College of Computer & Information Science — Fall 2006
Northeastern University — Handout 4
CSU690: Algorithms and Data — 19 September 2006

Mathematical Notation, Definitions, and Review

Inequalities

$f(n)$ is "*monotonically increasing*": $m \le n \Rightarrow f(m) \le f(n)$

$f(n)$ is "*monotonically decreasing*": $m \le n \Rightarrow f(m) \ge f(n)$

$f(n)$ is "*strictly increasing*": $m \le n \Rightarrow f(m) < f(n)$

$f(n)$ is "*strictly decreasing*": $m \le n \Rightarrow f(m) > f(n)$

$\lfloor x \rfloor = \max\{\, y : y \le x,\ y \text{ an integer} \,\}$ ("*floor*")

$\lceil x \rceil = \min\{\, y : y \ge x,\ y \text{ an integer} \,\}$ ("*ceiling*")

Fact: $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$

Fact: $\lceil \lceil n/a \rceil / b \rceil = \lceil n/ab \rceil$ and $\lfloor \lfloor n/a \rfloor / b \rfloor = \lfloor n/ab \rfloor$

$f(n)$ is "*polynomially bounded*" if $f(n) = O(n^k)$ (for some constant k) $= n^{O(1)}$

Exponentials

$\forall a \ne 0$, $\forall m, m$,

$a^0 = 1,$

$a^1 = a,$

$a^{-1} = 1/a,$

$(a^m)^n = a^{mn}$, and

$a^m a^n = a^{m+n}.$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ (Taylor's series)

$1 + x \le e^x \le 1 + x + x^2$ (for $|x| < 1$)

$\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$

Logarithms

Definition: $\log_b a = c$ if and only if $a = b^c$

$\forall a, b, c > 0$, $\forall n$,

$a = b^{\log_b a},$

$\log_c(ab) = \log_c a + \log_c b$,

$\log_b a^n = n \log_b a$,

$\log_b a = \frac{\log_c a}{\log_c b}$,

$\log_b(1/a) = \log_b a^{-1} = -\log_b a$,

$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$, and

$a^{\log_b n} = (b^{\log_b a})^{\log_b n} = b^{\log_b a \log_b n} = n^{\log_b a}$.

$\lg n = \log_2 n$ (binary logarithm)

$\ln n = \log_e n$ (natural logarithm)

$\lg^k n = (\lg n)^k$

$\lg \lg n = \lg(\lg n)$

$\lg n + k = (\lg n) + k$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots,\qquad |x| < 1$ (Taylor's series)

$\frac{x}{1+x} \leq \ln(1+x) \leq x, \qquad x > -1$

$f(n)$ is "*polylogarithmically bounded*" if $f(n) = O(\log^k n)$ (for some constant k) $= \log^{O(1)} n$

Factorial

$n! = 1$ if $n = 0$, and $n \cdot (n-1)!$ if $n > 0$

Alternatively, $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$

$n! = \sqrt{2\pi n}(\frac{n}{e})^n(1 + \Theta(\frac{1}{n}))$ (Stirling's approximation)

Binomial Coefficients

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\binom{n}{k} = \binom{n}{n-k}$

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$2^n = \sum_{k=0}^{n} \binom{n}{k}$

$\binom{n}{k} \geq (\frac{n}{k})^k$

$\binom{n}{k} \leq \frac{n^k}{k!} \leq (\frac{en}{k})^k$ (Stirling's formula)

Summations

A *series* is a *sequence* of partial sums of a summation, parametrized by the limit of summation.

$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ (*finite series*)

$\sum_{k=1}^{\infty} a_k \equiv a_1 + a_2 + \cdots = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ (*infinite series*)

An infinite series is *convergent* if the limit exists. Otherwise the series is *divergent.*

Linearity

If a series is finite or convergent (for n replaced by ∞), then

$$\sum_{k=1}^{n} (ca_k + b_k) = c \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Also, $\sum_{k=1}^{n} \Theta(f(k)) = \Theta(\sum_{k=1}^{n} f(k))$

Arithmetic Series

Definition: $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

$$\sum_{k=1}^{n} k = \frac{1}{2} n(n+1) = \Theta(n^2)$$

Geometric Series

Definition: $\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} \quad \forall x \neq 1$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \forall |x| < 1$$

Harmonic series

Definition: $H_n = \sum_{k=1}^{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

$H_n = \ln n + O(1)$

Asymptotic Notation

$O(g(n)) = \{ f(n) : \exists$ constants $n_0, c > 0$ such that $\forall n \geq n_0,\ 0 \leq f(n) \leq cg(n) \}$

$\Omega(g(n)) = \{ f(n) : \exists$ constants $n_0, c > 0$ such that $\forall n \geq n_0,\ cg(n) \leq f(n) \}$

$\Theta(g(n)) = \{ f(n) : \exists$ constants $n_0, c_1, c_2 > 0$ such that $\forall n \geq n_0,\ 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \}$

Alternatively, $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n)))$

$o(g(n)) = \{ f(n) : \forall$ constant $c > 0$, $\exists$ constant n_0 such that $\forall n \geq n_0,\ 0 \leq f(n) < cg(n) \}$

$\omega(g(n)) = \{ f(n) : \forall$ constant $c > 0$, $\exists$ constant n_0 such that $\forall n \geq n_0,\ cg(n) < f(n) \}$

$f(n) = O(g(n))$ means $f(n) \in O(g(n))$

$O(f(n)) = O(g(n))$ means $O(f(n)) \subseteq O(g(n))$

(and similarly for Ω, Θ, o, and ω)