Reducibility Among Fractional Stability Problems

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(Slides courtesy of Laura Poplawski)
Border Gateway Protocol

Varadhan, Govindan, and Estrin, 1996. Persistent Route Oscillations in Inter-Domain Routing.
Border Gateway Protocol

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Stable Paths Problem

Fractional Stable Paths Problem


- Total paths must sum to 1
- Can't use a path more than a node ahead of you uses its portion.
- Use highest preference allowed

Diagram:
- Total paths must sum to 1
- Can't use a path more than a node ahead of you uses its portion.
- Use highest preference allowed

Mathematical representation:
- Fractional translation of paths:
  - ABD > AD
  - BCD > BD
  - CAD > C

Diagram with fractions:
- A to B: 1/2
- A to C: 1/2
- B to D: 1/2
- C to D: 1/2
- D to B: 1/2
- D to C: 1/2
Stable Paths Problem as a Game

- Node’s strategy set is collection of paths to destination
- Preference relation among strategies: Strategy $P$ is preferred at least as much as strategy $P'$ if
  - either $P'$ is not feasible, or
  - both are feasible and path $P$ is more preferred than path $P'$
- Utility for feasible path given by its preference
- A stable solution is precisely a pure Nash equilibrium
- NP-complete to determine whether a given SPP instance has a stable paths solution [Griffin, Shepherd, Wilfong 02]
- In every FSPP instance, there exists a stable solution [Haxell-Wilfong 08]
Fractional Hypergraph Matching

- Hypergraphic Preference System:
  - A hypergraph G = (V,E)
  - Each vertex has a linear order over its incident edges

- Stable Matching:
  - Each vertex is in at most one edge
  - For each edge e, there exists a vertex v in e and an edge m in matching such that v prefers m over e

- Stable fractional matching: w: E → R
  - For each vertex, total weight of incident edges at most 1
  - For each edge e, there exists v in e such that sum of weights of edges that v prefers over e equals 1.

- A stable fractional matching always exists [Aharoni-Fleiner 03]
Complexity class PPAD

- Search problems for which existence proofs are based on parity arguments
  - Polynomial Parity Argument in a Directed graph [Papadimitriou 94]
- All problems poly-reducible to END OF THE LINE
  - Number of vertices $2^n$
  - Given poly-size predecessor and successor circuits and source vertex label
  - In-degree and out-degree at most one
  - At least one source implies at least one sink
PPAD-Hard Problems

• PPAD-complete problems:
  – Sperner’s Lemma, discrete versions of Brouwer’s Fixed Point Theorem, Borsuk-Ulam Theorem
  – Nash equilibria in matrix games

• Every matrix game has a mixed Nash equilibrium [Nash 51]

• There exist 3-player games with rational inputs in which every Nash equilibrium is irrational [Nash 51]

• For 4-player games, \( \varepsilon \)-Nash is PPAD-complete [Daskalakis, Goldberg, Papadimitriou 06]

• PPAD-completeness for 3- and 2-player games [Chen, Deng 06; Chen, Deng, Teng 06]
Main Results: A Slew of Reductions

[Image of a graph with nodes labeled as follows:
- Constant Degree Preference Game
- Fractional Bounded Budget Connection Game
- Fractional Stable Paths Problem
- Personalized Equilibrium
- Strong Kernel
- 3-D Brouwer
- Fractional Bounded Budget Connection Game
- ε-Approximate Personalized Equilibrium
- Scarf’s Lemma
- End Of the Line

References:
[Papadimitriou 91]
[Aharoni-Fleiner 03]
[Aharoni-Holzman 98]
[Scarf 67]
Structural and Hardness Results

- **Preference Game**: A very simple new game that captures the complexity of several stability problems
  - Easily reduces to FSPP, Stable Fractional Matching, Core of Balanced Games, Computational version of Scarf's Lemma.
- **Reduce** Brouwer’s fixed point problem to Preference Game:
  - No fully polynomial time approximation scheme for Preference Games, unless FP = PPAD.
- **Personalized Equilibrium**: A new notion for matrix games that generalizes several stability problems
  - The set of stable solutions can be expressed as the union of (an exponential number of) linear programs
    - Rational solutions always exist
    - Also useful in placing all the above problems in PPAD.
Outline of Talk

- The Preference Game
- Preference Game reduces to Fractional Stable Paths Problem (FSPP)
- PPAD-hardness:
  - Exact and $\varepsilon$-approximate equilibria
- Other Stability problems
The Preference Game - given

Alice

Bob

Charlie

David

Ed
The Preference Game - given
The Preference Game - given

To play: Assign a weight to each player, weights sum to 1
The Preference Game - given

**To play:** Assign a weight to each player, weights sum to 1

**Only rule:** cannot assign more weight to a player than that player assigns to herself.
The Preference Game - given

To play: Assign a weight to each player, weights sum to 1

Only rule: cannot assign more weight to a player than that player assigns to himself.

Goal: Assign weight to highest preference players.
The Preference Game

Bob

David, Alice, Charlie, me

Alice

Ed

David

Charlie
The Preference Game

Alice

Bob

Charlie

David

Ed

E = 1/2
C = 1/6
A = 1/3

B = 5/6
C = 1/6

D = Alice, Alice, Charlie, me

David, Alice, Charlie, me

C = 1/6
E = 1/2
B = 1/3

1/3
0
1/6

5/6
The Preference Game

Alice

Bob

Charlie

David

Ed

A = 1/3

B = 5/6

C = 1/6

D = 1/3

E = 1/2

A = 1/3

C = 1/6

B = 5/6

C = 1/6

E = 1/2

D = 1/3

David, Alice, Charlie, me

Charlie

1/6

David

0

Bob

5/6

1/3
The Preference Game

- Alice: $E = 1/2, C = 1/6, A = 1/3$
- Bob: $C = 1/6, B = 5/6$
- Charlie: $B = 5/6, C = 1/6$
- Ed: $A = 1/3, C = 1/6$
- David: $E = 1/2, D = 1/3$

Preferences:

- David > Alice > Charlie > me

0
Alice

Bob

Charlie

David

Ed

The Preference Game

A = 1/3

C = 1/6

E = 1/2

B = 5/6

David, Alice, Charlie, me

0

1/6

1/2

5/6

1/3

1/6
The Preference Game - Notation

• Each player $i$ assigns weight $w(i,j)$ to each player $j$

$$\sum_j w(i, j) = 1 \quad \forall i$$

$$w(i, j) \leq w(j, j) \quad \forall i, j$$

• Best Response: Cannot move weight from a lower preference to a higher preference
Reducing Preference Game to Fractional Stable Paths Problem
Reducing Preference Game to Fractional Stable Paths Problem
Reducing Preference Game to Fractional Stable Paths Problem
Equilibrium in Fractional Stable Paths Problem

Rules for Fractional Stable Paths Problem

- Total paths must sum to 1
- Can't use a path more than a node ahead of you uses its portion.
- Use highest preference paths

Rules for the Preference Game

- Place a total of weight 1
- Can’t place more weight on another player than that player places on itself
- Put weight on highest preferences possible
Computational 2D Brouwer

- Exponentially large grid: \( N = 2^n \)
- Given a circuit:

  The coordinate bits of a point

  Color = (0,1), (1,0), or (-1,-1)

- Find a 3-color triangle.
A reduction framework
(Daskalakis-Goldberg-Papadimitriou)

Coordinate players: Play "1" amount equal to the coordinate

Bit Players: Play "1" if the bit is 1

The coordinate bits of a point

Color = (0,1), (1,0), or (-1,-1)

circuit

Logic Gate Players: Play "1" if the result of the logic gate is 1

Result check players: Play "0" if all three colors are represented around a square
A reduction framework (Daskalakis-Goldberg-Papadimitriou)

• (Almost) sufficient to be able to create players that will compute:
  • And
  • Or
  • Not
  • Sum
  • Difference
  • Copy
  • Double
  • Half
  • Is Less Than
Overview of Reduction

- Encode values in Brouwer instance as the weight player assigns to itself.
- Coordinate player P: \( w(P) \) equals coordinate value.
- Need to extract bits from coordinate:
  - \( \text{SUM}(A,B) \): In equilibrium, \( \text{SUM}(A,B) \) plays the sum of what A and B play.
  - DIFF, LESS, COPY.
- LESS with error.
- Circuit simulation:
  - \( \text{OR}(A,B) \): In any equilibrium, if A, B play from \( \{0,1\} \), then play the OR of their values.
  - AND and NOT gadgets.
  - Output is one of \( \{(1,0), (0,1), (-1,-1)\} \).
- Compute circuit at a large (constant) number of points around \( (x,y) \).
- Compute average of these and add it back to the coordinate players.
A Preference Game Gadget

- Difference: $A - B$

Alice plays self with value $A$.

Bob plays self with value $B$. 
A Preference Game Gadget

- Difference: $A - B$

Alice plays herself with value $A$

Bob plays himself with value $B$

Charlie plays himself with value $(1-A)$

Alice, me > Alice, me
A Preference Game Gadget

- Difference: $A - B$

Alice

Plays self with value $A$

Charlie

Plays self with value $(1-A)$

Bob

Plays self with value $B$

Plays self with value $1 - (1-A) - B = (A$

David

Plays self with value $A$

Plays self with value $1 - (1-A) - B = (A$
Approximate Equilibria

- $\varepsilon$-Approximate Equilibrium: weight distribution $w$ such that
  - $\sum_j w(i,j) = 1$ for all $i$
  - For each $j$, $w(i,j)$ is at most $w(j,j) + \varepsilon$
  - For each $j$, either $|w(i,j) - w(j,j)|$ is at most $\varepsilon$ or we have $\sum_i \text{prefers } i \text{ over } j w(i,k)$ is at most $1 - \varepsilon$

- PPAD-hard to find $\varepsilon$-approximate equilibrium in time for inverse polynomial in $n$
  - Idea similar to [Chen-Deng-Teng 06]
  - Reduce from $n$-dimensional Brouwer
  - Each “cell” is a $n$-hypercube, colors assigned from $\{1,2,\ldots,n,n+1\}$
  - Seeking a panchromatic simplex inside a hypercube

- Main hurdle:
  - Errors introduced in Boolean gadgets
  - Prevent magnification of errors by strategically adding LESS gadgets after each logic step
Structure of Preference Game Equilibria

• Feasible solution:
  \[
  \sum_j w(i, j) = 1 \quad \forall i \\
  w(i, j) \leq w(j, j) \quad \forall i, j
  \]
• Feasible solution:

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• Feasible solution:

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\[
w(i, j) \leq w(j, j) \quad \forall i, j
\]

• Best response: For each player \( i \), there exists a “threshold” player \( k \) with

\[
w(i, j) = w(j, j) \quad \forall j >_i k
\]

\[
w(i, j) = 0 \quad \forall j <_i k
\]
Structure of Preference Game
Equilibria – Rational Solution

- Feasible solution:
  \[ \sum_j w(i, j) = 1 \quad \forall i \]
  \[ w(i, j) \leq w(j, j) \quad \forall i, j \]

- Best response: For each player \( i \), there exists a “threshold” player \( k \) with
  \[ w(i, j) = w(j, j) \quad \forall j \geq_i k \]
  \[ w(i, j) = 0 \quad \forall j <_i k \]

- For each \( i \), we have \( n \) possible values for threshold \( k \).
Structure of Preference Game Equilibria – Rational Solution

• Feasible solution:
  \[ \sum_{j} w(i, j) = 1 \quad \forall i \]
  \[ w(i, j) \leq w(j, j) \quad \forall i, j \]

• Best response: For each player \( i \), there exists a “threshold” player \( k \) with
  \[ w(i, j) = w(j, j) \quad \forall j >_{i} k \]
  \[ w(i, j) = 0 \quad \forall j <_{i} k \]

• For each \( i \), we have \( n \) possible values for threshold \( k \).

• \( n^n \) Linear Programs will cover all combinations.
Structure of Preference Game
Equilibria – Rational Solution

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  w(i, j) = 0 \quad \forall j <_i k
  \]

  - For each \( i \), we have \( n \) possible values for threshold \( k \).
  - \( n^n \) Linear Programs will cover all combinations.
  - Since the preference game has an equilibrium, one of these LPs is feasible.
Structure of Preference Game

Equilibria – Rational Solution

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\[ \sum_{j} w(i, j) = 1 \quad \forall i \]

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• For each \( i \), we have \( n \) possible values for threshold \( k \).

• \( n^n \) Linear Programs will cover all combinations.

• Since the preference game has an equilibrium, one of these LPs is feasible.

• If an LP is feasible, then it has a rational solution. Thus, the preference game has a rational solution.
Complexity of preference games

- PPAD hard, even to compute approximate equilibrium
- A rational equilibrium always exists
Complexity of Preference Games

- PPAD hard to compute approximate equilibrium
- A rational equilibrium always exists
- Implies membership in PPAD
  - Finding approximate equilibrium is in PPAD
  - Given a point $\varepsilon$ away from an LP feasible region
    \[ \Rightarrow \text{The LP is feasible} \]
  - Approximate equilibrium is $\varepsilon$ away from at least one of our exponentially many LPs.
  - Can modify our “union of many LPs” to get a single feasible LP.
Other Fractional Stability Problems

- Finding the core of a balanced cooperative game with non-transferrable utilities [Scarf 67]
- Computational version of Scarf’s Lemma [Scarf 67]
- Finding a fractional hypergraph matching [Aharoni-Fleiner 03]
- Finding a strong fractional kernel [Aharoni-Holzman 98]
- Finding an equilibrium in the fractional Bounded Budget Connection Game [Laoutaris et al 08]
- Personalized equilibrium: variant of correlated equilibrium; each player assigns personal weights to strategies subject to “projection constraints”
Fractional Hypergraphic Matching

Find the Core of a Balanced Cooperative Game with Non-Transferrable Utilities

Fractional Stable Paths Problem

3-D Brouwer

End Of the Line

[Papadimitriou 91]

Constant Degree Preference Game

Full version

Fractional Bounded Budget Connection Game

ε-Approximate Personalized Equilibrium

Strong Kernel

[Aharoni-Holzman 98]

Personalized Equilibrium

[Aharoni-Fleiner 03]

Scarfs Lemma

[Scarf 67]