Final

1. (4 x 5 = 20 points) Properties of language classes

For each of the following assertions, either prove that the assertion is true or provide a counterexample.

(a) The set of all non-regular languages is closed under concatenation.

(b) Let $L_1$ and $L_2$ be two regular languages over the alphabet $\Sigma = \{0, 1\}$. Then the language $L$ given by

$$\{w_1w_2\cdots w_{2n} : w_i \in \Sigma, w_1w_3\cdots w_{2n-1} \in L_1, w_2w_4\cdots w_{2n} \in L_2\}$$

is regular.

(c) If $R$ is any recognizable language and $D$ is any decidable language, then $R - D$ is recognizable.

(d) If $R$ is any recognizable language and $D$ is any decidable language, then $D - R$ is decidable.

2. (20 points) Integer equations

You are given a set of $m$ equations over $n$ variables. The $i$ equation is given by

$$\sum_{j=1}^{n} a_{ij}x_j = b_i,$$

where $a_{ij}$ and $b_i$ are integers for $1 \leq i \leq m$ and $1 \leq j \leq n$. You are asked to determine whether there exists an assignment of nonnegative integer values to each $x_j$ such that all the $m$ equations are true.

Formulate a language associated with the above problem, and show that it is NP-complete.

3. (15 points) Sorting in logarithmic space

Show that any sequence of $n$ integers, each presented in binary, can be sorted in logarithmic space. That is, describe a log-space transducer which takes as input a sequence $\sigma$ of $n$ integers and produces as output the elements of $\sigma$ in sorted order. (Assume that the input is given as $a_1\#a_2\#\cdots\#a_n$, where for $1 \leq i \leq n$, $a_i$ is an integer, presented in binary.)

4. (3 x 5 = 15 points) Classifying a simple language

Consider the following language.

$$\{b_n\#b_{n+1} : b_i \text{ is binary representation of } i \text{ with no preceding 0s}\}$$

It is easily seen to be decidable.
(a) Is it context-free? If yes, then is it also regular?

(b) Give the best upper bound you can on the deterministic space complexity of the language. For bonus points, also give a matching lower bound.

(c) What is the deterministic time complexity of the language?

5. (15 points) Probabilistic Polynomial Space

Define BPS to be the following complexity class. A language \( L \) is in BPS if and only if there exists a probabilistic polynomial-space Turing machine \( M \) such that

\[
\begin{align*}
  w \in L & \implies \Pr[M \text{ accepts } w] \geq 2/3, \\
  w \notin L & \implies \Pr[M \text{ accepts } w] \leq 1/3.
\end{align*}
\]

Show that BPS = PSPACE.

(Hint: One approach is the following. Consider the reachability problem in the configuration graph of a probabilistic polynomial-space machine \( M \) on input \( w \). Show the probability of reaching an accept configuration from the start configuration can be calculated in space polynomial in the length of \( w \).)

6. (15 points) Multiplying polynomials

You are given three polynomials over one variable, \( p(x), q(x), \) and \( r(x) \) and are asked to determine whether \( p(x) \cdot q(x) = r(x) \). Each polynomial is specified by giving the coefficient of each \( x^i \). In a model where the multiplication or addition of any two integers is unit cost, regardless of the size of the integer, describe how to solve this problem in \( O(n^2) \) time. Give a probabilistic algorithm that terminates in \( O(n) \) time and works correctly with probability at least 1/2.