Percolation Processes

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Outline

- Branching processes
  - Idealized model of epidemic spread
- Percolation theory
  - Epidemic spread in an infinite graph
- Erdös-Renyi random graphs
  - Model of random graphs and percolation over a complete graph
- Percolation on finite graphs
  - Epidemic spread in a finite graph

Branching Processes

- Natural probabilistic process studied in mathematics
- Widely used for modeling the spread of diseases, viruses, innovation, etc., in networks
- Basic model:
  - Disease originates at root of an infinite tree
  - Branching factor k: number of children per node
  - Probability p of transmission along each edge
- Question: What is the probability that the disease persists indefinitely?
- Theorem 1
  - If $pk < 1$, then probability = 0
  - If $pk > 1$, then probability > 0

Percolation Theory

- Infinite graph G
- Bond percolation:
  - Each edge is selected independently with probability p
  - As p increases from 0 to 1, the selected subgraph goes from the empty graph to G
- Question:
  - What is the probability that there is an infinite connected component?
  - Kolmogorov 0-1 law: Always 0 or 1
- What is the critical probability p_c at which we move from no infinite component to infinite component?

Percolation Theory

- Theorem 2: For the 2-D infinite grid, $p_c = \frac{1}{2}$ [Harris 60, Kesten 80]
- Not hard to see that $1/3 \leq p_c \leq 2/3$
  - The first inequality can be derived from the branching process analysis
- [Bollobas-Riordan 06] book
- [Bagchi-Kherani 08] notes

Erdös-Renyi Random Graphs

- Percolation over the complete graph K_n
- Critical probability and sharp threshold for various phenomena [Erdös-Renyi 59,60]
  - Emergence of giant component
    - $p = 1/n$
  - Connectivity
    - $p = \ln(n)/n$
- Every symmetric monotone graph property has a sharp threshold [Friedgut-Kalai 96]

Percolation on Finite Graphs

- Given arbitrary undirected graph G
  - At what probability will we have at least one connected component of size $\Omega(n)$?
- Given a uniform expanders family G_n with a uniform bound on degrees [Alon-Benjamini-Stacey 04]
  - $\Pr[G_n(p)$ contains more than one giant component$]$ tends to 0
  - For high-girth d-regular expanders, critical probability for (unique) giant component is 1/(d-1)
- Lots of open questions

Network Models and Phenomena

- In study of complex networks and systems, many properties undergo phase transition
  - Corresponds to critical probabilities and sharp thresholds in random graphs and percolation
- ER random graphs provide a useful model for developing analytical tools
- Various other random graph models
  - Specified degree distribution [Bender-Canfield 78]
  - Preferential attachment [Barabasi-Albert 99]
  - Power law graph models [Aiello-Chung-Lu 00]
  - Small-world models [Watts-Strogatz 98, Kleinberg 00]

Take Away Messages

- Spreading of information, diseases, etc. in massive networks well-captured by branching processes
  - Analysis yields useful rule-of-thumb bounds for many applications
- Many such phenomena experience phase transitions
  - Calculate critical probability and establish sharp thresholds
- Random graph models
  - A large collection of models starting from ER
  - Many motivated by real observations
  - Aimed at explaining observed phenomena and predict future properties
  - Certain algorithms may be more efficient on random graphs
- Tools:
  - Basic probability (Chernoff-type bounds)
  - Correlation inequalities to handle dependence among random variables
  - Generating functions and empirical methods to get reasonable estimates