Final

1. ($4 \times 5 = 20$ points) Properties of language classes

For each of the following assertions, either prove that the assertion is true or provide a counterexample.

(a) The set of all non-regular languages is closed under concatentation.

(b) Let L_1 and L_2 be two regular languages over the alphabet $\Sigma = \{0, 1\}$. Then the language L given by

$$\{w_1 w_2 \cdots w_{2n} : w_i \in \Sigma, w_1 w_3 \cdots w_{2n-1} \in L_1, w_2 w_4 \cdots w_{2n} \in L_2\}$$

is regular.

(c) If R is any recognizable language and D is any decidable language, then $R - D$ is recognizable.

(d) If R is any recognizable language and D is any decidable language, then $D - R$ is decidable.

2. (20 points) Integer equations

You are given a set of m equations over n variables. The i equation is given by

$$\sum_{j=1}^{n} a_{ij} x_j = b_i,$$

where a_{ij} and b_i are integers for $1 \leq i \leq m$ and $1 \leq j \leq n$. You are asked to determine whether there exists an assignment of nonnegative integer values to each x_j such that all the m equations are true.

Formulate a language associated with the above problem, and show that it is NP-complete.

3. (15 points) Sorting in logarithmic space

Show that any sequence of n integers, each presented in binary, can be sorted in logarithmic space. That is, describe a log-space transducer which takes as input a sequence σ of n integers and produces as output the elements of σ in sorted order. (Assume that the input is given as $a_1 \# a_2 \# \ldots \# a_n$, where for $1 \leq i \leq n$, a_i is an integer, presented in binary.)

4. ($3 \times 5 = 15$ points) Classifying a simple language

Consider the following language.

$$\{b_n \# b_{n+1} : b_i \text{ is binary representation of } i \text{ with no preceding 0s}\}$$

It is easily seen to be decidable.

(a) Is it context-free? If yes, then is it also regular?

(b) Give the best upper bound you can on the deterministic space complexity of the language. For bonus points, also give a matching lower bound.

(c) What is the deterministic time complexity of the language?

5. (15 points) Probabilistic Polynomial Space

Define BPS to be the following complexity class. A language L is in BPS if and only if there exists a probabilistic polynomial-space Turing machine M such that

$$\begin{aligned} w \in L &\Rightarrow \Pr[M \text{ accepts } w] \geq 2/3, \\ w \notin L &\Rightarrow \Pr[M \text{ accepts } w] \leq 1/3. \end{aligned}$$

Show that BPS = PSPACE.

(*Hint:* One approach is the following. Consider the reachability problem in the configuration graph of a probabilistic polynomial-space machine M on input w. Show the probability of reaching an accept configuration from the start configuration can be calculated in space polynomial in the length of w.)

6. (15 points) Multiplying polynomials

You are given three polynomials over one variable, $p(x)$, $q(x)$, and $r(x)$ and are asked to determine whether $p(x) \cdot q(x) = r(x)$. Each polynomial is specified by giving the coefficient of each x^i. In a model where the multiplication or addition of any two integers is unit cost, regardless of the size of the integer, describe how to solve this problem in $O(n^2)$ time. Give a probabilistic algorithm that terminates in $O(n)$ time and works correctly with probability at least 1/2.