Scribe: Jingjing Duan

Lecture Outline:

- Shannon Capacity
  - Pentagon
- Shannon's Upper Bound
- Lovasz's Tight Bound

1 Shannon Capacity of a Graph

Given a graph $G = (V, E)$, where

- Vertices represent symbols.
- Edges exist between two symbols when they can't be confused with each other.

The Shannon capacity of a graph G is defined as:

$$\mathrm{SC}(G) = \sup_n \log_2 \sqrt[n]{\omega(G^n)} = \lim_{n\to\infty} \log_2 \sqrt[n]{\omega(G^n)},$$

where $\omega(H)$ computes the largest size of a clique in a graph.

There's an upper bound on $SC(G)$. However, we don't know how to compute it in general graphs.

It is more convenient to define Shannon Capacity of a channel is also defined as:

$$\mathrm{SC}(G) = \sup_n \sqrt[n]{\omega(G^n)} = \lim_{n\to\infty} \sqrt[n]{\omega(G^n)}$$

Suppose $\chi(G)$ denotes the chromatic number of a graph G. It's easy to see that:

$$\omega(G) \le \mathrm{SC}(G) \le \chi(G)$$

The reason why $\omega(G) \le \mathrm{SC}(G)$ is that $\sup_n \sqrt[n]{\omega(G^n)} \ge \sqrt[1]{\omega(G^1)} = \omega(G)$. Since the chromatic number of a graph is no less than the clique number of it, we have $\omega(G) \le \chi(G)$.

To get the upper bound on Shannon Capacity, we claim that:

$$\omega(G) \le \lim_{n\to\infty} \sqrt[n]{\omega(G^n)} \le \lim_{n\to\infty} \sqrt[n]{\chi(G^n)} \le \chi(G)$$

In figure 1, the line above the horizontal line represents $\sqrt[n]{\chi(G^n)}$ and the line below the horizontal line represents $\sqrt[n]{\omega(G^n)}$. Finally they converge.

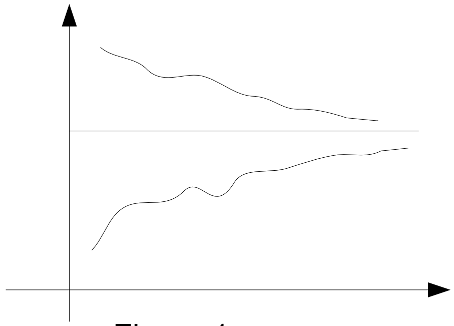

Figure 1

We denote by $\chi_f(G)$ the fractional chromatic number of G, which is the infimum of all fractions a/b such that, to each vertex of G, one can assign a b-element subset of $\{1, 2, 3, \ldots, a\}$ in such a way that adjacent vertices are assigned disjoint subsets.

A fractional clique is a map $f : V(G) \to [0, 1]$ such that, if S is any independent set of vertices in $V(G), \sum_{v \in S} f(v) \leq 1$. The fractional clique number $\omega_f(G)$ is equal to $\sup\{\sum_{v \in V(G)} f(v)\}$, where the supremum is taken over all fractional cliques f. This is just a combinatorial description of the parameter calculated by the real relaxation of the integer program that calculates $\omega(G)$, so $\omega_f(G) = \chi_f(G)$ by the duality theorem of linear programming.

The primal and the dual are as follows.

$$\begin{array}{c|c} \chi_f = \min 1^T x & \omega_f = \max y^T 1 \\ Ax \geq 1 & y^T A \leq 1 \\ x \geq 0 & y \geq 0 \end{array}$$

So, we have the following theorem.

Theorem 1. $\chi_f(G) = \omega_f(G)$

We now consider the fractional chromatic number of products of graphs.

Theorem 2. $\chi_f(G \times H) = \chi_f(G) \times \chi_f(H)$

Proof:

$\omega_f(G)$ is super-multiplicative, i.e.,

$$\omega_f(G \times H) \geq \omega_f(G) \times \omega_f(H)$$

$\chi_f(G)$ is sub-multiplicative, i.e.,

$$\chi_f(G \times H) \leq \chi_f(G) \times \chi_f(H)$$

According to the previous theorem, we also know that:

$$\chi_f(G \times H) = \omega_f(G \times H)$$

$$\chi_f(G) = \omega_f(G)$$
$$\chi_f(H) = \omega_f(H)$$

Combining them will lead to the conclusion that $\chi_f(G \times H) = \chi_f(G) \times \chi_f(H)$ □

Theorem 3. *$SC(G) \leq \chi_f(G)$*

Proof: Based on the previous theorem, it's not hard to show that:

$$\text{SC}(G) = \sup \sqrt[n]{\omega(G^n)} \leq \sup \sqrt[n]{\omega_f(G^n)} = \sup \sqrt[n]{\chi_f(G^n)} = \sqrt[n]{[\chi_f(G)]^n} = \chi_f(G)$$

□

Suppose graph G is a pentagon, then we know that $\chi_f(G) = 2.5$ and $\sqrt{\omega(G^2)} = \sqrt{5}$. Based on these two results, we know further that $\sqrt{5} \leq \text{SC}(G) \leq 2.5$.

Lovasz proved that Shannon Capacity of a pentagon is actually $\sqrt{5}$, strictly.

Theorem 4. *Suppose graph G is a pentagon. Then $SC(G) = \sqrt{5}$.*

Proof: An orthogonal representation of a graph G is defined by associating a vector $\overline{v_i}$ for each vertex v_i such that $v_i^T v_j = 0$ if $(i, j) \in E(G)$.

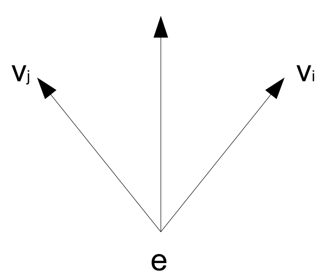

Figure 2

In figure 2, the angle is θ. Without loss of generality, we can assume that $||v_i|| = 1$. Since the vectors associated with a clique form an orthonormal basis we have

$$\sum_{v_i \in \text{clique}} (e \cdot v_i)^2 \leq 1$$

We say that an orthonormal representation is nice if $\forall v_i \in V(G), v_i \cdot e = \cos\theta$.

Thus the clique size of a graph G, $\omega(G) \leq \frac{1}{\cos^2\theta}$, where θ is the smallest angle associated with a nice orthonormal representation. It turns out that such an orthonormal representation is computable in polynomial time by semi-definite programming.

For a pentagon G, we will show that a nice orthonormal representation can be created for which $\cos\theta = 5^{\frac{-1}{4}}$.

If we have nice orthonormal representations for graph G and H, we can also construct a nice orthonormal representation for graph $G \times H$.

$$(v_{g_1} \circ v_{h_1}) \cdot (v_{g_2} \circ v_{h_2}) = v_{g_1} \cdot v_{g_2} \circ v_{h_1} \cdot v_{h_2}$$

We can see that the orthogonality is preserved.

Thus for the pentagon G, we can construct a nice representation of graph G^n for which $\cos\theta = 5^{\frac{-n}{4}}$. Thus, $\sqrt[n]{\omega(G^n)} \leq \sqrt{5}, \forall n$, which imply $\mathrm{SC}(G) \leq \sqrt{5}$.

We already know that $\mathrm{SC}(G) \leq \sqrt{5}$, so $\mathrm{SC}(G) = \sqrt{5}$. □