1 The Sparsest Cut Problem

Recall the Sparsest Cut Problem. For a graph $G = (V, E)$ and non-empty set $S \subset V$ of the vertices, $(S, \overline{S})$ denotes the set of edges of $G$ with exactly one end vertex in $S$. An edge set of this form is called an edge cut, or cut. The density of an edge cut $(S, \overline{S})$ is the ratio between the number of edges that are present in the cut, and the maximum number of edges that are possible between $S$ and $\overline{S}$: $d(S, \overline{S}) = |(S, \overline{S})|/(|S||\overline{S}|)$. An edge cut with minimum density is called a sparsest cut of the graph.

2 Previous Approximation Algorithms

- Eigenvalue approaches (Cheeger’70, Alon’85, Alon-Milman’85) Only yield factor $n$ approximation.
- $O(\log n)$-approximation via LP (multicommodity flows) - (Leighton-Rao’88)
- Embeddings of finite metric spaces into $\ell_1$ - (Linial, London, Rabinovich’94, AR’94) Geometric approach; more general result (but still $O(\log n)$ approximation); we studied this last lecture for the uniform case.

3 Arora-Rao-Vazirani algorithm - $O(\sqrt{\log n})$ approximation for the sparsest cut problem

The key idea underlying algorithms for graph partitioning is to spread out the vertices in some abstract space while not stretching the edges too much. Finding a good graph partition is then accomplished by partitioning this abstract space. [1]
The approach of the ARV algorithm is based on the geometry of metric spaces. In particular, this approach maps the vertices to points in a high-dimensional space such that the average square distance between vertices is a fixed constant, but the average squared distance between the endpoints of edges is minimized. Furthermore, we insist that these squared distances form a metric. This embedding, which we refer to as an $\ell_2^2$ representation, can be computed in polynomial time using semi-definite programming. It is shown in [1] that this $\ell_2^2$ representation embeds into $\ell_1$ with $O(\sqrt{\log n})$ distortion.

**Definition 1.** Given a graph $G = (V, E)$, the goal in the UNIFORM SPARSEST CUT problem is to determine the cut $(S, \overline{S})$ (where $|S| \leq |\overline{S}|$ without loss of generality) that minimizes

$$\frac{|E(S, \overline{S})|}{|S||\overline{S}|}$$

From previous lecture, we know that

$$
\min_S \frac{|E(S, \overline{S})|}{|S||\overline{S}|} = \min_{l_1} \frac{\sum_{i,j \in E} d_{l_1}(i, j)}{\sum_{i,j} d_{l_1}(i, j)} = \min_{v_i \in \{-1, +1\}} \frac{\sum_{i,j \in E} |v_i - v_j|^2}{\sum_{i,j} |v_i - v_j|^2}
$$

Here

$$
\sum_{i,j \in E} |v_i - v_j|^2 = 4E(S, \overline{S})
$$

$$
\sum_{i,j} |v_i - v_j|^2 = 4|S||\overline{S}|
$$

Relaxing $v_i$ to arbitrary vector $(v_1, v_2, \cdots, v_n \in \mathbb{R}^n)$, we obtain that the sparsest cut value is at least

$$
\min_{v_i \in \mathbb{R}^n} \frac{\sum_{i,j \in E} \|v_i - v_j\|^2_2}{\sum_{i,j} \|v_i - v_j\|^2_2}
$$

This can be written as the following SDP:

$$
\min_{v_i \in \mathbb{R}^n} \sum_{i,j \in E} \|v_i - v_j\|^2_2
$$

$$
s.t. \sum_{i,j} \|v_i - v_j\|^2_2 = n^2
$$

$n^2$ is used for simplicity, as this would make the average distance equal to 1. We can use value other than $n^2$ and then scale the distance accordingly.

The cut integrality gap is very large, therefore they added *triangle inequality*. 
**Definition 2.** ($\ell_2^2$ REPRESENTATION) An $\ell_2^2$ representation of a graph is an assignment of a point (vector) to each node, say $v_i$ assigned to node $i$, such that for all $i$, $j$, $k$:

$$|v_i - v_j|^2 + |v_j - v_k|^2 \geq |v_i - v_k|^2$$

(triangle inequality)

An $\ell_2^2$ representation of a graph is called a unit-$\ell_2^2$ representation if all points lie on the unit sphere (or equivalently, all vectors have unit length.)

Note that an $\ell_2^2$ representation implies that all the angles in any triangle formed by these vectors are acute angles.

This SDP is then solved and the collection of points returned can then be embedded in $\ell_1$ with average distortion $O(\sqrt{\log n})$. For embedding $\ell_2^2$ into $\ell_1$, we rely on the Master Structure Theorem.

**The Master Structure Theorem:** Given $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$, such that $\|v_i\|^2 \leq 1$ for all $i$, the squared distances form a metric (i.e. all the angles in any triangle formed by three points are acute)

$$\frac{1}{n^2} \sum_{i,j} \|v_i - v_j\|^2 = \Omega(1),$$

there exist $S, T$ such that $|S| = \Omega(n)$, $|T| = \Omega(n)$ and $d(S, T) = \Omega(\frac{1}{\sqrt{\log n}})$.

A tight example of $\Omega(\frac{1}{\sqrt{\log n}})$ is a hypercube with $n$ points in log $n$ dimensions.

4 Some Observations on High Dimensional Geometry

• What is the maximum number of points in $n$-dimensions such that no three points sub tend obtuse angle?
  – 2-dimensions 4 points (square)
  – 3-dimensions 8 points (cube)
  – ...
  – n-dimensions $2^n$ points

• High-Dimensional Spheres in Cubes

In high dimensions, 99% of the mass of the sphere is concentrated at its center.

Here is a puzzle [2]. Take a cube (in dimension $n$) with side length 1. Consider a sphere of radius $1/2$ around each vertex of the cube. Now construct a central sphere around the center of the cube such that it touches each of the spheres that are centered at the vertices. What
is the volume of this central sphere? In particular, as $n$ increases, does this sphere form a small fraction of the cube? Or does it contain most of the cube?

Somewhat surprisingly, the central sphere's volume goes to infinity, as $n$ increases! To see this, note that the center of the cube is distance $\sqrt{n}/2$ from each of the cube vertices. So the radius of the central sphere is $(\sqrt{n} - 1)/2$, yielding a volume of infinity, as $n$ tends to infinity. In fact, for even small values of $n$ (greater than 9), part of the central sphere bulges outside the cube.

References
