Lecture Outline:

- The Sparsest Cut Problem
- Previous Approximation Algorithms
- Arora-Rao-Vazirani Algorithm - $O(\sqrt{\log n})$ Approximation for the Sparsest Cut Problem
- Some Observations on High Dimension Geometry

In this lecture, we have introduced the $O(\sqrt{\log n})$ approximation algorithm for the sparsest cut problem. Good references for the material covered in this lecture include a seminal paper on graph partioning by Sanjeev Arora, Satish Rao and Umesh Vazirani [1] and lecture notes from Avner Megan's class at University of Toronto.

1 The Sparsest Cut Problem

Recall the Sparsest Cut Problem. For a graph $G = (V, E)$ and non-empty set $S \subset V$ of the vertices, $(S, \overline{S})$ denotes the set of edges of G with exactly one end vertex in S. An edge set of this form is called an *edge cut*, or cut. The *density* of an edge cut $(S, \overline{S})$ is the ratio between the number of edges that are present in the cut, and the maximum number of edges that are possible between S and $\overline{S}$: $d(S, \overline{S}) = |(S, \overline{S})|/(|S||\overline{S}|)$. An edge cut with minimum density is called a sparsest cut of the graph.

2 Previous Approximation Algorithms

- Eigenvalue approaches (Cheeger'70, Alon'85, Alon-Milman'85)
  Only yield factor n approximation.
- $O(logn)$-approximation via LP (multicommodity flows) - (Leighton-Rao'88)
- Embeddings of finite metric spaces into ℓ_1 - (Linial, London, Rabinovich'94, AR'94)
  Geometric approach; more general result (but still $O(logn)$ approximation); we studied this last lecture for the uniform case.

3 Arora-Rao-Vazirani algorithm - $O(\sqrt{\log n})$ approximation for the sparsest cut problem

The key idea underlying algorithms for graph partitioning is to spread out the vertices in some abstract space while not stretching the edges too much. Finding a good graph partition is then accomplished by partitioning this abstract space. [1]

The approach of the ARV algorithm is based on the geometry of metric spaces. In particular, this approach maps the vertices to points in a high-dimensional space such that the average square distance between vertices is a fixed constant, but the average squared distance between the end-points of edges is minimized. Furthermore, we insist that these squared distances form a metric. This embedding, which we refer to as an ℓ_2^2 representation, can be computed in polynomial time using semi-definite programming. It is shown in [1] that this ℓ_2^2 representation embeds into ℓ_1 with $O(\sqrt{\log n})$ distortion.

Definition 1. Given a graph $G = (V, E)$, the goal in the UNIFORM SPARSEST CUT problem is to determine the cut $(S, \overline{S})$ (where $|S| \leq |\overline{S}|$ without loss of generality) that minimizes

$$\frac{|E(S, \overline{S})|}{|S||\overline{S}|}$$

From previous lecture, we know that

$$\begin{aligned} & \min_{S} \frac{|E(S, \overline{S})|}{|S||\overline{S}|} \\ = & \min_{l_1} \frac{\sum_{i,j \epsilon E} d_{l_1}(i,j)}{\sum_{i,j} d_{l_1}(i,j)} \\ = & \min_{v_i \epsilon \{-1,+1\}} \frac{\sum_{i,j \epsilon E} |v_i - v_j|^2}{\sum_{i,j} |v_i - v_j|^2} \end{aligned}$$

Here

$$\sum_{i,j \epsilon E} |v_i - v_j|^2 = 4E(S, \overline{S})$$
$$\sum_{i,j} |v_i - v_j|^2 = 4|S||\overline{S}|$$

Relaxing v_i to arbitrary vector $(v_1, v_2, \cdots, v_n \in \Re^n)$, we obtain that the sparsest cut value is at least

$$\min_{v_i \epsilon \Re^m} \frac{\sum_{i,j \epsilon E} \parallel v_i - v_j \parallel_2^2}{\sum_{i,j} \parallel v_i - v_j \parallel_2^2}$$

This can be written as the following SDP:

$$\begin{aligned} & \min_{v_i \epsilon \Re^m} \sum_{i,j \epsilon E} \parallel v_i - v_j \parallel_2^2 \\ s.t. \quad & \sum_{i,j} \parallel v_i - v_j \parallel_2^2 = n^2 \end{aligned}$$

n^2 is used for simplicity, as this would make the average distance equal to 1. We can use value other than n^2 and then scale the distance accordingly.

The cut integrality gap is very large, therefore they added *triangle inequality.*

Definition 2. (ℓ_2^2 REPRESENTATION) *An ℓ_2^2 representation of a graph is an assignment of a point (vector) to each node, say v_i assigned to node i, such that for all i, j, k:*

$$|v_i - v_j|^2 + |v_j - v_k|^2 \geq |v_i - v_k|^2 \quad \textit{(triangle inequality)}$$

An ℓ_2^2 representation of a graph is called a unit-ℓ_2^2 representation if all points lie on the unit sphere (or equivalently, all vectors have unit length.)

Note that an ℓ_2^2 representation implies that all the angles in any triangle formed by these vectors are acute angles.

This SDP is then solved and the collection of points returned can then be embedded in ℓ_1 with average distortion $O(\sqrt{\log n})$. For embedding ℓ_2^2 into ℓ_1, we rely on the Master Structure Theorem.

The Master Structure Theorem: Given $v_1, v_2, \cdots, v_n \in \Re^m$, such that $\|v_i\|^2 \leq 1$ for all i, the squared distances form a metric (i.e. all the angles in any triangle formed by three points are acute)

$$\frac{1}{n^2} \sum i, j \|v_i - v_j\|^2 = \Omega(1),$$

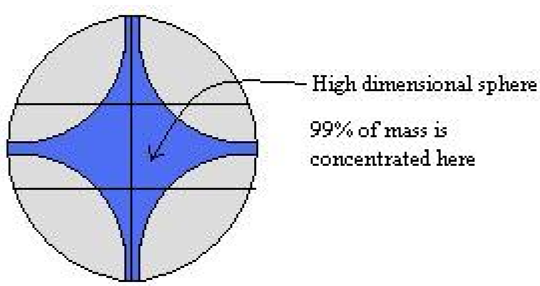
there exist S, T such that $|S| = \Omega(n)$, $|T| = \Omega(n)$ and $d(S,T) = \Omega(\frac{1}{\sqrt{\log n}})$.

A tight example of $\Omega(\frac{1}{\sqrt{\log n}})$ is a hypercube with n points in log n dimensions.

4 Some Observations on High Dimensional Geometry

- What is the maximum number of points in n-dimensions such that no three points subtend obtuse angle?
  - 2-dimensions 4 points (square)
  - 3-dimensions 8 points (cube)
  - ... ...
  - n-dimensions 2^n points
- High-Dimensional Spheres in Cubes

  In high dimensions, 99% of the mass of the sphere is concentrated at its center.

  Here is a puzzle [2]. Take a cube (in dimension n) with side length 1. Consider a sphere of radius 1/2 around each vertex of the cube. Now construct a central sphere around the center of the cube such that it touches each of the spheres that are centered at the vertices. What is the volume of this central sphere? In particular, as n increases, does this sphere form a small fraction of the cube? Or does it contain most of the cube?

Somewhat surprisingly, the central sphere's volume goes to infinity, as n increases! To see this, note that the center of the cube is distance $\sqrt{n}/2$ from each of the cube vertices. So the radius of the central sphere is $(\sqrt{n}-1)/2$, yielding a volume of infinity, as n tends to infinity. In fact, for even small values of n (greater than 9), part of the central sphere bulges outside the cube.

References

[1] Sanjeev Arora, Satish Rao and Umesh Vazirani. *Expander Flows, Geometric Embeddings and Graph Partitioning.* April 2007.

[2] http://www.math.hmc.edu/funfacts/ffiles/20007.2.shtml