Spanning and Sparsifying

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Sparse Approximations of Graphs

- How well can an undirected graph G = (V,E) be approximated by a sparse graph H?
- Distance-based approximation:
  - Distance in H should be at least distance in G
  - Stretch = distance in H/distance in G
  - Minimize stretch
- Cut-based approximation:
  - Capacity of cut in H should be at most capacity in G
  - Congestion = capacity in G/capacity in H
  - Minimize congestion

What kind of sparse graphs?

- Minimize number of edges while satisfying a required stretch or congestion constraint
- Achieve very small stretch or congestion with O(n) or Õ(n) edges
- Require that sparse graph H be a tree
- Require that H be a spanning tree of G

Applications

- Many problems can be solved much better and/or much faster on sparsifier H
  - Many NP-hard optimization problems can be solved optimally on trees
  - The running time of many flow/cut problems depend significantly on the number of edges
- If H is a good approximation of G, then solution for H or some refinement is likely a good solution for G

Distance-based Approximations

- Spanners [Peleg-Schaeffer 89]:
  - Sparse subgraphs that approximate all pairwise distances
- (α,β)-spanner:
  - For each pair (u,v), $dist_H(u,v)$ is at most $\alpha \times dist_G(u,v) + \beta$
- Multiplicative spanner: Focus on α
  - $(1+\varepsilon, (\log(k)/\varepsilon)^{\log(k)})$-spanners exist with $O(kn^{1+1/k})$ edges [Elkin-Peleg 04,Thorup-Zwick 06]
- Additive spanner: Focus on β
  - Conjecture: (1,2k-2)-spanners with $O(n^{1+1/k})$ edges exist [Woodruff 06, Baswana-Kavitha-Mehlhorn-Pettie 05,…]

Approximations using Trees

- What can be approximated using the sparsest graphs?
- Maximum stretch of any deterministic selection is $\Omega(n)$
- Probabilistic approximation:
  - There exists a probability distribution of trees such that the expected stretch for any pair is $O(\log(n))$
  - [Bartal 96, 98], …, [Fakcharoenphol-Rao-Talwar 03]
  - $\tilde{O}(\log(n))$ achievable even if tree required to be a spanning tree [Elkin-Emek-Spielman-Teng 05, Abraham-Bartal-Neiman 09,…]

Major Tool in Network Optimization

- When objective function is a linear combination of distances:
  - Can reduce the problem on a general graph to that of solving on a tree
  - Cost of O(log(n)) in approximation
- Metrical Task Systems [Bartal 96]
- Group Steiner Tree [Garg-Konjevod-Ravi 98]

Cut-Based Approximations

- Is there a tree whose cuts well-approximate the ones in the graph?
- [Räcke 03, 08] showed for two notions of probabilistic approximation that this is true
- There exists a probability distribution over capacitated trees such that the expected congestion of every cut is O(log(n))
- Applications:
  - Minimum bisection
  - Minimum k-multicut
  - Oblivious routing

Graph Sparsifiers

- Is there a sparse subgraph H of G, with capacities on edges, such that:
  - For every cut, capacity in H is within $(1 \pm \varepsilon)$ factor of the capacity in G
- Graph sparsifiers with O(n) edges exist!
  - [Benczur-Karger 02]
  - [Spielman-Srivastava 08] provide stronger spectral sparsifiers
  - [Fung-Hariharan-Harvey-Panigrahy 11] provide a general framework

Applications of Sparsifiers

- Faster (1 ± ε)-approximation algorithms for flow-cut problems
  - Maximum flow and minimum cut [Benczur-Karger 02, ..., Madry 10]
  - Graph partitioning [Khandekar-Rao-Vazirani 09]
- Improved algorithms for linear system solvers [Spielman-Teng 04, 08]

Graph Sparsification via Random Sampling

- Sample each edge with a certain probability
- Uniform probability will not always work
- Assign weight of selected edge to be the inverse of its probability
  - Ensures that in expectation, we are all set!
- Real challenge:
  - How do we guarantee approximation for every one of the exponential number of cuts?
- Three key components:
  - Non-uniform probability chosen captures "importance" of the cut (several measures have been proposed)
  - Distribution of the number of cuts of a particular size [Karger 99]
  - Chernoff bounds