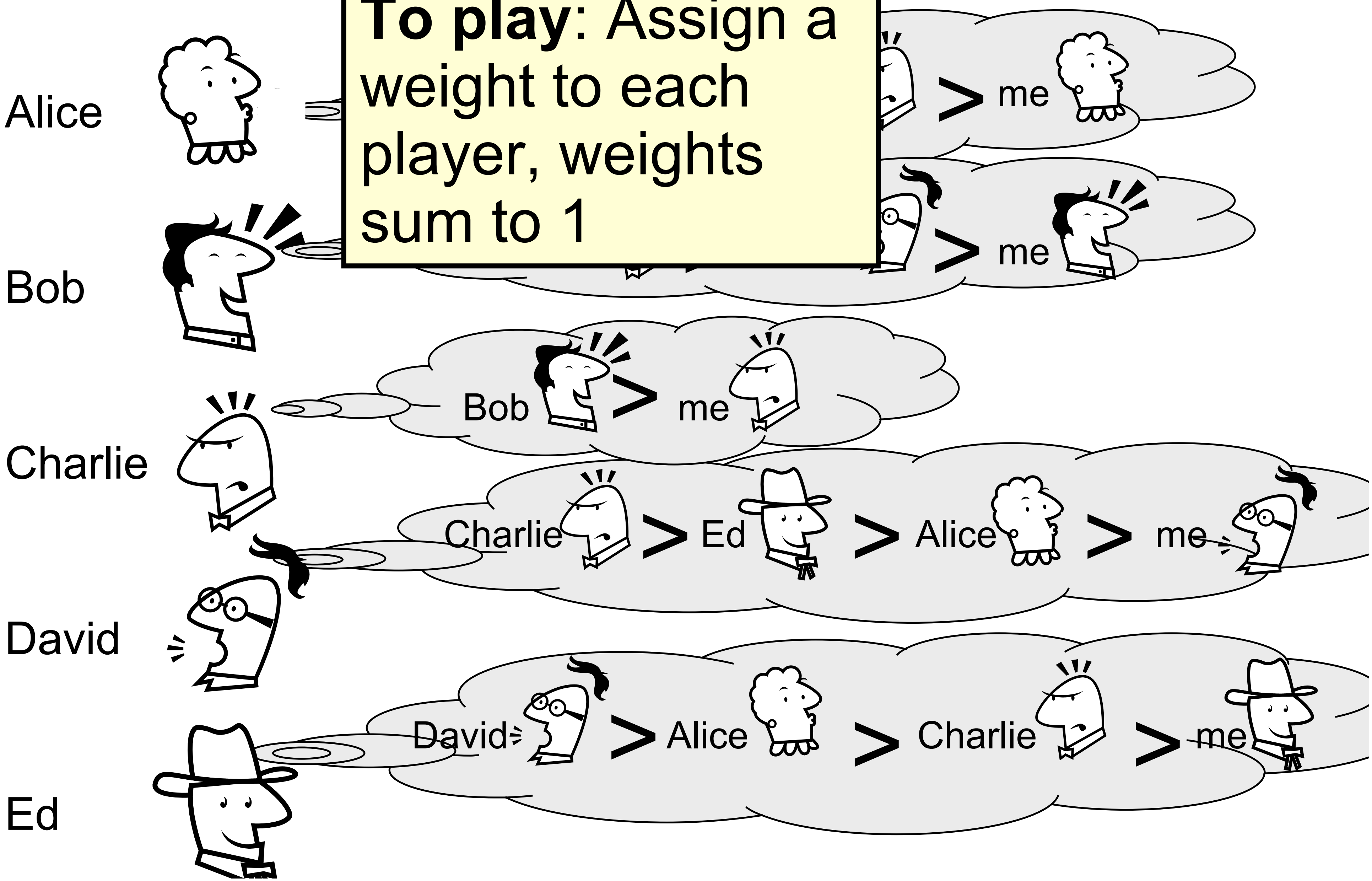


The Preference Game - given

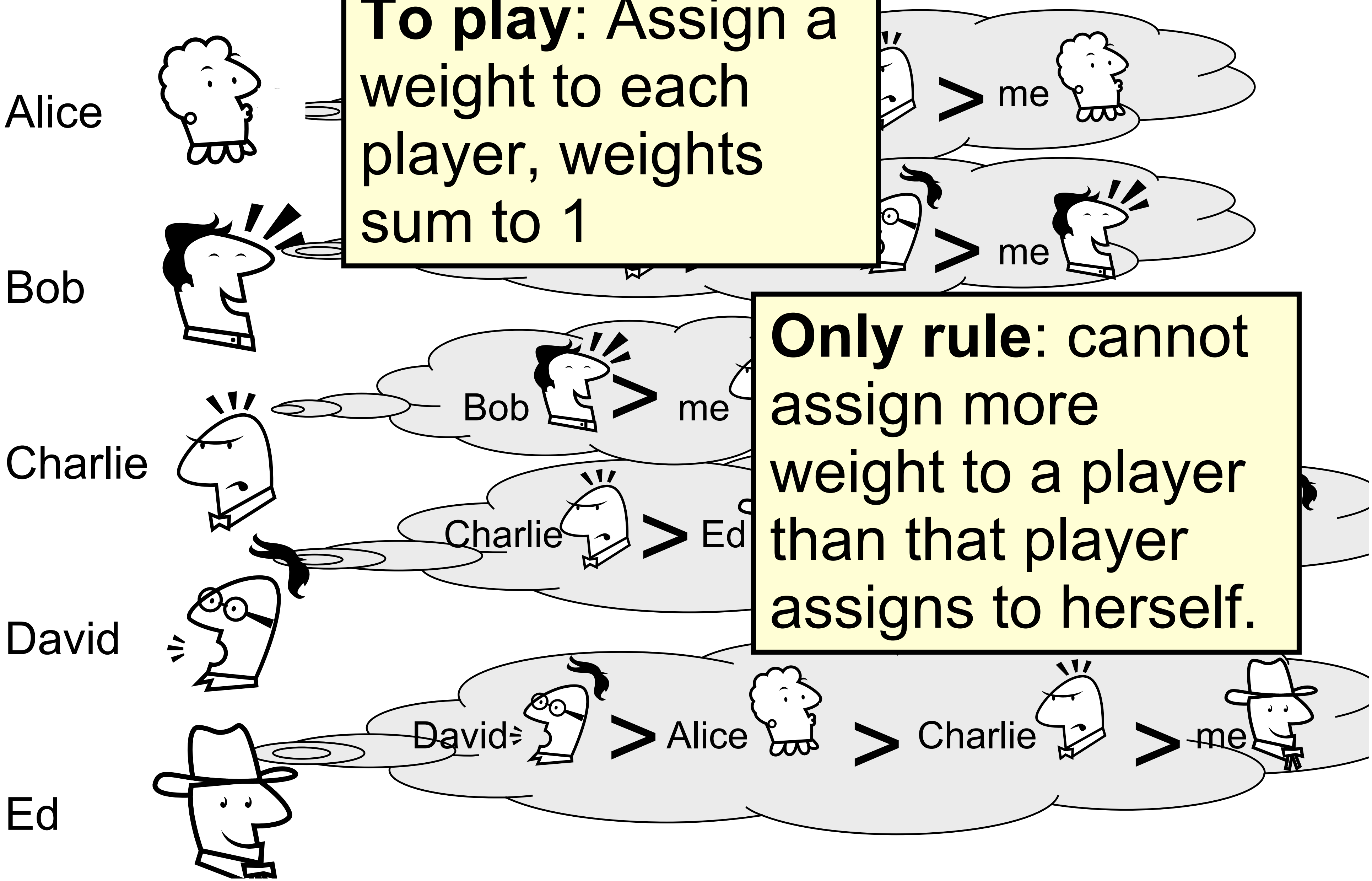
To play: Assign a weight to each player, weights sum to 1



The Preference Game - given

To play: Assign a weight to each player, weights sum to 1

Only rule: cannot assign more weight to a player than that player assigns to herself.



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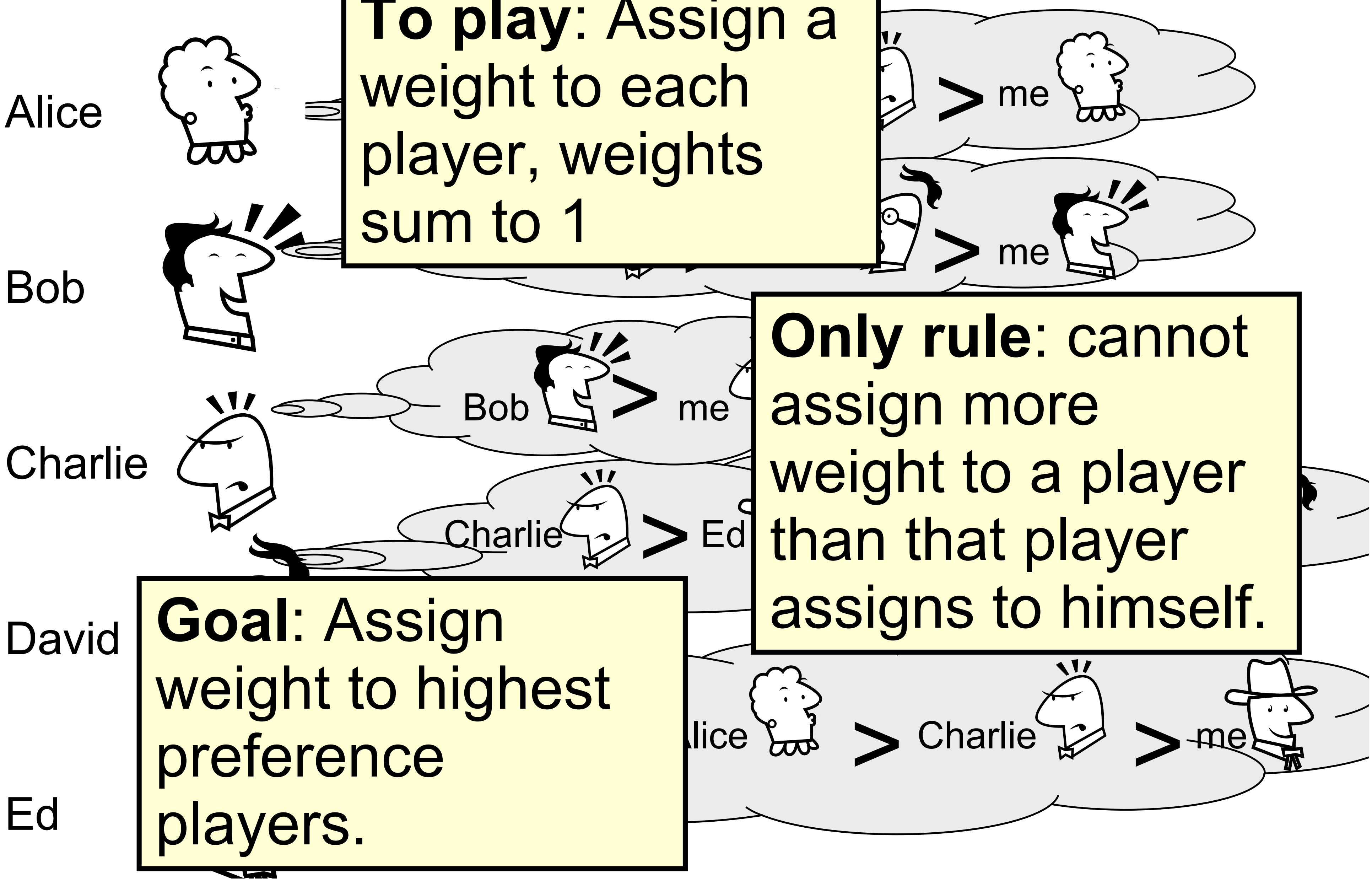
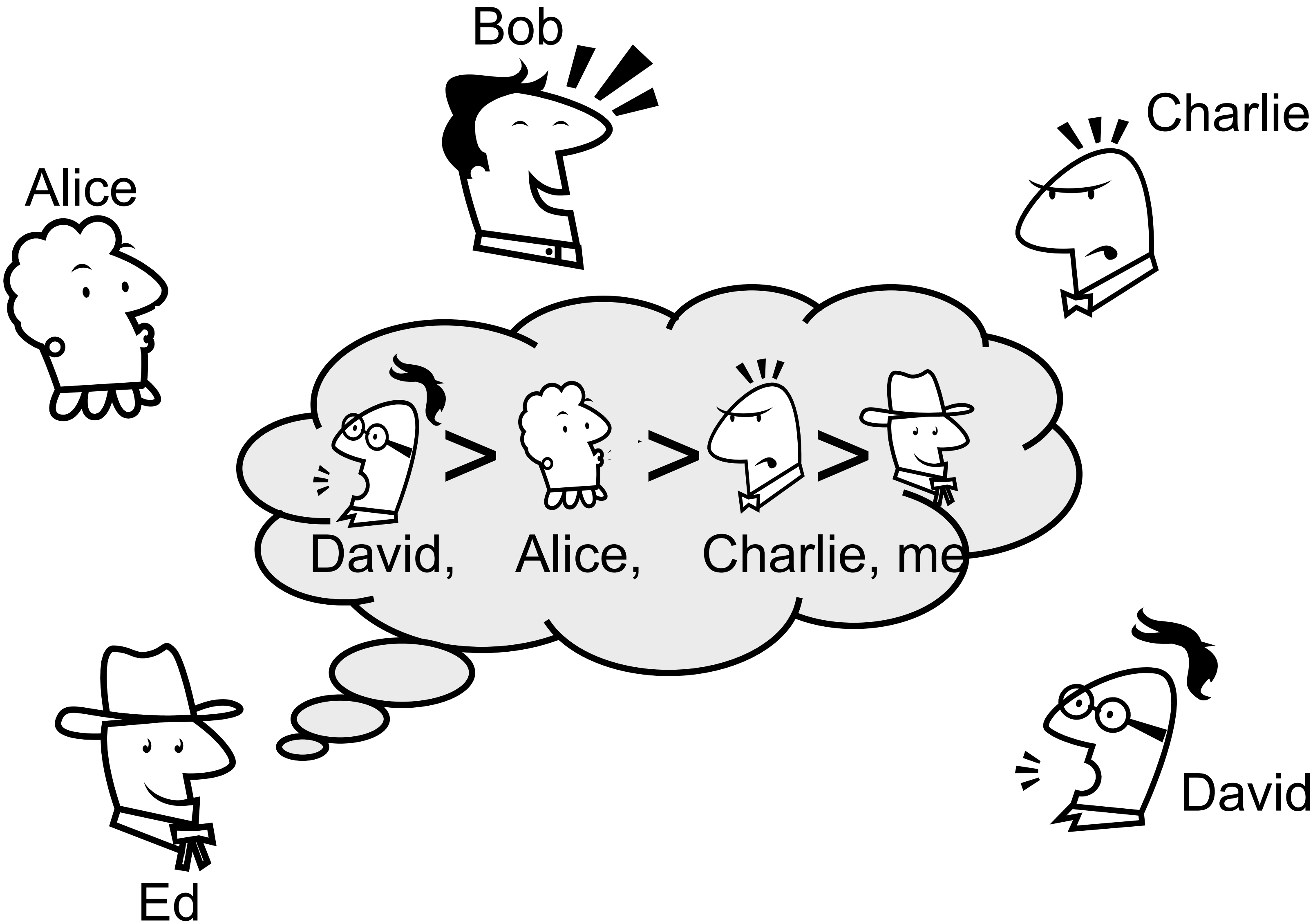


To play: Assign a weight to each player, weights sum to 1

Only rule: cannot assign more weight to a player than that player assigns to himself.

Goal: Assign weight to highest preference players.

The Preference Game



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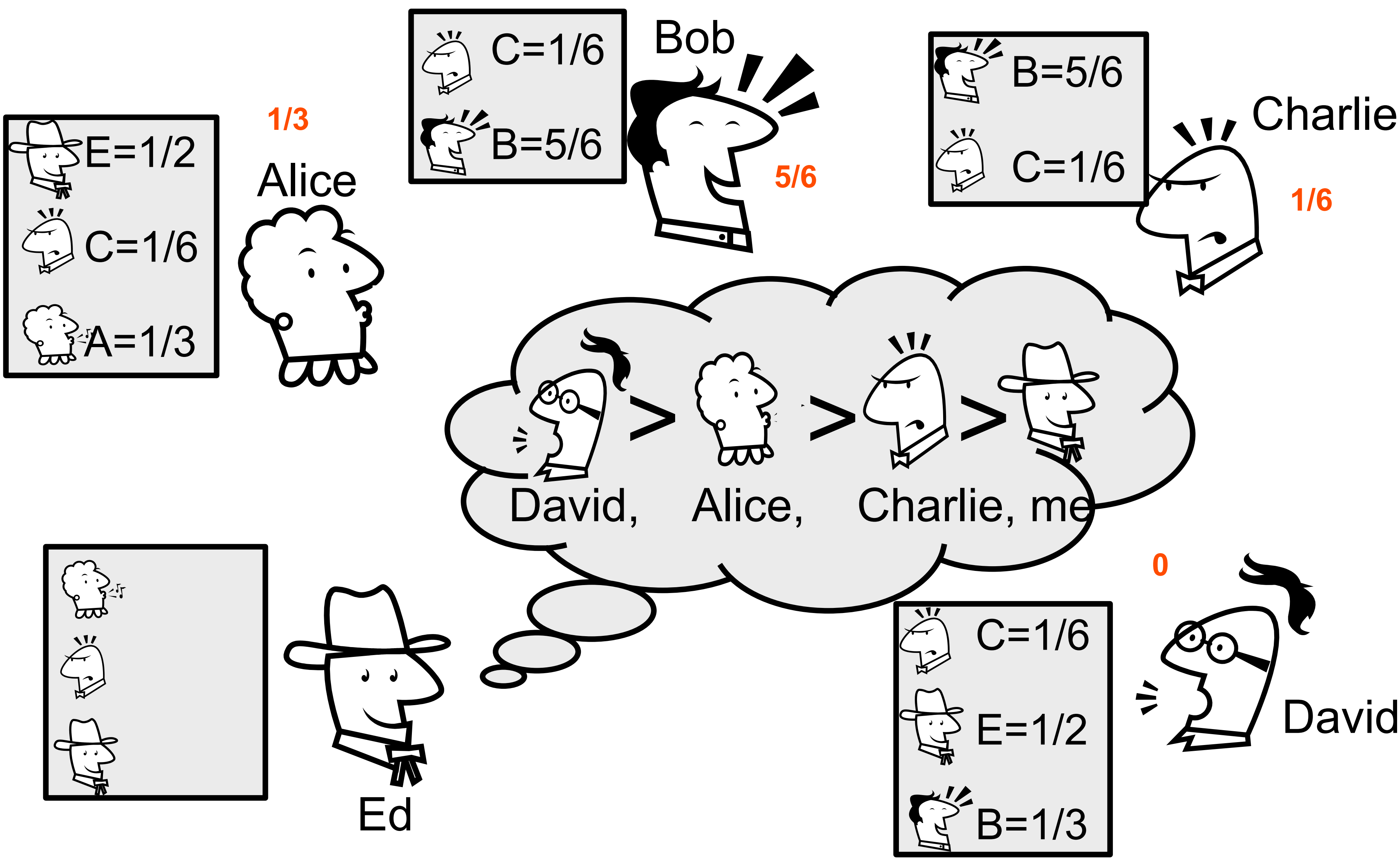

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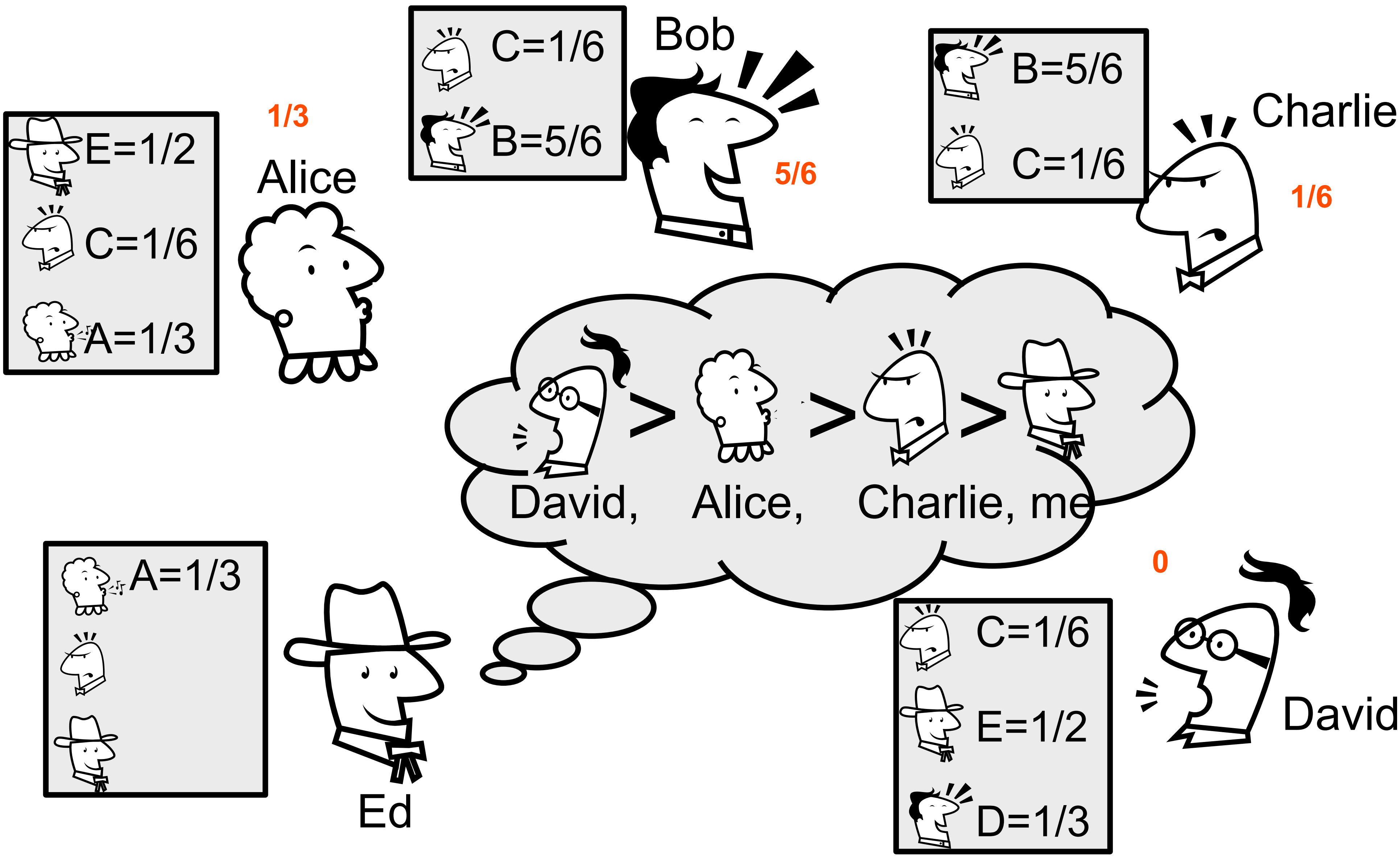

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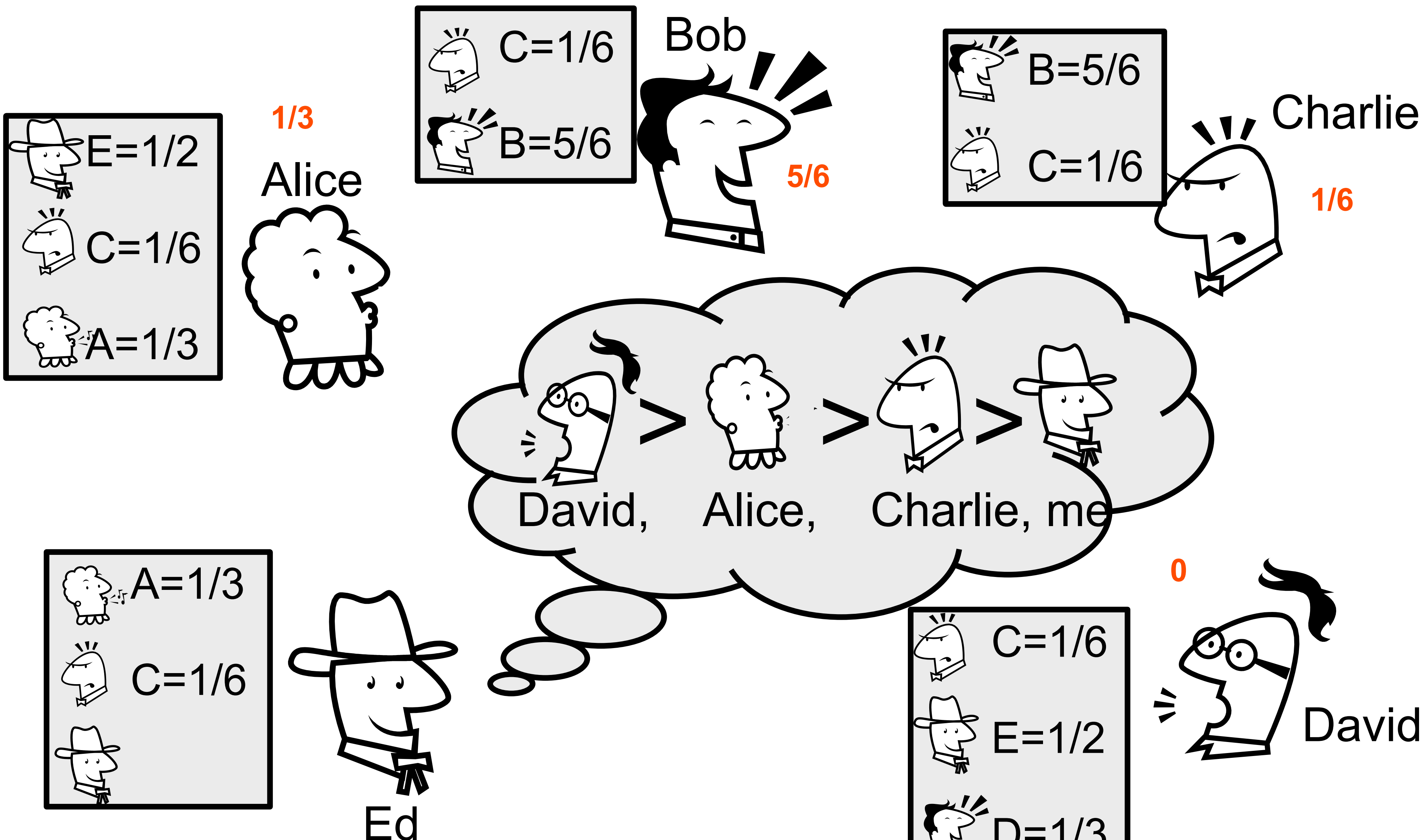

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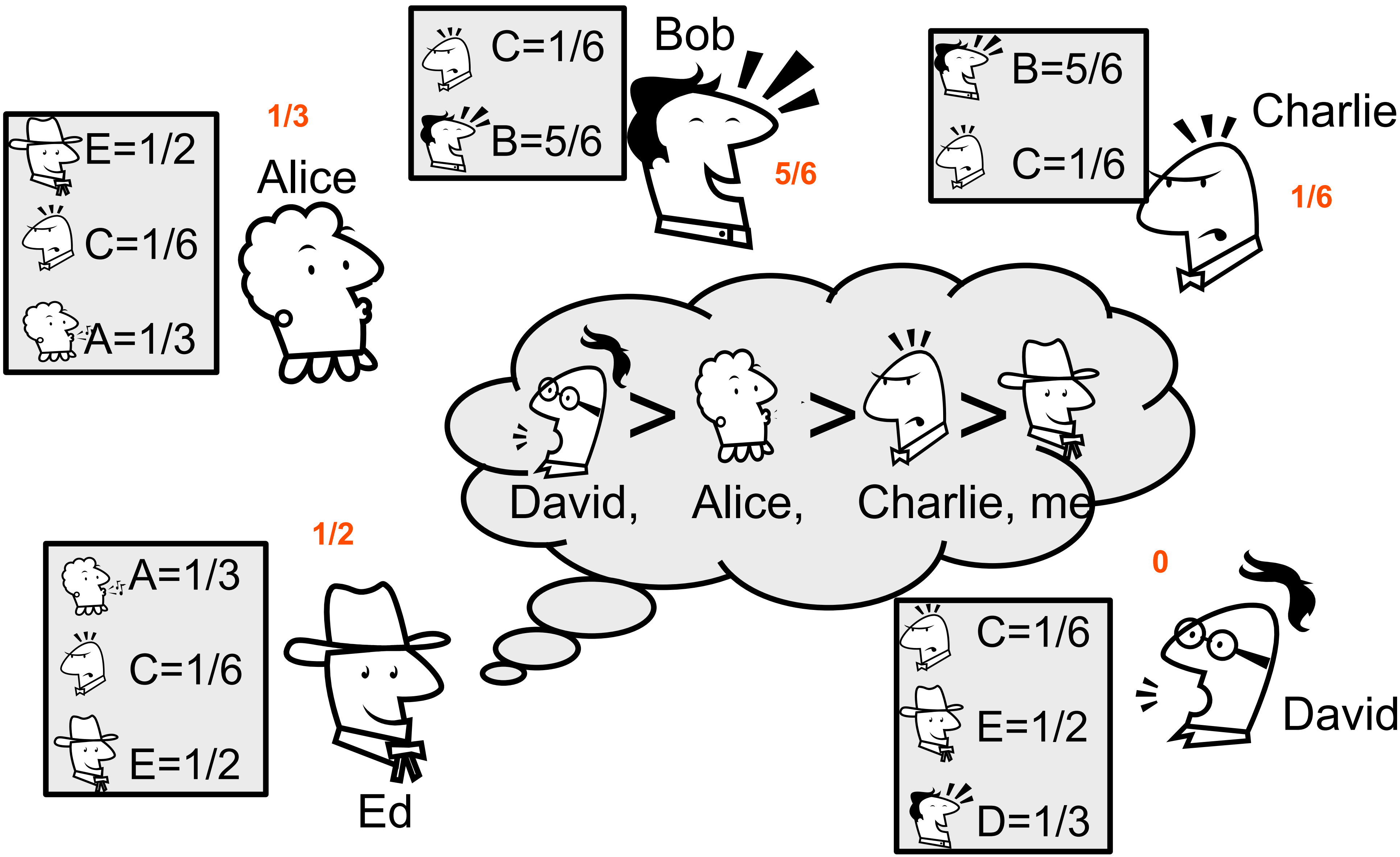


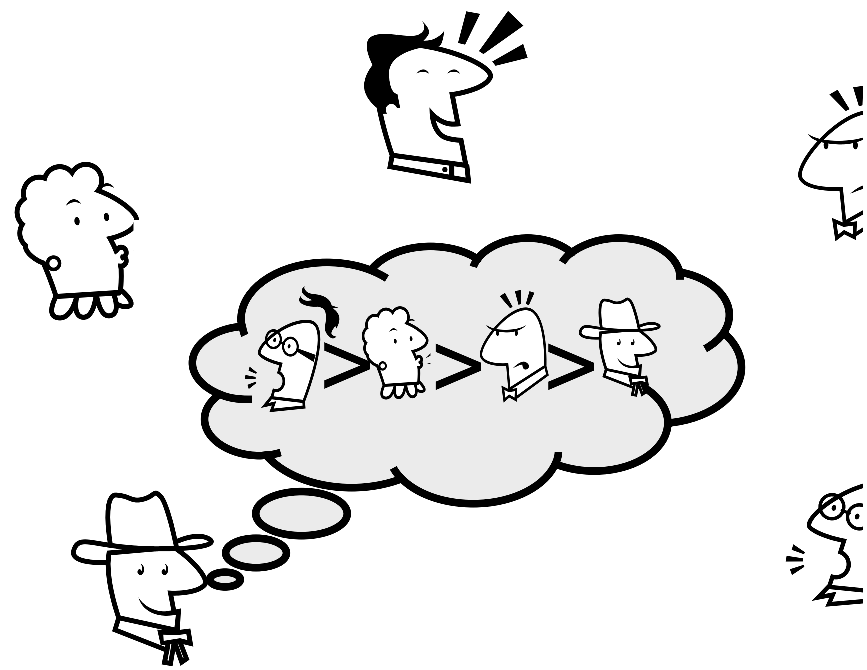
The Preference Game - Notation

- Each player *i* assigns weight *w(i,j)* to each player *j*

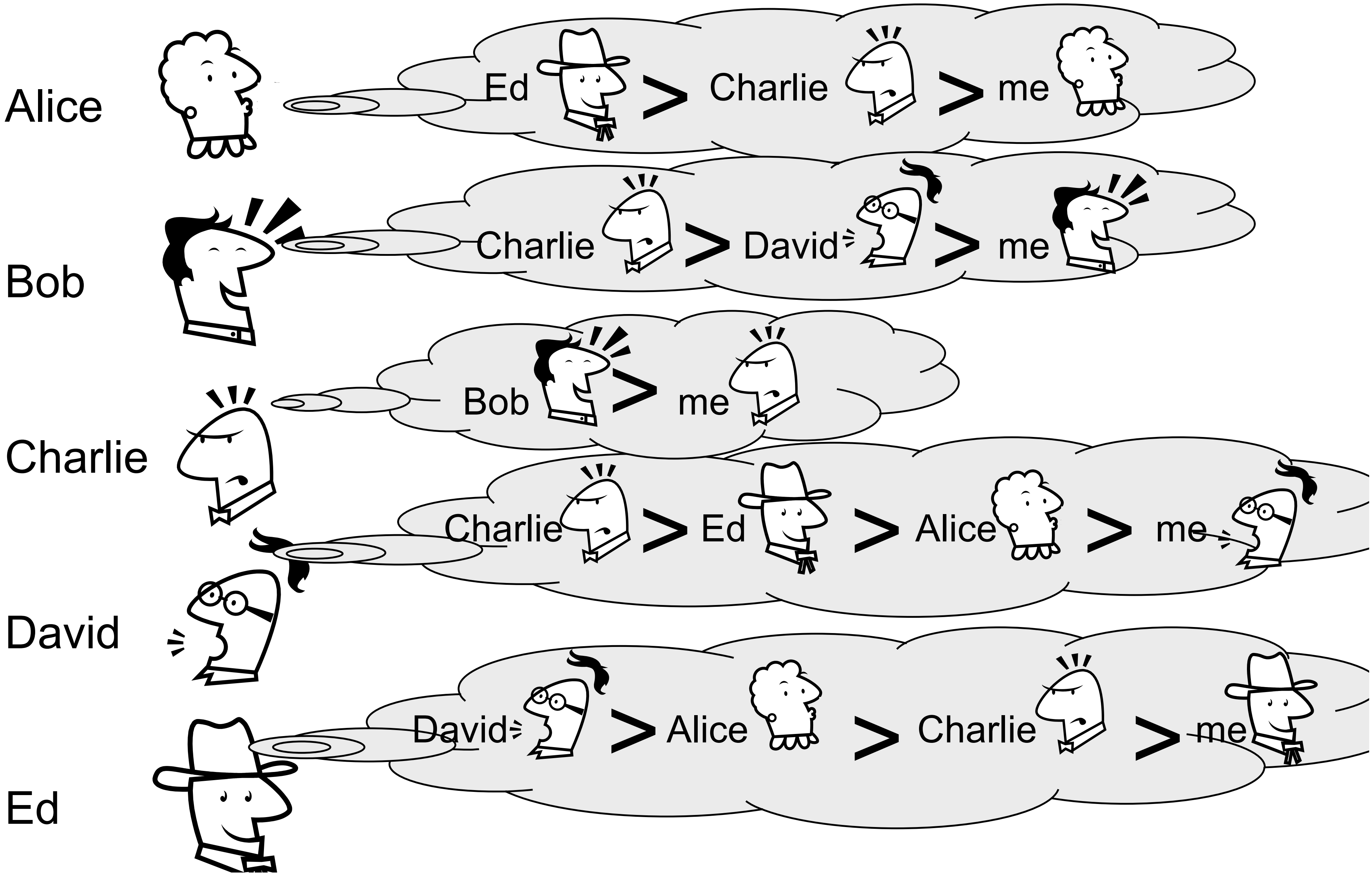
$$\sum_j w(i,j) = 1 \qquad \forall i$$

$$w(i,j) \le w(j,j) \quad \forall i,j$$

- Best Response: Cannot move weight from a lower preference to a higher preference



Reducing Preference Game to Fractional Stable Paths Problem



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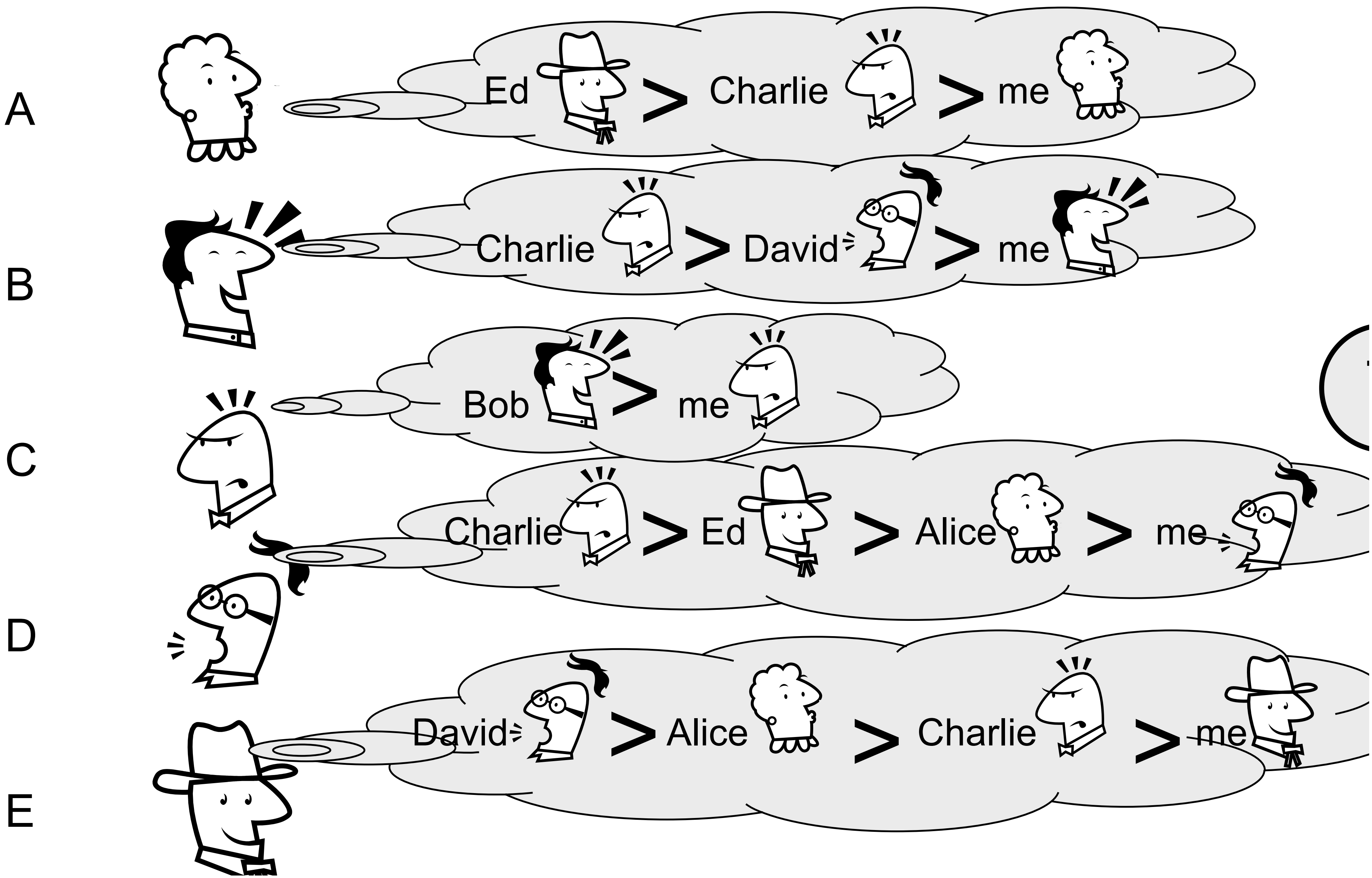

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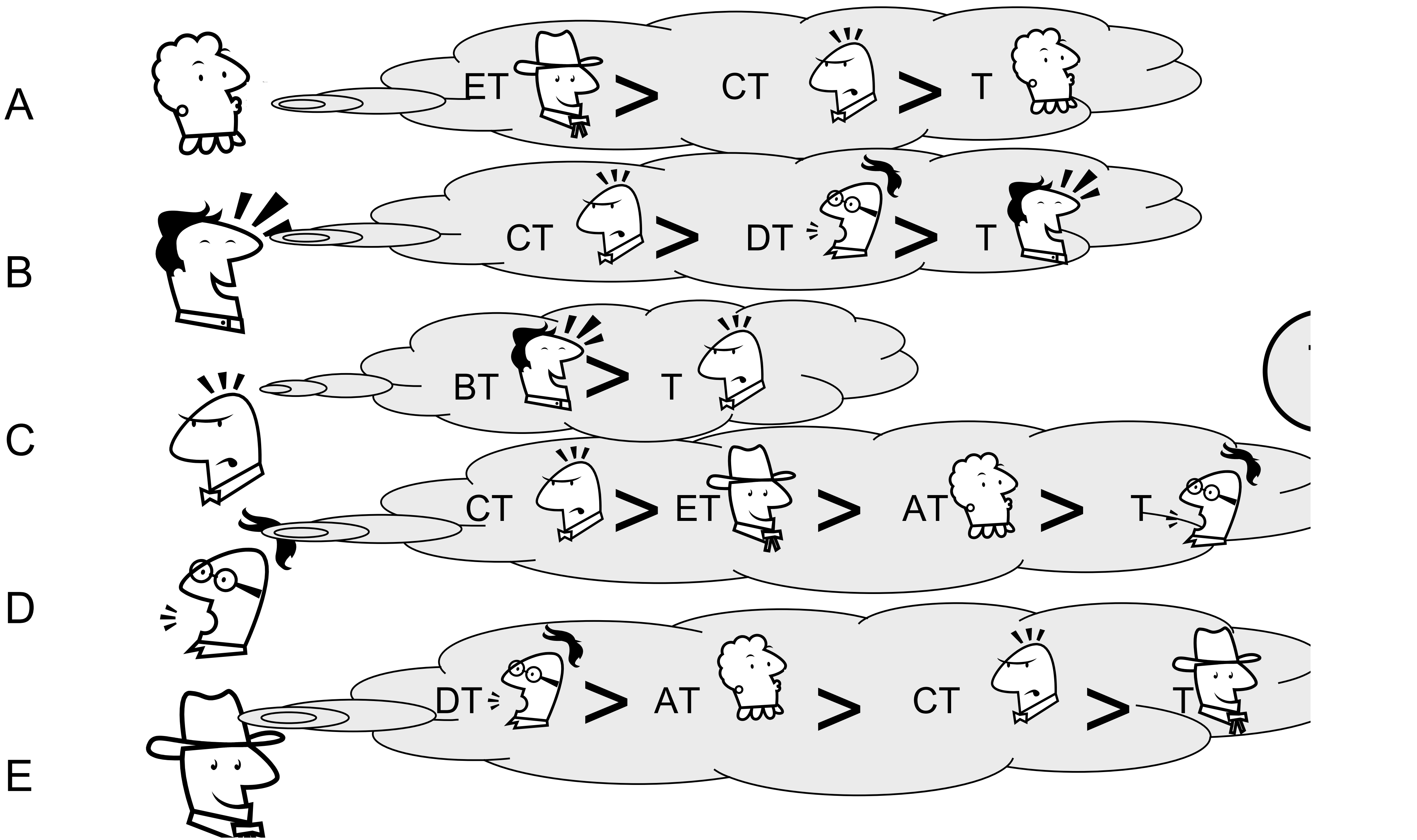


Equilibrium in Fractional Stable Paths Proble ⇔ Equilibrium in Preference Game

Rules for Fractional Stable Paths Problem

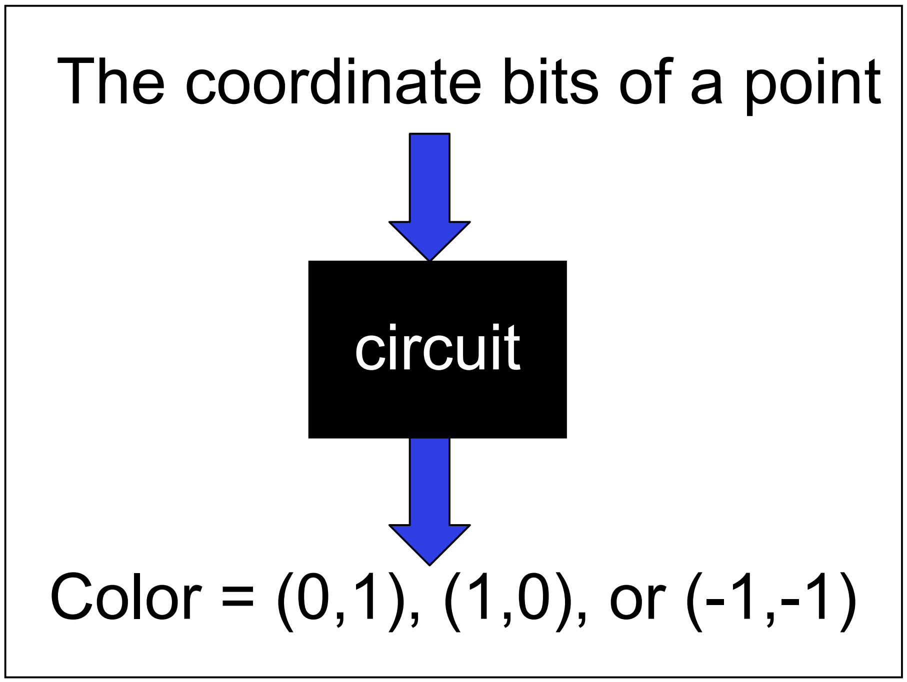
- Total paths must sum to 1
- Can't use a path more than a node ahead of you uses its portion.
- Use highest preference paths

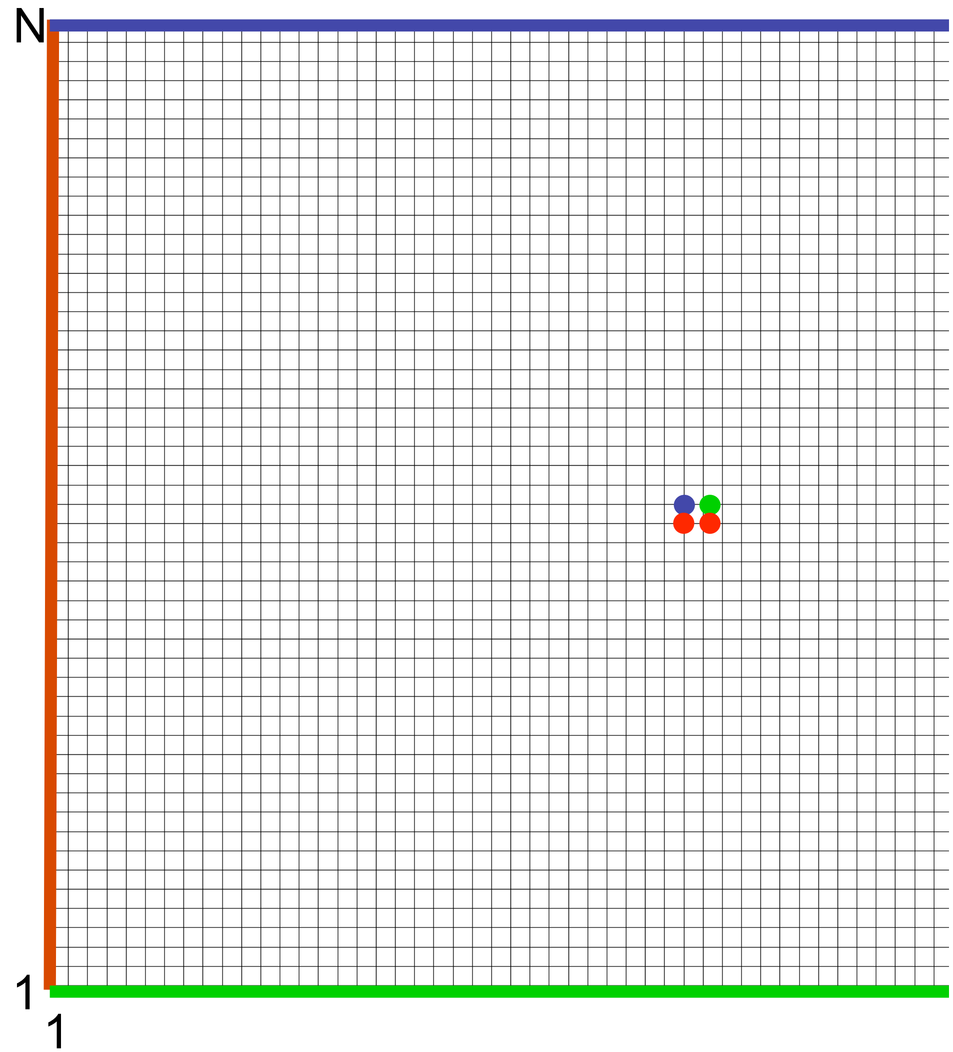
Rules for the Preference Game

- Place a total of weight 1
- Can't place more weight on another player than that player places on itself
- Put weight on highest preferences possible

Computational 2D Brouwer

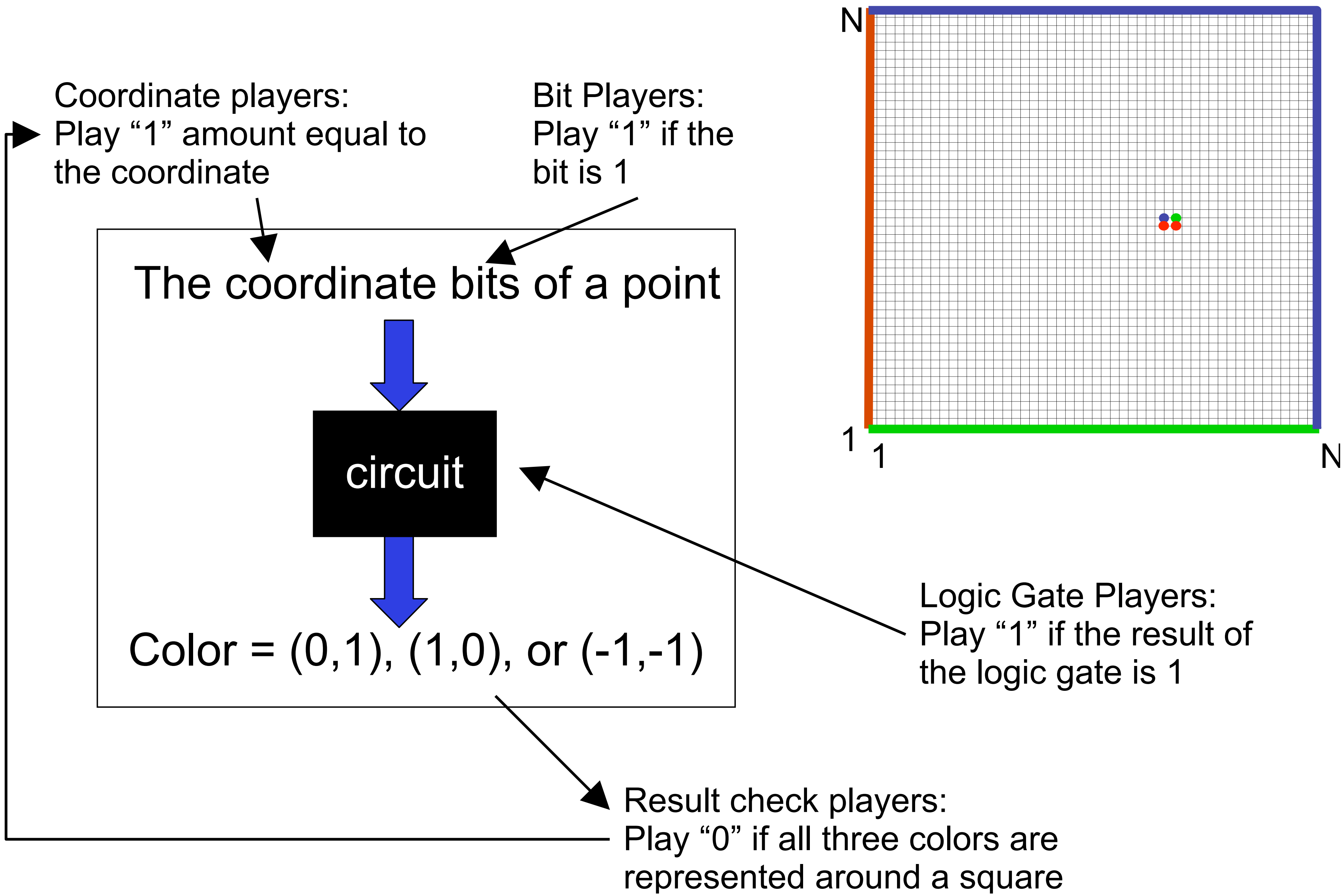
- Exponentially large grid: N = 2^n
- Given a circuit:





- Find a 3-color triangle.

A reduction framework (Daskalakis-Goldberg-Papadimitriou)



A reduction framework
(Daskalakis-Goldberg-Papadimitriou)

- (Almost) sufficient to be able to create players that will compute:
  - And
  - Or
  - Not
  - Sum
  - Difference
  - Copy
  - Double
  - Half
  - Is Less Than

Overview of Reduction

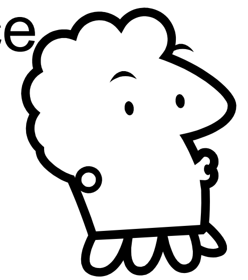
- Encode values in Brouwer instance as the weight player assigns to itself
- Coordinate player P: w(P) equals coordinate value
- Need to extract bits from coordinate
  - SUM(A,B): In equilibrium, SUM(A,B) plays the sum of what A and B play
  - DIFF, LESS, COPY
- LESS with error

- Circuit simulation:
  - OR(A,B): In any equilibriun A, B play from {0,1}, then p the OR of their values
  - AND and NOT gadgets
  - Output is one of {(1,0), (0,1 (-1,-1)}
- Compute circuit at a larg (constant) number of points around (*x*,*y*)
- Compute average of thes and add it back to the coordinate players

A Preference Game Gadget

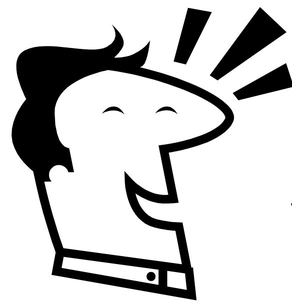
- Difference: A - B

Alice



Plays self with value A

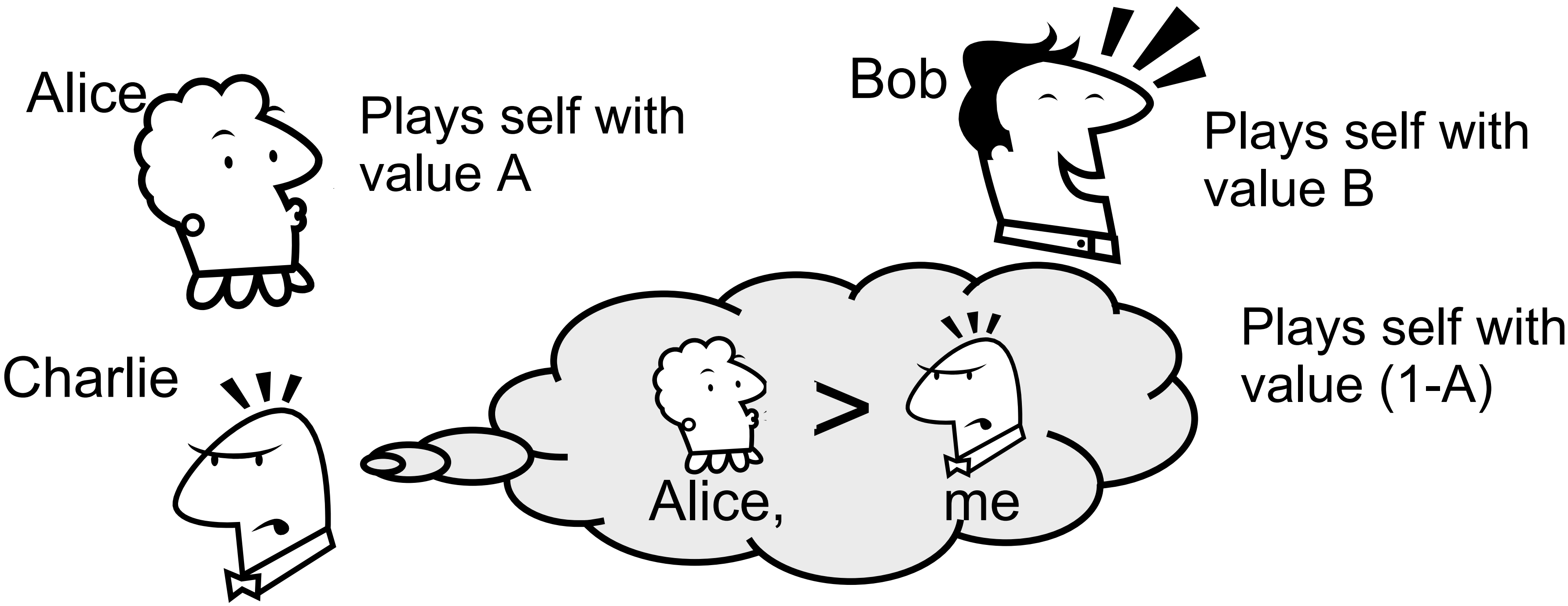
Bob



Plays self with value B

A Preference Game Gadget

- Difference: A - B



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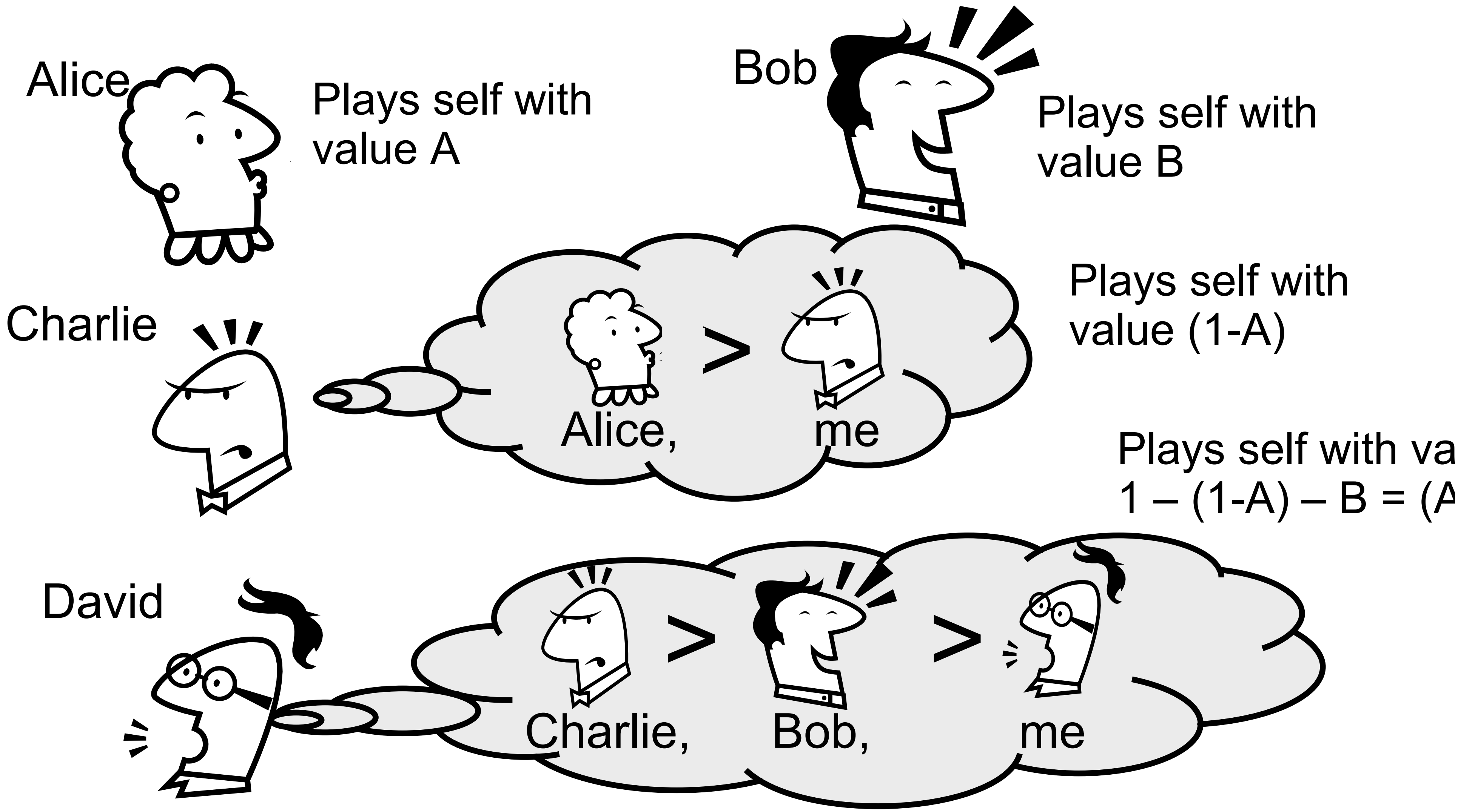

Approximate Equilibria

- ε-Approximate Equilibrium: weight distribution w such th
  - $\sum_j w(i,j) = 1$ for all i
  - For each j, w(i,j) is at most w(j,j) + ε
  - For each j, either |w(i,j) - w(j,j)| is at most ε or we have $\sum_{i \text{ prefers k over j}} w(i,k)$ is at most 1 - ε
- PPAD-hard to find ε-approximate equilibrium in time for inverse polynomial in n
  - Idea similar to [Chen-Deng-Teng 06]
  - Reduce from n-dimensional Brouwer
  - Each "cell" is a n-hypercube, colors assigned from {1,2,…,n,n+
  - Seeking a panchromatic simplex inside a hypercube
- Main hurdle:
  - Errors introduced in Boolean gadgets
  - Prevent magnification of errors by strategically adding LESS gadgets after each logic step

Structure of Preference Game Equilib

- Feasible solution:

$$\sum_j w(i,j) = 1 \qquad \forall i$$

$$w(i,j) \le w(j,j) \quad \forall i,j$$

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- Best response: For each player *i*, there exists a "threshold" player *k* with

$$w(i,j) = w(j,j) \quad \forall j >_i k$$

$$w(i,j) = 0 \qquad \forall j <_i k$$

Structure of Preference Game Equilibria – Rational Solution

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- n^n Linear Programs wil cover all combinations.

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- If an LP is feasible, then it has a rational solutior Thus, the preference game has a rational solution.

Complexity of preference games

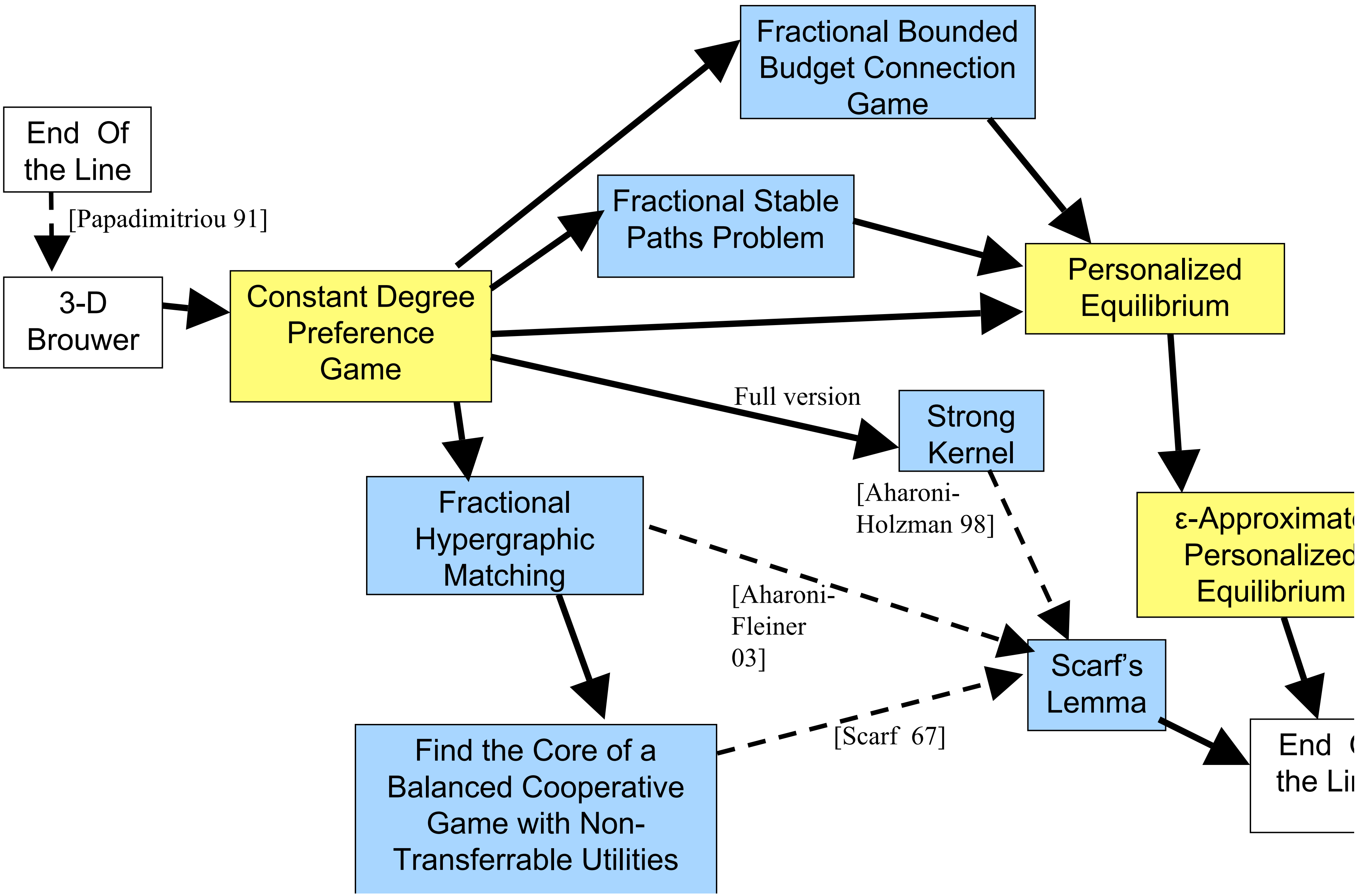
- PPAD hard, even to compute approximate equilibrium
- A rational equilibrium always exists

Complexity of Preference Game

- PPAD hard to compute approximate equilibrium
- A rational equilibrium always exists
- Implies membership in PPAD
  - Finding approximate equilibrium is in PPAD
  - Given a point ε away from an LP feasible region
    ⇒ The LP is feasible
  - Approximate equilibrium is ε away from at least one of our exponentially many LPs.
  - Can modify our "union of many LPs" to get a single feasible LP.

Other Fractional Stability Problem

- Finding the core of a balanced cooperative game with non-transferrable utilities [Scarf 67]
- Computational version of Scarf's Lemma [Scarf 67]
- Finding a fractional hypergraph matching [Aharoni-Fleiner 03]
- Finding a strong fractional kernel [Aharoni-Holzman 9
- Finding an equilibrium in the fractional Bounded Budg Connection Game [Laoutaris et al 08]
- Personalized equilibrium: variant of correlated equilibrium; each player assigns personal weights to strategies subject to "projection constraints"



Fractional Bounded Budget Connection Game
End Of the Line
[Papadimitriou 91]
3-D Brouwer
Constant Degree Preference Game
Fractional Stable Paths Problem
Personalized Equilibrium
Full version
Strong Kernel
[Aharoni-Holzman 98]
Fractional Hypergraphic Matching
ε-Approximat
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[Aharoni-Fleiner 03]
Scarf's Lemma
[Scarf 67]
Find the Core of a Balanced Cooperative Game with Non-Transferrable Utilities
End
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