1. (15 points) Unary regular languages

Show that a language \( L \) over the unary alphabet \( \{0\} \) is regular if and only if there exist integers \( r, r_1, r_2, \ldots, r_j, r_{j+1}, \ldots, r_k \) (for some finite \( j \) and \( k \)) such that

\[
L = \left( \bigcup_{1 \leq i \leq j} 0^{r_i} \right) \bigcup \left( \bigcup_{j \leq i \leq k} (0^r)^* \cdot 0^{r_i} \right).
\]

2. (15 points) Input-uniform Turing machines

We say that a deterministic single-tape Turing machine is \textit{input-uniform} if the position of its head at the \( i \)th step of its computation depends only on \( i \) and \( |x| \), not \( x \) itself.

Prove or disprove: The set of languages recognized by input-uniform Turing machines is exactly the set of Turing-recognizable languages.

3. (4 \times 10 = 40 points) Classifying languages

Each of the following three parts gives a definition, description, or some properties of a language. In each case, tell whether the language is:
A  regular
B  context-free but not regular
C  Turing-decidable but not context-free
D  Turing-recognizable but not Turing-decidable
E  not Turing-recognizable
F  there is insufficient information to tell.

Justify your answers with proofs, constructions, algorithms, or examples as needed.

If, for example, you decide that a language is context-free but not regular, you must give a context-free grammar that generates the language or a pushdown automaton that accepts the language and also prove that the language is not regular.

(a) The set of all Turing machine descriptions \( \langle M \rangle \), where \( M \) is a nondeterministic Turing machine and there exists a string \( x \) such that \( M \) halts on \( x \) on all computation paths.

(b) \( \{ \langle M \rangle : \text{the language recognized by } M \text{ is finite} \} \)

(c) \( \{0^{f_1}1^{f_2}0^{f_3}1^{f_4}\ldots0^{f_k} : k, f_1, \ldots, f_k \text{ are positive integers and } f_i \text{ is the } i\text{th Fibonacci number} \} \)

(d) The complement of the language of part (c).

4. (3 + 12 = 15 points) Pirates Undercover

My daughter has a fun puzzle called Pirates Undercover\(^1\). The puzzle consists of a \( 2 \times 2 \) game board, 4 puzzle pieces, and several puzzle challenges. Each of the four cells of the game board has some pictures on it; each picture is one of five types – a pirate ship, a white sailboat, a rowboat, an island, and a cave. The pictures and their positions on the cell differ from cell to cell. Each of the four puzzle pieces can be placed on a cell in multiple (but a finite number of) orientations, and each orientation hides a subset of the pictures. Each puzzle challenge consists of a count of the number of pictures of each kind. The goal is to place one puzzle piece on each cell so that the collection of all visible pictures match the prescribed count.

Let us generalize the above puzzle. Suppose, we are given a game board with \( n \) cells. Each cell has some pictures, each picture being of one of \( k \) types. We are also given \( n \) puzzle pieces; each puzzle piece can be placed on a cell in a finite number of orientations (say \( \ell \)), each orientation hiding some pictures and showing some. Given a count for each of the \( k \) picture types, we are asked whether there exists a placement of the puzzle pieces on the game board (one piece on each cell) such the for each picture type, the number of visible pictures equals the count for the type.

(a) Formulate the above problem as a language and show that it is in NP.

(b) Show that the language is NP-hard.

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\(^1\)http://www.toys4minds.com/pirates-undercover-game-educational-insights.html
5. (5 + 10 = 15 points) Business Games

Consider the following game between two companies in a tough zoning environment. The game is specified by an undirected graph $G = (V,E)$, a function $r : V \rightarrow Z$, and an integer $R$. The graph $G$ is the graph of divisions in a city, each vertex being a division and each edge representing two neighboring divisions. Each vertex $v$ in the graph has an integer $r(v)$ associated with it, indicating the revenue that a firm will obtain by setting up a store at $v$. Zoning laws require (a) at most one of the companies can open a store at any division $v$, and (b) the set of all store locations (together, by both the firms) forms an independent set (i.e., if $A$ or $B$ opens a store at a division $v$, then neither of the companies can open a store at any of the divisions adjacent to $v$).

The game proceeds in a sequence of steps. Firm $A$ goes first, opens a store at a particular division. Then firm $B$ opens a store, satisfying the constraints mentioned above. Then $A$ goes next, and so on. The goal of $B$ is to open stores so as to attain a total revenue of at least a given bound $R$.

Let $L$ be the set of all games $\langle G, r, R \rangle$ in which $B$ achieves a revenue of $R$.

(a) Show that $L$ is in PSPACE.

(b) Show that $L$ is PSPACE-hard.

(Hint: Reduce from the Formula Game discussed in class and also defined in Sipser’s text (Chapter 8). You may assume that the SAT formula in the formula game is a 3-SAT formula. Associate vertices of the graph with the variables and clauses in the formula. Make sure that you assign the revenues such that the moves the players make in the competition game correspond to assigning truth values to the variables in the formula in the appropriate order.)