Problem Set 1 (due Friday, January 25)

1. (7 points) [Sipser 1.48] A regular language

Let $\Sigma = \{0, 1\}$ and let

$$D = \{w|w \text{ contains an equal number of occurrences of the substrings 01 and 10}\}.$$

Thus, $101 \in D$ because 101 contains one 10 and one 01, but $1010 \notin D$ because 1010 contains two 10s and one 01. Show that D is a regular language.

2. ($4 \times 5 = 20$ points) A non-regular language

Consider the language $F = \{a^i b^j c^k | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$.

(a) Show that F is not regular,

(b) Show that F satisfies the conditions of the pumping lemma.

(c) Explain why parts (a) and (b) do not contradict the pumping lemma.

(d) Show that F is context-free.

(Parts (a) through (c) are from [Sipser 1.54].)

3. (8 points) Designing a Turing machine

Design a Turing machine for the language of all strings with an equal number of 0s and 1s. The input alphabet is $\{0, 1, \}$.

4. ($10 + 3 + 2 = 15$ points) Input-read-only Turing machines

Define the *input-read-only* Turing machine as a single tape deterministic Turing machine that is not allowed to write on the portion of the tape containing the input string.

(a) Prove that input-read-only Turing machines can only recognize regular languages.

(b) Does the claim of part (a) hold if we allow the input-read-only Turing machine to have *two heads*? Briefly justify your answer. Note that we maintain the restrictions of a single tape and that the portion of the input cannot be written into.

The definition of a two-head Turing machine is fairly natural. The transition function of a two-head Turing machine takes as input a state and two symbols (one corresponding to each head) and returns a state, two symbols to be written (one corresponding to each head), and two moves (one corresponding to each head). We can assume an arbitrary tie-breaking mechanism for the scenario when two heads attempt to write on the same cell. Initially, both the heads are at the leftmost end of the tape.

(c) Does the claim of part (a) hold if we allow the input-read-only Turing machine to have two tapes? Briefly justify your answer.

5. ($4 \times 5 = 20$ points) Classification of languages

Each of the following four parts gives a definition, description, or some properties of a language. In each case, tell whether the language is:

A regular

B context-free but not regular

C Turing-decidable but not context-free

D not Turing-decidable

E there is insufficient information to tell.

Justify your answers with proofs, constructions, algorithms, or examples as needed.

If, for example, you decide that a language is context-free but not regular, you must give a context-free grammar that generates the language or a pushdown automaton that accepts the language and also prove that the language is not regular.

(a) $L = \{0^i 1^j 2^k \mid \text{ either } i + j = k \text{ or } i + k = j, \text{ and } i, j, k \geq 0\}$. The input alphabet is $\{0, 1, 2\}$.

(b) $L = \{0^p \,|p \text{ is prime}\}$. The input alphabet is $\{0\}$.

(c) $L = \{01^m 0 \,|\, m \text{ is a multiple of } 7\}$. The input alphabet is $\{0, 1\}$.

(d) L is the set of balanced strings of three types of parentheses, ( ) and [ ] and {, }. (For example, [{[()()]}({})] is balanced, while {([)]} is not balanced.)