Problem Set 3 (due Friday, December 14)

Grading. This problem set has a total of 100 points. All the problems concern specific problems we covered in the lectures. From a grading perspective, the total points allocated for problem sets is 200 points. So you should plan on attempting at least 200 points in all, over all problem sets. Any score you earn beyond the basis of 200 points will be bonus points for you.

Collaboration. You are welcome to collaborate on solving these problems. But you must write the solutions on your own. Please cite any collaborators and/or references that help you in arriving at your solutions.

1. (15 points) Local improvement algorithm for Max-Cut

Recall that in the Max-Cut problem, we are given an undirected graph $G = (V, E)$ and we would like to partition the vertices into two sets V_1 and V_2 so as to maximize the number of edges with one end-point in V_1 and the other end-point in V_2. In this exercise, we consider a simple local improvement algorithm for the problem discussed in class.

Initially, partition the vertices arbitrarily. Consider the following local improvement step. Pick a node u, say in V_1, that has fewer edges to vertices in V_2 than to other vertices in V_1; if such a node u exists, then move u from V_1 to V_2.

The local improvement algorithm executes the above step as long as there exists a node that has fewer edges in the cut than edges to other vertices in its set.

Prove that the above algorithm is a polynomial-time 2-approximation algorithm for Max-Cut.

2. (15 points) Capacity of a binary erasure channel

Consider an erasure channel in which an input bit (0 or 1) is erased (output as '?') by the channel independently with probability p. What is the capacity of this channel?

3. (15 points) Supermultiplicative and submultiplicative functions

Let f be a function mapping positive integers to positive reals. We say that f is supermultiplicative if $f(m+n) \geq f(m) \cdot f(n)$, and submultiplicative if $f(m+n) \leq f(m) \cdot f(n)$. Prove that if f is supermultiplicative, then

$$\sup_n f(n)^{1/n} = \lim_{n\to\infty} f(n)^{1/n}.$$

Prove that if f is submultiplicative, then

$$\inf_n f(n)^{1/n} = \lim_{n\to\infty} f(n)^{1/n}.$$

You may assume that $f(n)^{1/n}$ is bounded; that is, there exists a real M such that for any positive integer n, $f(n)^{1/n} \leq M$.

We used the above two facts for establishing that the fractional chromatic number of a graph is equal to the fractional clique number.

4. (15 points) Embedding a hypercube in a sphere

Consider an embedding of the 2^k-point hypercube in the k-dimensional sphere. Show that any angle formed by three points of the hypercube is at most 90 degrees.

5. (15 points) Random vector in high dimensions

Let r be a unit vector, chosen uniformly at random from an m-dimensional unit sphere. Prove the following two inequalities.

$$\begin{aligned} \Pr\left[|r \cdot e_z| < \frac{t}{\sqrt{m}}\right] &\leq \Theta(t) \\ \Pr\left[|r \cdot e_z| > \frac{t}{\sqrt{m}}\right] &\leq e^{-\Theta(t^2)}. \end{aligned}$$

6. (25 points) Conditions in the Master Structure Theorem

The Master Structure Theorem (in Arora-Rao-Vazirani's sparsest cut result) says that given n points $v_1, \ldots, v_n$, in $\mathbf{R}^m$ such that (a) the average ℓ_2^2 distance among them is $\Omega(1)$, (b) the ℓ_2^2 distances form a metric, and (c) $||v_i|| \leq 1$, there exist subset S and T of the points such that $|S| = \Omega(n)$, $|T| = \Omega(n)$ and $d(S,T) = \Omega(1/\sqrt{\log n})$.

Show that the desired claim does not hold if any of the three conditions is dropped.