Lecture Outline:

- Primal-dual schema
- Primal-dual for set cover

This lecture describes the primal-dual schema, and also an application of this method to design approximation algorithms.

1 Primal-dual schema

In a previous lecture we have introduced the concept of duality for LP problems. A common form of the LP problem with its duality is showed below:

$$\begin{array}{rr|l} & \textit{Primal} & \textit{Dual} \\ & \min \sum_i c_i x_i & \max \sum_j b_j y_j \\ s.t. & \sum_i a_{ij} x_i \geq b_j & s.t. \quad \sum_j a_{ij} y_j \leq c_i \\ & x_i \geq 0 & y_j \geq 0 \end{array}$$

From the principle of duality, a feasible solution of the dual in fact sets a lower bound for the primal problem. We here use a concept of **complementary slackness** to help us illustrate the connection between the two solutions of the primal and duality problem:

Definition 1. The complementary slackness condition is as follows:

$$\begin{aligned} \textit{Primal} &: \quad x_i > 0 \Rightarrow \sum_j a_{ij} y_j = c_i \\ \textit{Dual} &: \quad y_j > 0 \Rightarrow \sum_i a_{ij} x_i = b_j \end{aligned}$$

Then the following theorem will show this connection:

Theorem 1. *If (x, y) satisfies* **complementary slackness**, *then x and y are optimal solutions for primal and dual problems, respectively.*

Proof: From the forms of the primal and duality, if complementary slackness is satisfied, we will have

$$\begin{aligned}
\sum_i c_i x_i &= \sum_i (\sum_j a_{ij} y_j) * x_i \\
\sum_j b_j y_j &= \sum_j (\sum_i a_{ij} x_i) * y_j
\end{aligned}$$

It is easy to see that the RHS of the two equations are equal. So we have

$$\sum_i c_i x_i = \sum_j b_j y_j$$

implying that x and y are both optimal by duality.

□

So if we obtain an LP solution that satifies the complementary slackness, then the solution is optimal. For integer LPs in general, however, it is unlikely that the optimal solution is integral. We apply the complementary slackness approach to approximation algorithms by defining **relaxed slackness** as follows:

Definition 2.

$$\begin{aligned}
\textit{Primal} &: \quad x_i > 0 \Rightarrow \frac{c_i}{\alpha} \le \sum_j a_{ij} y_j \le c_i \\
\textit{Dual} &: \quad y_j > 0 \Rightarrow b_j \le \sum_i a_{ij} x_i \le \beta b_j
\end{aligned}$$

And accordingly, we will fit 1 into the approximate situation with the description below:

Theorem 2. *If (x, y) satisfies* **relaxed complementary slackness**, *then x and y are $\alpha\beta$-optimal for both primal and dual problem.*

Proof: Based on the definition of relaxed complementary slackness, we will have

$$\sum_i c_i x_i \le \alpha \sum_i (\sum_j a_{ij} y_j) * x_i \tag{1}$$

$$\sum_j b_j y_j \ge \frac{1}{\beta} \sum_j (\sum_i a_{ij} x_i) * y_j \tag{2}$$

Then we can get

$$\frac{\text{cost of x}}{\text{cost of y}} = \frac{\sum_i c_i x_i}{\sum_j b_j y_j} \le \alpha\beta$$

which indicates that the duality yields a $\alpha\beta$-approximation solution. □

Based on the property of the relaxed complementary slackness, the primal-dual schema for LP solution can be done in an iterative process: start with a dual feasible solution, then we try to improve this dual problem, until the improved solution satisfies the relaxed complementary slack conditions.

2 Primal-dual for set cover

In this section we will apply the primal-dual schema to the set cover problem. Recall the primal problem of set cover with universe U of elements and collection C of sets, and its dual:

$$
\begin{array}{rlrl}
& \textit{Primal} & & \textit{Dual} \\
& \min \sum_{S \in C} c(S) x_S & & \max \sum_{e \in U} y_e \\
s.t. & \sum_{S\colon e \in S} x_S \geq 1,\ \forall e \in U & s.t. & \sum_{e \in s} y_e \leq c(S),\ \forall S \in C \\
& x_j \geq 0,\ \forall j & & y_e \geq 0,\ \forall e
\end{array}
$$

As mentioned above, we will take an iterative process to get the approximate solution. First of all, we start with a primal unfeasible but dual feasible solution

$$x = 0,\ y = 0$$

The iterative algorithm is as following:

- While any element is uncovered
  1. Pick any uncovered elements e
  2. Increase y_e until some set S become tight, i.e., $\sum_{e \in S} y_e = c(S)$.
  3. Add S to solution.

According to the definition of the relaxed complementary slackness, we will have

$$x_S > 0 \Rightarrow \sum_{e \in S} y_e = c(S) \qquad \alpha = 1 \tag{3}$$

$$y_e > 0 \Rightarrow \sum_{S\colon e \in S} x_S \leq f_e \qquad \beta = \max_e f_e \tag{4}$$

Here we let f_e be the number of the sets that contain e. So from the above analysis we see that the primal-dual algorithm will yield an f-approximate solution, where f is the maximum frequency of any element.