Percolation Processes

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Outline

• Branching processes
  – Idealized model of epidemic spread
• Percolation theory
  – Epidemic spread in an infinite graph
• Erdös-Renyi random graphs
  – Model of random graphs and percolation over a complete graph
• Percolation on finite graphs
  – Epidemic spread in a finite graph
Branching Processes

• Natural probabilistic process studied in mathematics
• Widely used for modeling the spread of diseases, viruses, innovation, etc., in networks
• Basic model:
  – Disease originates at root of an infinite tree
  – Branching factor $k$: number of children per node
  – Probability $p$ of transmission along each edge
• Question: What is the probability that the disease persists indefinitely?
  • Theorem 1
    – If $pk < 1$, then probability = 0
    – If $pk > 1$, then probability > 0
Percolation Theory

• Infinite graph G
• Bond percolation:
  – Each edge is selected independently with probability $p$
  – As $p$ increases from 0 to 1, the selected subgraph goes from the empty graph to $G$
• Question:
  – What is the probability that there is an infinite connected component?
  – Kolmogorov 0-1 law: Always 0 or 1
• What is the critical probability $p_c$ at which we move from no infinite component to infinite component?
Percolation Theory

• **Theorem 2**: For the 2-D infinite grid, $p_c = \frac{1}{2}$
  [Harris 60, Kesten 80]

• Not hard to see that $\frac{1}{3} \leq p_c \leq \frac{2}{3}$
  – The first inequality can be derived from the
    branching process analysis

• [Bollobas-Riordan 06] book

• [Bagchi-Kherani 08] notes
Erdös-Renyi Random Graphs

• Percolation over the complete graph \( K_n \)
• Critical probability and sharp threshold for various phenomena [Erdös-Renyi 59,60]
  – Emergence of giant component
    • \( p = \frac{1}{n} \)
  – Connectivity
    • \( p = \frac{\ln(n)}{n} \)
• Every symmetric monotone graph property has a sharp threshold [Friedgut-Kalai 96]
Percolation on Finite Graphs

• Given arbitrary undirected graph $G$
  – At what probability will we have at least one connected component of size $\Omega(n)$?

• Given a uniform expanders family $G_n$ with a uniform bound on degrees [Alon-Benjamini-Stacey 04]
  – $\Pr[G_n(p) \text{ contains more than one giant component}]$ tends to 0
  – For high-girth d-regular expanders, critical probability for (unique) giant component is $1/(d-1)$

• Lots of open questions
Network Models and Phenomena

• In study of complex networks and systems, many properties undergo phase transition
  – Corresponds to critical probabilities and sharp thresholds in random graphs and percolation
• ER random graphs provide a useful model for developing analytical tools
• Various other random graph models
  – Specified degree distribution [Bender-Canfield 78]
  – Preferential attachment [Barabasi-Albert 99]
  – Power law graph models [Aiello-Chung-Lu 00]
  – Small-world models [Watts-Strogatz 98, Kleinberg 00]
Take Away Messages

• Spreading of information, diseases, etc. in massive networks well-captured by branching processes
  – Analysis yields useful rule-of-thumb bounds for many applications
• Many such phenomena experience phase transitions
  – Calculate critical probability and establish sharp thresholds
• Random graph models
  – A large collection of models starting from ER
  – Many motivated by real observations
  – Aimed at explaining observed phenomena and predict future properties
  – Certain algorithms may be more efficient on random graphs
• Tools:
  – Basic probability (Chernoff-type bounds)
  – Correlation inequalities to handle dependence among random variables
  – Generating functions and empirical methods to get reasonable estimates